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On an optimal quadrature formula in the sense of Sard

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Abstract In this paper we construct an optimal quadrature formula in the sense of Sard in the Hilbert space $K_2(P_2)$. Using S.L. Sobolev's method we obtain new optimal quadrature formula of such type and give explicit expressions for the corresponding optimal coefficients. Furthermore, we investigate order of the convergence of the optimal formula and prove an asymptotic optimality of such a formula in the Sobolev space $L_2^{(2)}(0,1)$. The obtained optimal quadrature formula is exact for the trigonometric functions $\sin x$ and $\cos x$. Also, we include a few numerical examples in order to illustrate the application of the obtained optimal quadrature formula.

Keywords Optimal quadrature formulas · Error functional · Extremal function · Hilbert space · Optimal coefficients

Mathematics Subject Classification (2000) MSC 65D32

1 Introduction and preliminaries

We consider the following quadrature formula

$$\int_0^1 \varphi(x)\,\mathrm{d}x \cong \sum_{\nu=0}^{N} C_\nu \varphi(x_\nu), \tag{1.1}$$

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with an error functional given by

$$\ell(x) = \chi_{[0,1]}(x) - \sum_{\nu=0}^{N} C_\nu \delta(x - x_\nu), \tag{1.2}$$

where C_ν and x_ν ($\in [0,1]$) are coefficients and nodes of the formula (1.1), respectively, $\chi_{[0,1]}(x)$ is the characteristic function of the interval $[0,1]$, and $\delta(x)$ is Dirac's delta-function. We suppose that the functions $\varphi(x)$ belong to the Hilbert space

$$K_2(P_2) = \{\varphi : [0,1] \to \mathbb{R} \mid \varphi' \text{ is absolutely continuous and } \varphi'' \in L_2(0,1)\},$$

equipped with the norm

$$\|\varphi\,|K_2(P_2)\| = \left\{ \int_0^1 \left(P_2\left(\frac{\mathrm{d}}{\mathrm{d}x}\right) \varphi(x) \right)^2 \mathrm{d}x \right\}^{1/2}, \tag{1.3}$$

where $P_2\left(\frac{\mathrm{d}}{\mathrm{d}x}\right) = \frac{\mathrm{d}^2}{\mathrm{d}x^2} + 1$ and $\int_0^1 \left(P_2\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)\varphi(x)\right)^2 \mathrm{d}x < \infty$.
The equality (1.3) is semi-norm and $\|\varphi\| = 0$ if and only if $\varphi(x) = c_1 \sin x + c_2 \cos x$.

It should be noted that for a linear differential operator of order n, $L \equiv P_n(\mathrm{d}/\mathrm{d}x)$, Ahlberg, Nilson, and Walsh in the book [1, Chapter 6] investigated the Hilbert spaces in the context of generalized splines. Namely, with the inner product

$$\langle \varphi, \psi \rangle = \int_0^1 L\varphi(x) \cdot L\psi(x)\,\mathrm{d}x,$$

$K_2(P_n)$ is a Hilbert space if we identify functions that differ by a solution of $L\varphi = 0$. Also, such a type of spaces of periodic functions and optimal quadrature formulae were discussed in [6].

The corresponding error of the quadrature formula (1.1) can be expressed in the form

$$R_N(\varphi) = \int_0^1 \varphi(x)\,\mathrm{d}x - \sum_{\nu=0}^{N} C_\nu \varphi(x_\nu) = (\ell, \varphi) = \int_{\mathbb{R}} \ell(x)\varphi(x)\,\mathrm{d}x \tag{1.4}$$

and it is a linear functional in the conjugate space $K_2^*(P_2)$ to the space $K_2(P_2)$.

By the Cauchy-Schwarz inequality

$$|(\ell, \varphi)| \le \|\varphi\,|K_2(P_2)\| \cdot \|\ell\,|K_2^*(P_2)\|$$

the error (1.4) can be estimated by the norm of the error functional (1.2), i.e.,

$$\left\|\ell\,|K_2^*(P_2)\right\| = \sup_{\|\varphi\,|K_2(P_2)\|=1} |(\ell, \varphi)|\,.$$

In this way, the error estimate of the quadrature formula (1.1) on the space $K_2(P_2)$ can be reduced to finding a norm of the error functional $\ell(x)$ in the conjugate space $K_2^*(P_2)$.

Obviously, this norm of the error functional $\ell(x)$ depends on the coefficients C_ν and the nodes x_ν, $\nu = 0, 1, \ldots, N$. The problem of finding the minimal norm of the error functional $\ell(x)$ with respect to coefficients C_ν and nodes x_ν is called as *Nikolskii problem*, and the obtained formula is called *optimal quadrature formula in the sense of Nikolskii*. This problem first considered by S.M. Nikolskii [15], and continued by many authors (see e.g. [3–6, 16, 32] and references therein). A minimization of the norm of the error functional $\ell(x)$ with respect only to coefficients C_ν, when nodes are fixed, is called as *Sard's problem*. The obtained formula is called the *optimal quadrature formula in the sense of Sard*. This problem was first investigated by A. Sard [18].

