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SOME INTERPOLATORY RULES FOR THE APPROXIMATIVE EVALUATION OF COMPLEX CAUCHY PRINCIPAL VALUE INTEGRALS

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ABSTRACT

For the numerical approximation of complex Cauchy principal value integral

$$\fint_{z_0-h}^{z_0+h} \frac{f(z)}{z-\zeta}\,dz$$

along directed line segment from z_0-h to z_0+h, where ζ is an interior point on the path of integration, some interpolatory rules have been constructed. The asymptotic error estimates for the rules have been derived and the rules have been numerically tested.

1. INTRODUCTION

Singular integrals of the Cauchy type occur abundantly

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in applied mathematics, particularly in the theory of aerodynamics and in scattering theory. There has been substantial research work for the numerical approximation of the real Cauchy principal value (CPV) integral of the type

$$I(g,a) = \fint_{-1}^{1} (x - a)^{-1} g(x)dx,$$

where $a \in (-1,1)$ and g is a continuous function on $[-1,1]$. Some of the popular methods for the numerical evaluation of the real CPV integral $I(g,a)$ are due to Price [12] (see also [4, p. 149]), Hunter [6], Chawla and Jayarajan [3], Paget and Elliott [11], Ioakimidis and Theocaris [7], Elliot and Paget [5], Theocaris and Kazantzakis [13], Monegato [10], etc.

Complex CPV integrals of the type

$$\text{(1)} \qquad J(f,\zeta) = \fint_{z_0-h}^{z_0+h} f(z)/(z-\zeta)dz$$

along the directed line segment L, from the point z_0-h to z_0+h containing the point ζ as an interior point, where f is an analytic function in a domain Ω containing L, occur very often in contour integration, which, in turn, is an essential tool in applied mathematics. As far as it is known, the numerical evaluation of the complex CPV integral $J(f,\zeta)$ has not received sufficient attention. Only recently, Acharya and Das [1] epmloying the transformation $z \to z_0 + th$, $t \in [-1,1]$, have developed some transformed rules analogous to the ones in [12] for the numerical approximation of the complex CPV integral $J(f,\zeta)$. It may be pointed out that using this transformation we could have also tran transformed rules analogous to the rules in [6] and [3].

The object of this paper is to obtain interpolatory rules for the numerical approximation of the complex CPV integral $J(f,\zeta)$ given by (1), so that the points ζ, z_0, $z_0+i^{m-1}kh$, $m = 1(1)4$, are chosen as the points of interpolation, where k is a real parameter in $(0,1]$ and $i = \sqrt{-1}$. It is noteworthy here that Tošić [14], using the points z_0, $z_0+i^{m-1}kh$, $m = 1(1)4$, has formulated a general rule R_G given by

(2) $$I = \int_L \phi(z)dz \approx \tilde{R}_G(\phi) = h[2(1 - 1/5k^4)\phi(z_0) + (1/6k^2 + 1/10k_0^4)(\phi(z_0+kh) + \phi(z_0 - kh)) + (-1/6k^2 + 1/10k^4)(\phi(z_0 + ikh) + \phi(z_0 - ikh))]$$

for the approximation of the contour integral I. Giving the values $k = 1$, $\sqrt{0.6}$ and $(3/7)^{1/4}$, the Birkhoff-Young rule [2], Gauss--Legendre rule [8] and the modified rule due to Tošić [14] for the approximation of the contour integral I given by (2), are obtained.

2. FORMULATION OF THE RULES

Let the point ζ be an interior point of L. The following two cases: (i) $\zeta \neq z_0$; (ii) $= z$ deserve separate treatment.