There are several methods of construction of optimal quadrature formulas in the sense of Sard (see e.g. [3,26]). In the space $L_2^{(m)}(a,b)$, based on these methods, Sard's problem was investigated by many authors (see, for example, [2,3,5,7–9,12–14,19,20, 22–28,30,31] and references therein). Here, $L_2^{(m)}(a,b)$ is the Sobolev space of functions, with a square integrable m-th generalized derivative.

It should be noted that a construction of optimal quadrature formulas in the sense of Sard, which are exact for solutions of linear differential equations, was given in [9, 13], using the Peano kernel method, including several examples for some number of nodes.

An optimal quadrature formula in the sense of Sard was constructed in [21], using Sobolev's method in the space $W_2^{(m,m-1)}(0,1)$, with the norm defined by

$$\left\|\varphi\,|W_2^{(m,m-1)}(0,1)\right\| = \left\{\int_0^1 \left(\varphi^{(m)}(x)+\varphi^{(m-1)}(x)\right)^2 \mathrm{d}x\right\}^{1/2}.$$

In this paper we give the solution of Sard's problem in the space $K_2(P_2)$, using Sobolev's method for an arbitrary number of nodes N. Namely, we find the coefficients C_ν (and the error functional $\mathring{\ell}$) such that

$$\left\|\mathring{\ell}\,|K_2^*(P_2)\right\| = \inf_{C_\nu}\left\|\ell\,|K_2^*(P_2)\right\|. \tag{1.5}$$

Thus, in order to construct an optimal quadrature formula in the sense of Sard in $K_2(P_2)$, we need consequently to solve the following two problems:

Problem 1. *Calculate the norm of the error functional $\ell(x)$ for the given quadrature formula* (1.1).

Problem 2. *Find such values of the coefficients C_ν such that the equality* (1.5) *be satisfied with fixed nodes x_ν.*

The paper is organized as follows. In Section 2 we determine the extremal function which corresponds to the error functional $\ell(x)$ and give a representation of the norm of the error functional (1.2). Section 3 is devoted to a minimization of $\|\ell\|^2$ with respect to the coefficients C_ν. We obtain a system of linear equations for the coefficients of the optimal quadrature formula in the sense of Sard in the space $K_2(P_2)$. Moreover, the existence and uniqueness of the corresponding solution is proved. Explicit formulas for coefficients of the optimal quadrature formula of the form (1.1) are found in Section 4. In Section 5 we calculate the norm of the error functional (1.2) of the optimal quadrature formula (1.1). Furthermore, we give an asymptotic analysis of this norm. Finally, in Section 6 some numerical results are presented.

2 The extremal function and representation of the error functional $\ell(x)$

In order to solve Problem 1, i.e., to calculate the norm of the error functional (1.2) in the space $K_2^*(P_2)$, we use a concept of the extremal function for a given functional. The function $\psi_\ell(x)$ is called the *extremal* for the functional $\ell(x)$ (cf.[27]) if the following equality is fulfilled

$$(\ell,\psi_\ell) = \left\|\ell\,|K_2^*(P_2)\right\|\cdot\left\|\psi_\ell\,|K_2(P_2)\right\|. \tag{2.1}$$

Since $K_2(P_2)$ is a Hilbert space, the extremal function $\psi_\ell(x)$ in this space can be found using the Riesz theorem about general form of a linear continuous functional on Hilbert spaces. Then, for the functional $\ell(x)$ and for any $\varphi \in K_2(P_2)$ there exists such a function $\psi_\ell \in K_2(P_2)$, for which the following equality

$$(\ell, \varphi) = \langle \psi_\ell, \varphi \rangle \tag{2.2}$$

holds, where

$$\langle \psi_\ell, \varphi \rangle = \int_0^1 \left(\psi_\ell''(x) + \psi_\ell(x) \right) \left(\varphi''(x) + \varphi(x) \right) \, \mathrm{d}x \tag{2.3}$$

is an inner product defined on the space $K_2(P_2)$.

Further, we will investigate the solution of the equation (2.2).

Let first $\varphi \in \mathring{C}^{(\infty)}(0,1)$, where $\mathring{C}^{(\infty)}(0,1)$ is a space of infinity-differentiable and finite functions in the interval $(0,1)$. Then from (2.3), an integration by parts gives

$$\langle \psi_\ell, \varphi \rangle = \int_0^1 \left(\psi_\ell^{(4)}(x) + 2\psi_\ell''(x) + \psi_\ell(x) \right) \varphi(x) \, \mathrm{d}x. \tag{2.4}$$

According to (2.2) and (2.4) we conclude that

$$\psi_\ell^{(4)}(x) + 2\psi_\ell''(x) + \psi_\ell(x) = \ell(x). \tag{2.5}$$

Thus, when $\varphi \in \mathring{C}^{(\infty)}(0,1)$ the extremal function $\psi_\ell(x)$ is a solution of the equation (2.5). But, we have to find the solution of (2.2) when $\varphi \in K_2(P_2)$.