(i) Case $\zeta \neq z_0$. Let k be a real parameter such that $k \in (0,1]$ and $z_0 \pm kh \neq \zeta$. If $P_5(z)$ is the Lagrange polynomial of degree ≤ 5, interpolating to f at ζ, z_0, $z_m = z_0 + i^{m-1}kh$ $(m = 1(1)4)$, then we have

$$J(f,\zeta) \approx Q(f,\zeta) = \sum_{m=0}^{4} \frac{f_m}{z_m - } \oint_L \ell_m(z)dz + f(\zeta) \oint_L \frac{\psi(z)}{z-\zeta}\, dz,$$

where

(3) $$\begin{cases} f_0 = f(z_0),\ f_m = f(z_0 + i^{m-1}kh),\ m = 1(1)4; \\ \ell_m(z) = \prod\limits_{\substack{j=0\\ j\neq m}}^{4} (z - z_j) \Big/ \prod\limits_{\substack{j=0\\ j\neq m}}^{4} (z_m - z_j); \\ \psi(z) = \prod\limits_{j=0}^{4} (z - z_j) \Big/ \prod\limits_{j=0}^{4} (\zeta - z_j). \end{cases}$$

As each of $\ell_m(z)$ is analytic in $\mathbb{C}$

(4) $$\oint_L \ell_m(z)dz = \int_L \ell_m(z)dz.$$

Setting the expressions for $\ell_m(z)$ given in (3) and (4), we evaluate each of the integrals. Lastly, using the definition of CPV integral and setting the expression for $\psi(z)$ given in (3), we evaluate $\int_L (\psi(z)/(z-\zeta))dz$. Hence the rule $Q(\cdot,\zeta)$ for approximating the complex CPV integral $J(f,\zeta)$ is given by

$$(5)\qquad J(f,\zeta) \approx Q(f,\zeta) = Cf(\zeta) + 2h(1 - 1/5k^4)\frac{f_0}{z_0-\zeta} +$$
$$+ h\left[\left(\frac{1}{10k^4} + \frac{1}{6k^2}\right)\left(\frac{f_2}{z_1-\zeta} + \frac{f_3}{z_3-\zeta}\right) + \left(\frac{1}{10k^4} - \frac{1}{6k^2}\right)\left(\frac{f_2}{z_2-\zeta} + \frac{f_4}{z_4-\zeta}\right)\right],$$

where

$$\nu = (\zeta - z_0)/h, \quad C = \ln\left\{\frac{(1-\nu)}{(1+\nu)}\right\} + \frac{2}{\nu} + 2\left(\frac{1}{5\nu} + \frac{\nu}{3}\right)/(\nu^4 - k^4).$$

However, if $z_0 + kh = z_1 = \zeta$ (or $z_0 - kh = z_3 = \zeta$) we use the osculatory interpolation by the Hermite polynomial $H_5(x)$, interpolating to f at z_0, z_1, z_1, z_2, z_3, z_4 (or z_0, z_1, z_2, z_3, z_3, z_4). Then we obtain

$$(6)\qquad J(f,z_1) \approx \bar{Q}(f,z_1) = h\left(\frac{1}{10k^4} + \frac{1}{6k^2}\right)f_1' + 2\left(\frac{1}{5k^5} - \frac{1}{k}\right)f_0 +$$
$$+ \left(\frac{2}{k} - \frac{1}{4k^5} - \frac{1}{12k^3} + \ln\frac{1-k}{1+k}\right)f_1 + \frac{1+i}{4k^3}\left(\frac{1}{3} - \frac{1}{5k^2}\right)f_2 -$$
$$- \frac{1}{4k^3}\left(\frac{1}{5k^2} + \frac{1}{3}\right)f_3 + \frac{1-i}{4k^3}\left(\frac{1}{3} - \frac{1}{5k^2}\right)f_4.$$

If we put $S(f;k) = \bar{Q}(f, z_0 + kh)$ then for $z_0 - kh = z_3 = \zeta$, we have $J(f,z_3) \approx S(f;-k)$.