Since the space $\mathring{C}^{(\infty)}(0,1)$ is dense in $K_2(P_2)$, then functions from $K_2(P_2)$ can be uniformly approximated as closely as desired by functions from the space $\mathring{C}^{(\infty)}(0,1)$. For $\varphi \in K_2(P_2)$ we consider the inner product $\langle \psi_\ell, \varphi \rangle$. Now, an integration by parts gives

$$\begin{aligned} \langle \psi_\ell, \varphi \rangle = &\left(\psi_\ell''(x) + \psi_\ell(x) \right) \varphi'(x) \Big|_0^1 - \left(\psi_\ell'''(x) + \psi_\ell'(x) \right) \varphi(x) \Big|_0^1 \\ &+ \int_0^1 \left(\psi_\ell^{(4)}(x) + 2\psi_\ell''(x) + \psi_\ell(x) \right) \varphi(x) \, \mathrm{d}x. \end{aligned}$$

Hence, taking into account arbitrariness $\varphi(x)$ and uniqueness of the function $\psi_\ell(x)$ (up to functions $\sin x$ and $\cos x$), taking into account (2.5), it must be fulfilled the following equation

$$\psi_\ell^{(4)}(x) + 2\psi_\ell''(x) + \psi_\ell(x) = \ell(x), \tag{2.6}$$

with boundary conditions

$$\psi_\ell''(0) + \psi_\ell(0) = 0, \quad \psi_\ell''(1) + \psi_\ell(1) = 0, \tag{2.7}$$

$$\psi_\ell'''(0) + \psi_\ell'(0) = 0, \quad \psi_\ell'''(1) + \psi_\ell'(1) = 0. \tag{2.8}$$

Thus, we conclude, that the extremal function $\psi_\ell(x)$ is a solution of the boundary value problem (2.6)–(2.8).

Taking the convolution of two functions f and g, i.e.,

$$(f * g)(x) = \int_{\mathbb{R}} f(x-y) g(y) \, \mathrm{d}y = \int_{\mathbb{R}} f(y) g(x-y) \, \mathrm{d}y, \tag{2.9}$$

we can state the following result:

Theorem 2.1 *The solution of the boundary value problem* $(2.6)-(2.8)$ *is the extremal function* $\psi_\ell(x)$ *of the error functional* $\ell(x)$ *and it has the following form*

$$\psi_\ell(x) = (G * \ell)(x) + d_1 \sin x + d_2 \cos x,$$

where d_1 *and* d_2 *are arbitrary real numbers, and*

$$G(x) = \frac{1}{4}\,\mathrm{sign}\, x\,(\sin x - x\cos x) \tag{2.10}$$

is the solution of the equation

$$\psi_\ell^{(4)}(x) + 2\psi_\ell''(x) + \psi_\ell(x) = \delta(x). \tag{2.11}$$

Proof It is known that the general solution of a non-homogeneous differential equation can be represented as a sum of its particular solution and the general solution of the corresponding homogeneous equation. In our case, the general solution of the corresponding homogeneous equation for (2.6) is given by

$$\psi_\ell^h(x) = d_1 \sin x + d_2 \cos x + d_3 x \sin x + d_4 x \cos x, \tag{2.12}$$

where d_k, $k = 1, 2, 3, 4$, are arbitrary constants. It is not difficult to verify that a particular solution of the differential equation (2.6) can be expressed as a convolution of the functions $\ell(x)$ and $G(x)$ defined by (2.9). The function $G(x)$ is the fundamental solution of the equation (2.6), and it is determined by (2.10).

According to the general rule for finding fundamental solutions of a linear differential operators (cf. [29, p. 88]), in our case for the operator $\frac{\mathrm{d}^4}{\mathrm{d}x^4} + 2\frac{\mathrm{d}^2}{\mathrm{d}x^2} + 1$ we get (2.10)

Thus, we have the following general solution of the equation (2.6)

$$\psi_\ell(x) = (\ell * G)(x) + d_1 \sin x + d_2 \cos x + d_3 x \sin x + d_4 x \cos x. \tag{2.13}$$

In order that in the space $K_2(P_2)$ the function $\psi_\ell(x)$ will be unique (up to functions $\sin x$ and $\cos x$), it has to satisfy the conditions (2.7) and (2.8), where derivatives are taken in a generalized sense. In computations we need first three derivatives of the function $G(x)$:

$$G'(x) = \frac{\mathrm{sign}\, x}{4}\, x\, \sin x,\ \ G''(x) = \frac{\mathrm{sign}\, x}{4}(\sin x + x\, \cos x),\ \ G'''(x) = \frac{\mathrm{sign}\, x}{4}(2\cos x - x\, \sin x),$$

where we used the following formulas from the theory of generalized functions [29],

$$(\mathrm{sign}\, x)' = 2\delta(x), \quad \delta(x)f(x) = \delta(x)f(0).$$

Further, using the well-known formula

$$\frac{\mathrm{d}}{\mathrm{d}x}(f * g)(x) = (f' * g)(x) = (f * g')(x),$$

we get

$$\begin{aligned}
\psi_\ell'(x) &= (\ell * G')(x) + d_3 \sin x + (d_1 + d_3 x)\cos x + d_4 \cos x - (d_2 + d_4 x)\sin x,\\
\psi_\ell''(x) &= (\ell * G'')(x) + 2d_3 \cos x - (d_1 + d_3 x)\sin x - 2d_4 \sin x - (d_2 + d_4 x)\cos x,\\
\psi_\ell'''(x) &= (\ell * G''')(x) - 3d_3 \sin x - (d_1 + d_3 x)\cos x - 3d_4 \cos x + (d_2 + d_4 x)\sin x.
\end{aligned}$$