(ii) Case $\zeta = z_0$. Using the osculatory interpolation by the Hermite polynomial $H_5(x)$, interpolating to f at z_0, z_0, z_1, z_2, z_3, z_4, we have the following rule:

$$(7)\qquad J(f,z_0) \approx \tilde{Q}(f,z_0) = 2f_0'\left(1 - \frac{1}{5k^4}\right)h + \left(\frac{1}{10k^5} + \frac{1}{6k^3}\right)(f_1 - f_3)$$
$$- i\left(\frac{1}{10k^5} - \frac{1}{6k^3}\right)(f_2 - f_4).$$

The rule $Q(\cdot, z_0)$, given by (7), is not derivative free.

Following Tošić [15] and defining the operator of the finite central difference δ_θ as

$$\delta_\theta f(z_0) = f(z_0 + k|h|e^{i\theta}) - f(z_0 - k|h|e^{i\theta}),$$

we have the following approximation for the derivative $f'(z_0)$ which utilizes four function evaluations:

$$f_0' \approx \frac{1}{4kh}(f_1 - f_3) - \frac{i}{4kh}(f_2 - f_4),$$

where f_m, $m = 1(1)4$, are given by (3).

Using the approximation for f_0' in formula (7), we have the following derivative free rule for the numerical approximation for the complex CPV integral

(8) $$J(f,z_0) \approx \tilde{Q}(f,z_0) = \frac{1}{2k}\Big[\Big(1 + \frac{1}{3k^2}\Big)(f_1 - f_3) - i\Big(1 - \frac{1}{3k^2}\Big)(f_2 - f_4)\Big].$$

It may be pointed out that if the point ζ lies outside L and $\zeta \neq z_0 \pm ikh$, then the CPV integral $J(f,\zeta)$ reduces to the definite integral I given by (2), where $\phi(z) = f(z)/(z-\zeta)$. In this case dropping the term $Cf(\zeta)$ in (5) we note that rule $Q(f,\zeta)$ boils down to the rule given by Tošić [14] for the numerical evaluation of the integral $\int_F \phi(z)dz$. Hence, we may regard the rule $Q(\cdot,\zeta)$, given by (5), as a modification of the rule due to Tošić [14].

3. ERROR ANALYSIS

Let

(9) $$E(f,\zeta) = Q(f,\zeta) - J(f,\zeta) \text{ and } \tilde{E}(f,z_0) = Q(f,z_0) - J(f,z_0)$$

be the errors associated with the rules $Q(f,\zeta)$ and $\tilde{Q}(f,z_0)$, respectively.

To find the expression giving $E(f,\zeta)$, we assume f to be analytic in the disc

$$\Omega_R = \{z: |z-\zeta| < R\},$$

where R is a number satisfying

$$R > \max\{|z_0 + h - \zeta|, |z_0 - h - \zeta|\}.$$

So the points of interpolation, as well as the points $z_0 \pm h$, are interior to the disc Ω_R. Using Taylor's expansion

$$f(z) = \sum_{n=0}^{\infty} a_n (z-\zeta)^n, \qquad a_n = \frac{1}{n!} f^{(n)}(\xi),$$

in (9) and, simplifying, we obtain

$$(10) \qquad E(f,\zeta) = \frac{2}{3}h^7\{a_7(k^4 - \frac{3}{7}) + a_8 h\nu(3 - 7k^4) +$$

$$+ a_9 h^2(\frac{3}{5}k^4 + 28k^4\nu^2 - \frac{1}{3} - 12\nu^2) + \ldots\}.$$

Form (10) we have the following:

THEOREM 1. *If* $k \neq \nu$ *and* f *is analytic in a certain domain* $D \subset L$, *then* $E(f,\zeta) = O(h^7)$ *except for* $k = (3/7)^{1/4}$ *when* $E(f,\zeta) = O(h^9)$.