Then, using these expressions and (2.13), as well as expressions for $G^{(k)}(x)$, $k = 0,1,2,3$, the boundary conditions (2.7) and (2.8) reduce to

$$\begin{cases} (\ell(y), \sin y) + 4d_3 = 0, \\ \sin 1 \cdot (\ell(y), \cos y) - \cos 1 \cdot (\ell(y), \sin y) + 4d_3 \cos 1 - 4d_4 \sin 1 = 0, \\ (\ell(y), \cos y) + 4d_4 = 0, \\ \cos 1 \cdot (\ell(y), \cos y) + \sin 1 \cdot (\ell(y), \sin y) - 4d_3 \sin 1 - 4d_4 \cos 1 = 0. \end{cases}$$

Hence we have $d_3 = 0$, $d_4 = 0$, and therefore

$$(\ell(y), \sin y) = 0, \quad (\ell(y), \cos y) = 0. \tag{2.14}$$

Substituting these values into equality (2.13) we get the assertion of this theorem. Thus, Theorem 2.1 is proved. □

The equalities (2.14) mean that our quadrature formula will be exact for functions $\sin x$ and $\cos x$.

Now, using Theorem 2.1, we immediately obtain a representation of the norm of the error functional

$$\left\| \ell | K_2^*(P_2) \right\|^2 = (\ell(x), \psi_\ell(x)) = \sum_{\nu=0}^{N} \sum_{\gamma=0}^{N} C_\nu C_\gamma \, G(x_\nu - x_\gamma)$$

$$-2 \sum_{\nu=0}^{N} C_\nu \int_0^1 G(x - x_\nu) \, \mathrm{d}x + \int_0^1 \int_0^1 G(x-y) \, \mathrm{d}x \, \mathrm{d}y. \tag{2.15}$$

Thus, Problem 1 is solved. Further in Sections 3 and 4 we deal with Problem 2.

3 Existence and uniqueness of optimal coefficients

Let the nodes x_ν of the quadrature formula (1.1) be fixed. The error functional (1.2) satisfies the conditions (2.14). Norm of the error functional $\ell(x)$ is a multidimensional function of the coefficients C_ν ($\nu = 0, 1, \ldots, N$). For finding its minimum under the conditions (2.14), we apply the Lagrange method. Namely, we consider the function

$$\Psi(C_0, C_1, \ldots, C_N, d_1, d_2) = \|\ell\|^2 - 2d_1 \, (\ell(x), \sin x) - 2d_2 \, (\ell(x), \cos x)$$

and its partial derivatives equating to zero, so that we obtain the following system of linear equations

$$\sum_{\gamma=0}^{N} C_\gamma G(x_\nu - x_\gamma) + d_1 \sin x_\nu + d_2 \cos x_\nu = f(x_\nu), \quad \nu = 0, 1, \ldots, N, \tag{3.1}$$

$$\sum_{\gamma=0}^{N} C_\gamma \sin x_\gamma = 1 - \cos 1, \quad \sum_{\gamma=0}^{N} C_\gamma \cos x_\gamma = \sin 1, \tag{3.2}$$

where $G(x)$ is determined by (2.10) and

$$f(x_\nu) = \int_0^1 G(x - x_\nu) \, \mathrm{d}x.$$

The system (3.1)–(3.2) has the unique solution and it gives thee minimum to $\|\ell\|^2$ under the conditions (3.2).

The uniqueness of the solution of the system (3.1)–(3.2) is proved following [28, Chapter I]. For completeness we give it here.

First, we put $\mathbf{C} = (C_0, C_1, \ldots, C_N)$ and $\mathbf{d} = (d_1,\, d_2)$ for the solution of the system of equations (3.1)–(3.2), which represents a stationary point of the function $\Psi\,(\mathbf{C}, \mathbf{d})$.