We then derive the expressions for the error $\tilde{E}(f,z_0)$ given in (9). For this, it is assumed that f is analytic in the disc

$$\Omega_{R'} = \{z: |z - z_0| < R'\},$$

where $R' > |h|$. Using the Taylor series expression

$$(11) \qquad f(z) = \sum_{n=0}^{\infty} \tilde{a}_n (z - z_0)^n, \quad \tilde{a}_n = \frac{1}{n!} f^{(n)}(z_0),$$

in (9), and simplifying, we obtain

$$\tilde{E}(f,z_0) = h^7\{\tfrac{2}{3}\tilde{a}_7(k^4 - \tfrac{3}{7}) + \tfrac{2}{3}(\tfrac{3}{5}k^4 - \tfrac{1}{3})\tilde{a}_9h^2 + \ldots\}.$$

From (11) we have the following:

THEOREM 2. *If* f *is analytic in the disc*

$$\Omega_{R'} = \{z: |z - z_0| < R', R' > |h|\},$$

then $\tilde{E}(f,z_0) = O(h^7)$ *except for* $k = (3/7)^{1/4}$ *when* $\tilde{E}(f,z_0) = O(h^9)$.

We shall finally discuss the error $\tilde{E}'(f,z_0)$ given by

$$\tilde{E}'(f,z_0) = \tilde{Q}'(f,z_0) - J(f,z_0), \tag{12}$$

where $\tilde{Q}'(\cdot,z_0)$ is the rule given by (7). For this, we also assume f to be analytic in the disc $\Omega_{R'}$. Using the Taylor expansion given by (11), expression (12) becomes

$$\tilde{E}'(f,z_0) = h^5\{2\tilde{a}_5(\tfrac{1}{5} - k^4) + \tfrac{2}{3}\tilde{a}_7(\tfrac{3}{7} - k^4)h^4 + \ldots\}. \tag{13}$$

So now we have the following:

THEOREM 3. *If* f *is analytic in the disc* $\Omega_{R'}$ *, then* $\tilde{E}'(f,z_0) = O(h^5)$ *except for* $k = (1/5)^{1/4}$, *when* $\tilde{E}'(f,z_0) = O(h^7)$.

4. NUMERICAL RESULTS

For numerical verification of the above obtained rules we shall consider the following CPV integral

$$J(e^z,\zeta) = ⨍_{-i}^{i} \frac{e^z}{(z - \zeta)}\,dz, \tag{14}$$

for $\zeta = i/4$, 0 and 1.1i. The exact value of (14) is given by

$$J(e^z,\zeta) = K(\nu) = C(\nu) + iS(\nu),$$

where

$$C(\nu) = \fint_{-1}^{1} \frac{\cos t}{t-\nu}\,dt\,, \qquad S(\nu) = \fint_{-1}^{1} \frac{\sin t}{t-\nu}\,dt$$

and $\zeta = i\nu$. Using the sine and cosine integrals we obtain

$$C(\nu) = -\,(\mathrm{Si}(1+\nu) + \mathrm{Si}(1-\nu))\sin\nu\, -$$

$$-\,(\mathrm{Ci}(1+\nu) - \mathrm{Ci}(1-\nu))\cos\nu$$

and

$$S(\nu) = (\mathrm{Si}(1+\nu) + \mathrm{Si}(1-\nu))\cos\nu\, -$$

$$-\,(\mathrm{Ci}(1+\nu) - \mathrm{Ci}(1-\nu))\sin\nu .$$

So we have

$$K(1/4) = -\,0.7368529056\,\ldots + i1.7453593365\ldots,$$

$$K(0) = i1.8921661460\ldots,$$

$$K(1.1) = -\,2.3456862027\ldots - i1.1943193677\ldots\,.$$

Functions Si and Ci were evaluated by means of the Chebyshev expansion from Luke [9, Ch. 4.7].

All the calculations were performed in D-arithmetic on a PDP 11/40 computer.

Table 1 depicts that, as expected, the rule $Q(\cdot,\zeta)$ yields the most accurate value when $k = (3/7)^{1/4}$ (Numbers in parentheses, in the third column, indicate decimal exponents). It is evident from (10) that the algebraic precision of the rule $Q(\cdot,\zeta)$ is 6, for all values of k except for $k = (3/7)^{1/4}$ when it becomes 8. This is because setting $k = (3/7)^{1/4}$ in (10), the first non-vanishing term, is $-\,16/315\, a_9 h^9$, and as a result the rule $Q(\cdot,\zeta)$ becomes exact for all polynomials up to 8. The value for k = 0,25 was obtained by means of formula (6).