Setting $C_\nu = \overline{C}_\nu + C_{1\nu}$, $\nu = 0, 1, \ldots, N$, (2.15) and the system (3.1)–(3.2) become

$$\|\ell\|^2 = \sum_{\nu=0}^{N}\sum_{\gamma=0}^{N} \overline{C}_\nu \overline{C}_\gamma G(x_\nu - x_\gamma) - 2\sum_{\nu=0}^{N} (\overline{C}_\nu + C_{1,\nu}) \int_0^1 G(x - x_\nu)\,\mathrm{d}x$$
$$+ \sum_{\nu=0}^{N}\sum_{\gamma=0}^{N} \left(2\overline{C}_\nu C_{1,\gamma} + C_{1,\nu} C_{1,\gamma}\right) G(x_\nu - x_\gamma) + \int_0^1\int_0^1 G(x - y)\,\mathrm{d}x\,\mathrm{d}y \quad (3.3)$$

and

$$\sum_{\gamma=0}^{N} \overline{C}_\gamma G(x_\nu - x_\gamma) + d_1 \sin x_\nu + d_2 \cos x_\nu = F(x_\nu), \quad \nu = 0, 1, \ldots, N, \quad (3.4)$$

$$\sum_{\gamma=0}^{N} \overline{C}_\gamma \sin x_\gamma = 0, \quad \sum_{\gamma=0}^{N} \overline{C}_\gamma \cos x_\gamma = 0, \quad (3.5)$$

respectively, where $F(x_\nu) = f(x_\nu) - \sum\limits_{\gamma=0}^{N} C_{1\gamma} G(x_\nu - x_\gamma)$ and $C_{1\gamma}$, $\gamma = 0, 1, \ldots, N$, are partial solutions of the system (3.2).

Hence, we directly get, that the minimization of (2.15) under the conditions (2.14) by C_ν is equivalent to the minimization of expression (3.3) by $\overline{C}_\nu$ under the conditions (3.5). Therefore, it is sufficient to prove that the system (3.4)–(3.5) has the unique solution with respect to $\overline{\mathbf{C}} = (\overline{C}_0, \overline{C}_1, \ldots, \overline{C}_N)$ and $\mathbf{d} = (d_1,\, d_2)$ and this solution gives the conditional minimum for $\|\ell\|^2$. From the theory of the conditional extremum, we need the positivity of the quadratic form

$$\Phi(\overline{\mathbf{C}}) = \sum_{\nu=0}^{N}\sum_{\gamma=0}^{N} \frac{\partial^2 \Psi}{\partial \overline{C}_\nu \partial \overline{C}_\gamma} \overline{C}_\nu \overline{C}_\gamma \quad (3.6)$$

on the set of vectors $\overline{\mathbf{C}} = (\overline{C}_0, \overline{C}_1, \ldots, \overline{C}_N)$, under the condition

$$S\overline{\mathbf{C}} = 0, \quad (3.7)$$

where S is the matrix of the system of equations (3.5)

$$S = \begin{pmatrix} \sin x_0 & \sin x_1 & \cdots & \sin x_N \\ \cos x_0 & \cos x_1 & \cdots & \cos x_N \end{pmatrix}.$$

Now, we show, that in this case the condition is satisfied.

Theorem 3.1 *For any nonzero vector $\overline{\mathbf{C}} \in \mathbb{R}^{N+1}$, lying in the subspace $S\overline{\mathbf{C}} = 0$, the function $\Phi(\overline{\mathbf{C}})$ is strictly positive.*

Proof Using the definition of the function $\Psi(\mathbf{C},\mathbf{d})$ and the previous equations, (3.6) reduces to

$$\Phi\left(\overline{\mathbf{C}}\right)=2\sum_{\nu=0}^{N}\sum_{\gamma=0}^{N}G(x_\nu-x_\gamma)\overline{C}_\nu\overline{C}_\gamma. \tag{3.8}$$

Consider now a linear combination of delta functions

$$\delta_{\overline{\mathbf{C}}}(x)=\sqrt{2}\sum_{\nu=0}^{N}\overline{C}_\nu\delta(x-x_\nu). \tag{3.9}$$

By virtue of the condition (3.7), this functional belongs to the space $K_2^*(P_2)$. So, it has an extremal function $u_{\overline{\mathbf{C}}}(x)\in K_2(P_2)$, which is a solution of the equation

$$\left(\frac{\mathrm{d}^4}{\mathrm{d}x^4}+2\frac{\mathrm{d}^2}{\mathrm{d}x^2}+1\right)u_{\overline{\mathbf{C}}}(x)=\delta_{\overline{\mathbf{C}}}(x). \tag{3.10}$$

As $u_{\overline{\mathbf{C}}}(x)$ we can take a linear combination of shifts of the fundamental solution $G(x)$:

$$u_{\overline{\mathbf{C}}}(x)=\sqrt{2}\sum_{\nu=0}^{N}\overline{C}_\nu G(x-x_\nu),$$

and we can see that

$$\left\|u_{\overline{\mathbf{C}}}|K_2(P_2)\right\|^2=\left(\delta_{\overline{\mathbf{C}}},u_{\overline{\mathbf{C}}}\right)=2\sum_{\nu=0}^{N}\sum_{\gamma=0}^{N}\overline{C}_\nu\overline{C}_\gamma G(x_\nu-x_\gamma)=\Phi\left(\overline{\mathbf{C}}\right).$$

Thus, it is clear that for a nonzero $\overline{\mathbf{C}}$ the function $\Phi\left(\overline{\mathbf{C}}\right)$ is strictly positive and Theorem 3.1 is proved. ◻

If the nodes $x_0,x_1,\ldots,x_N$ are selected such that the matrix S has the right inverse, then the system of equations (3.4)–(3.5) has the unique solution, as well as the system of equations (3.1)–(3.2).

Theorem 3.2 *If the matrix S has the right inverse matrix, then the main matrix Q of the system of equations* (3.4)–(3.5) *is nonsingular.*

Proof We denote by M the matrix of the quadratic form $\frac{1}{2}\Phi\left(\overline{\mathbf{C}}\right)$, given in (3.8). It is enough to consider the homogenous system of linear equations

$$Q\begin{pmatrix}\overline{\mathbf{C}}\\ \mathbf{d}\end{pmatrix}=\begin{pmatrix}M & S^*\\ S & 0\end{pmatrix}\begin{pmatrix}\overline{\mathbf{C}}\\ \mathbf{d}\end{pmatrix}=0 \tag{3.11}$$

and prove that it has only the trivial solution.

Let $\overline{\mathbf{C}}$, $\mathbf{d}$ be a solution of (3.11). Consider the function $\delta_{\overline{\mathbf{C}}}(x)$, defined before by (3.9). As an extremal function for $\delta_{\overline{\mathbf{C}}}(x)$ we take the following function

$$u_{\overline{\mathbf{C}}}(x)=\sqrt{2}\sum_{\nu=0}^{N}\overline{C}_\nu G(x-x_\nu)+d_1\sin x+d_2\cos x.$$

This is possible, because $u_{\overline{\mathbf{C}}}$ belongs to the space $K_2(P_2)$ and it is a solution of the equation (3.10). The first $N+1$ equations of the system (3.11) mean that $u_{\overline{\mathbf{C}}}(x)$ takes the value zero at all nodes x_ν, $\nu = 0, 1, \dots, N$. Then, for the norm of the functional $\delta_{\overline{\mathbf{C}}}(x)$ in $K_2^*(P_2)$, we have

$$\left\| \delta_{\overline{\mathbf{C}}} | K_2^*(P_2) \right\|^2 = \left(\delta_{\overline{\mathbf{C}}}, u_{\overline{\mathbf{C}}} \right) = \sqrt{2} \sum_{\nu=0}^{N} \overline{C}_\nu \, u_{\overline{\mathbf{C}}}(x_\nu) = 0,$$

which is possible only when $\overline{\mathbf{C}} = 0$. Taking into account this, from the first $N+1$ equations of the system (3.11) we obtain $S^*\mathbf{d} = 0$. Since the matrix S is a right-inversive (by the hypotheses of this theorem), we conclude that S^* has the left inverse matrix, and therefore $\mathbf{d} = 0$, i.e., $d_1 = d_2 = 0$, which completes the proof. $\square$

According to (2.15) and Theorems 3.1 and 3.2, it follows that at fixed values of the nodes x_ν, $\nu = 0, 1, \dots, N$, the norm of the error functional $\ell(x)$ has the unique minimum for some concrete values of $C_\nu = \overset{\circ}{C}_\nu$, $\nu = 0, 1, \dots, N$. As we mentioned in the first section, the quadrature formula with such coefficients $\overset{\circ}{C}_\nu$ is called *optimal quadrature formula in the sense of Sard*, and $\overset{\circ}{C}_\nu$, $\nu = 0, 1, \dots, N$, are the *optimal coefficients*. In the sequel, for convenience the optimal coefficients $\overset{\circ}{C}_\nu$ will be denoted only as C_ν.

4 Coefficients of optimal quadrature formula in the sense of Sard

In this section we solve the system (3.1)–(3.2) and find an explicit formula for the coefficients C_ν. We use a similar method, offered by Sobolev [26] for finding optimal coefficients in the space $L_2^{(m)}(0,1)$. Here, we mainly use a concept of functions of a discrete argument and the corresponding operations (see [27] and [28]). For completeness we give some of definitions.

Let nodes x_ν are equal spaced, i.e., $x_\nu = \nu h$, $h = 1/N$. Assume that $\varphi(x)$ and $\psi(x)$ are real-valued functions defined on the real line $\mathbb{R}$.

Definition 4.1 The function $\varphi(h\nu)$ is a *function of discrete argument* if it is given on some set of integer values of ν.

Definition 4.2 The *inner product* of two discrete functions $\varphi(h\nu)$ and $\psi(h\nu)$ is given by

$$[\varphi, \psi] = \sum_{\nu=-\infty}^{\infty} \varphi(h\nu) \cdot \psi(h\nu),$$

if the series on right hand side converges absolutely.