Similarly, the algebraic degree of the rule $\tilde{Q}(\cdot,z_0)$ is 6 for all values of k, except for $k = (3/7)^{1/4}$, when it is 8. In view of this, it is observed from Table 2 that the rule $Q(\cdot,z_0)$ corresponding to the value $k = (3/7)^{1/4}$ is most accurate. Here $z_0 = 0$.

Table 1

k	$Q(e^z, i/4)$	$\lvert E(e^z, i/4)\rvert$
0.1	− 0.73685466 + i1.74541535	5.60(−5)
0.25	− 0.73685464 + i1.74541485	5.55(−5)
0.50	− 0.73685440 + i1.74540717	4.79(−5)
$(0.6)^{1/2}$	− 0.73685318 + i1.74536818	8.85(−6)
$(3/7)^{1/4}$	− 0.73685290 + i1.74535919	1.43(−7)
1.00	− 0.73685056 + i1.74528429	7.51(−5)

Table 2

k	$\tilde{Q}(e^z,0)$	$\lvert\tilde{E}(e^z,0)\rvert$	$\tilde{Q}'(e^z,0)$	$\lvert\tilde{E}'(e^z,0)\rvert$
0.1	i1.89222221	5.61(−5)	i1.888891	3.28(−3)
0.5	i1.89221402	4.79(−5)	i1.889922	2.24(−3)
$(1/5)^{1/4}$	i1.89219599	2.98(−5)	i1.892196	2.98(−5)
$(0.6)^{1/2}$	i1.89217500	8.85(−6)	i1.894842	2.68(−3)
$(3/7)^{1/4}$	i1.89216600	1.44(−7)	i1.895976	3.81(−3)
1.0	i1.89209103	7.51(−5)	i1.905429	1.33(−2)

However, it is observed from (13) that the algebraic degree of the derivative free rule $\tilde{Q}'(\cdot,z_0)$ is 4 (even when $k = (3/7)^{1/4}$), except for $k = (1/5)^{1/4}$, when it is 6. Thus we notice that the derivative free rule $\tilde{Q}'(\cdot,z_0)$ is most accurate when the parameter $k = (1/5)^{1/4}$. This fact is also observed from Table 2.

The definite integral (14), for $\zeta = 1.1i$, has been evaluated by rule (5) and by the modified formula $\tilde{R}_G$, given by (2), for $k = 0.1$, 0.5, $\sqrt{0.6}$, $(3/7)^{1/4}$ and 1. The corresponding absolute errors are displayed in Table 3. It is observed that rule (2) is unsuitable if the singularity of ϕ (which is $z = \zeta$ in the present case). is near the path integration L. It is evident, from Table 3, that in such cases rule (5) is preferable to rule (2).

Table 3

k	Rule (5)	Rule (2)
0.1	5.57(-5)	4.47(-1)
0.5	4.75(-5)	4.44(-1)
$(0.6)^{1/2}$	8.79(-6)	2.33(-1)
$(3/7)^{1/4}$	1.38(-7)	1.67(-1)
1.0	7.46(-5)	1.14(0)

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REZIME

NEKA INTERPOLACIONA PRAVILA ZA PRIBLIŽNO IZRAČUNAVANJE KOMPLEKSNIH KOŠIJEVIH GLAVNIH VREDNOSTI INTEGRALA

Data su neka interpolaciona pravila za numeričku aproksimaciju kompleksne Košijeve glavne vrednosti integrala

$$\int_{z_0-h}^{z_0+h} \frac{f(z)}{z-\zeta}\, dz$$

duž linijskog segmenta od z_0-h do z_0+h, čija je unutrašnja tačka ζ . Izvedena je asimptotska ocena greške za data pravila a pravila su numerički testirana.