Definition 4.3 The *convolution* of two functions $\varphi(h\nu)$ and $\psi(h\nu)$ is the inner product

$$\varphi(h\nu) * \psi(h\nu) = [\varphi(h\gamma), \psi(h\nu - h\gamma)] = \sum_{\gamma=-\infty}^{\infty} \varphi(h\gamma) \cdot \psi(h\nu - h\gamma).$$

Suppose that $C_\nu = 0$ when $\nu < 0$ and $\nu > N$. Using these definitions, the system (3.1)–(3.2) can be rewritten in the convolution form

$$G(h\nu) * C_\nu + d_1 \sin(h\nu) + d_2 \cos(h\nu) = f(h\nu), \quad \nu = 0, 1, \dots, N, \tag{4.1}$$

$$\sum_{\nu=0}^{N} C_\nu \sin(h\nu) = 1 - \cos 1, \quad \sum_{\nu=0}^{N} C_\nu \cos(h\nu) = \sin 1, \tag{4.2}$$

where

$$f(h\nu) = \frac{1}{4}[4 - (2 + 2\cos 1 + \sin 1)\cos(h\nu) - (2\sin 1 - \cos 1)\sin(h\nu) + \sin 1 \cdot (h\nu)\cos(h\nu) - (1 + \cos 1)\cdot (h\nu)\sin(h\nu)]. \tag{4.3}$$

Now, we consider the following problem:

Problem A. *For a given $f(h\nu)$ find a discrete function C_ν and unknown coefficients d_1, d_2, which satisfy the system* (4.1) $-$ (4.2).

Further, instead of C_ν we introduce the functions $v(h\nu)$ and $u(h\nu)$ by

$$v(h\nu) = G(h\nu) * C_\nu \quad \text{and} \quad u(h\nu) = v(h\nu) + d_1 \sin(h\nu) + d_2 \cos(h\nu).$$

In such a statement it is necessary to express C_ν by the function $u(h\nu)$. For this we have to construct such an operator $D(h\nu)$, which satisfies the equation

$$D(h\nu) * G(h\nu) = \delta(h\nu), \tag{4.4}$$

where $\delta(h\nu)$ is equal to 0 when $\nu \neq 0$ and is equal to 1 when $\nu = 0$, i.e., $\delta(h\nu)$ is a discrete delta-function.

In connection with this, a discrete analogue $D(h\nu)$ of the differential operator $\frac{\mathrm{d}^4}{\mathrm{d}x^4} + 2\frac{\mathrm{d}^2}{\mathrm{d}x^2} + 1$, which satisfies (4.4) was constructed in [11] and some properties were investigated.

Following [11] we have:

Theorem 4.1 *The discrete analogue of the differential operator* $\frac{\mathrm{d}^4}{\mathrm{d}x^4} + 2\frac{\mathrm{d}^2}{\mathrm{d}x^2} + 1$ *satisfying the equation* (4.4) *has the form*

$$D(h\nu) = \frac{2}{\sin h - h\,\cos h}\begin{cases} A_1\,\lambda_1^{|\nu|-1}, & |\nu| \geq 2, \\ 1 + A_1, & |\nu| = 1, \\ \dfrac{2h\cos 2h - \sin 2h}{\sin h - h\cos h} + \dfrac{A_1}{\lambda_1}, & \nu = 0, \end{cases} \tag{4.5}$$

where

$$A_1 = \frac{4h^2 \sin^4 h\,\lambda_1^2}{(\lambda_1^2 - 1)(\sin h - h\cos h)^2}$$

and

$$\lambda_1 = \frac{2h - \sin 2h - 2\sin h\sqrt{h^2 - \sin^2 h}}{2(h\cos h - \sin h)}$$

is a zero of the polynomial

$$Q_2(\lambda) = \lambda^2 + \frac{2h - \sin(2h)}{\sin h - h\cos h}\lambda + 1, \tag{4.6}$$

and $|\lambda_1| < 1$ and h is a small parameter.

Theorem 4.2 *The discrete analogue $D(h\nu)$ of the differential operator $\frac{\mathrm{d}^4}{\mathrm{d}x^4}+2\frac{\mathrm{d}^2}{\mathrm{d}x^2}+1$ satisfies the following equalities:*

1) $D(h\nu) * \sin(h\nu) = 0,$
2) $D(h\nu) * \cos(h\nu) = 0,$
3) $D(h\nu) * (h\nu)\sin(h\nu) = 0,$
4) $D(h\nu) * (h\nu)\cos(h\nu) = 0,$
5) $D(h\nu) * G(h\nu) = \delta(h\nu).$

Here $G(h\nu)$ is the function of discrete argument, corresponding to the function $G(x)$ defined by (2.10)*, and $\delta(h\nu)$ is the discrete delta-function.*

Then, taking into account (4.4) and Theorems 4.1 and 4.2, for optimal coefficients we have

$$C_\nu = D(h\nu) * u(h\nu). \tag{4.7}$$

Thus, if we find the function $u(h\nu)$, then the optimal coefficients can be obtained from (4.7). In order to calculate the convolution (4.7) we need a representation of the function $u(h\nu)$ for all integer values of ν. According to (4.1) we get that $u(h\nu) = f(h\nu)$ when $h\nu \in [0,1]$. Now, we need a representation of the function $u(h\nu)$ when $\nu < 0$ and $\nu > N$.

Since $C_\nu = 0$ for $h\nu \notin [0,1]$, then $C_\nu = D(h\nu) * u(h\nu) = 0$, $h\nu \notin [0,1]$. Now, we calculate the convolution $v(h\nu) = G(h\nu) * C_\nu$ when $h\nu \notin [0,1]$.

Let $\nu < 0$, then, taking into account equalities (2.10) and (4.2), we have

$$\begin{aligned} v(h\nu) &= G(h\nu) * C_\nu = \sum_{\gamma=-\infty}^{\infty} C_\gamma\, G(h\nu - h\gamma) \\ &= \sum_{\gamma=0}^{N} C_\gamma \frac{\mathrm{sign}\,(h\nu - h\gamma)}{4}\big(\sin(h\nu - h\gamma) - (h\nu - h\gamma)\cos(h\nu - h\gamma)\big) \\ &= -\frac{1}{4}[(\sin(h\nu) - h\nu\cos(h\nu))\ \sin 1 - (\cos(h\nu) + h\nu\sin(h\nu))(1 - \cos 1) \\ &\quad + \cos(h\nu)\sum_{\gamma=0}^{N} C_\gamma\, h\gamma\cos(h\gamma) + \sin(h\nu)\sum_{\gamma=0}^{N} C_\gamma\, h\gamma\sin(h\gamma)]. \end{aligned}$$

Denoting $b_1 = \frac{1}{4}\sum_{\gamma=0}^{N} C_\gamma\, h\gamma\sin(h\gamma)$ and $b_2 = \frac{1}{4}\sum_{\gamma=0}^{N} C_\gamma\, h\gamma\cos(h\gamma)$, we get for $\nu < 0$

$$\begin{aligned} v(h\nu) &= -\frac{1}{4}[(\sin(h\nu) - h\nu\cos(h\nu))\ \sin 1 - (\cos(h\nu) + h\nu\sin(h\nu))(1 - \cos 1) \\ &\quad +4b_1\sin(h\nu) + 4b_2\cos(h\nu)], \end{aligned}$$

and for $\nu > N$

$$\begin{aligned} v(h\nu) &= \frac{1}{4}[(\sin(h\nu) - h\nu\cos(h\nu))\ \sin 1 - (\cos(h\nu) + h\nu\sin(h\nu))(1 - \cos 1) \\ &\quad +4b_1\sin(h\nu) + 4b_2\cos(h\nu)]. \end{aligned}$$

Now, setting

$$d_1^- = d_1 - b_1, \quad d_2^- = d_2 - b_2, \quad d_1^+ = d_1 + b_1, \quad d_2^+ = d_2 + b_2$$

we formulate the following problem:

PROBLEM B. *Find the solution of the equation*

$$D(h\nu) * u(h\nu) = 0, \quad h\nu \notin [0,1], \tag{4.8}$$

in the form

$$u(h\nu) = \begin{cases} -\frac{\sin 1}{4}(\sin(h\nu) - h\nu\cos(h\nu)) + \frac{1-\cos 1}{4}(\cos(h\nu) \\ \qquad +h\nu\sin(h\nu)) + d_1^-\sin(h\nu) + d_2^-\cos(h\nu), & \nu < 0, \\ f(h\nu), & 0 \le \nu \le N, \\ \frac{\sin 1}{4}(\sin(h\nu) - h\nu\cos(h\nu)) - \frac{1-\cos 1}{4}(\cos(h\nu) \\ \qquad +h\nu\sin(h\nu)) + d_1^+\sin(h\nu) + d_2^+\cos(h\nu), & \nu > N, \end{cases} \tag{4.9}$$

where d_1^-, d_2^-, d_1^+, d_2^+ *are unknown coefficients.*

It is clear that

$$d_1 = \frac{1}{2}\left(d_1^+ + d_1^-\right), \quad b_1 = \frac{1}{2}\left(d_1^+ - d_1^-\right), \quad d_2 = \frac{1}{2}\left(d_2^+ + d_2^-\right), \quad b_2 = \frac{1}{2}\left(d_2^+ - d_2^-\right).$$

These unknowns d_1^-, d_2^-, d_1^+, d_2^+ can be found from the equation (4.8), using the function $D(h\nu)$. Then, the explicit form of the function $u(h\nu)$ and optimal coefficients C_ν can be obtained. Thus, in this way PROBLEM B, as well as PROBLEM A, can be solved.

However, instead of this, using $D(h\nu)$ and $u(h\nu)$ and taking into account (4.7), we find here expressions for the optimal coefficients C_ν, $\nu = 1, \ldots, N-1$. For this purpose we introduce the following notations

$$p = \frac{2}{\sin h - h\cos h},$$

$$m = \sum_{\gamma=1}^{\infty} \frac{A_1 p}{\lambda_1}\lambda_1^\gamma \Big[-\frac{1}{4}(\sin(-h\gamma) + h\gamma\cos(h\gamma))\sin 1$$
$$+\frac{1}{4}(\cos(h\gamma) + h\gamma\,\sin(h\gamma))(1-\cos 1) - d_1^-\,\sin(h\gamma) + d_2^-\,\cos(h\gamma) - f(-h\gamma)\Big],$$

$$n = \sum_{\gamma=1}^{\infty} \frac{A_1 p}{\lambda_1}\lambda_1^\gamma \Big[\frac{1}{4}(\sin((N+\gamma)h) - (N+\gamma)h\cos((N+\gamma)h))\sin 1 - \frac{1}{4}(\cos((N+\gamma)h)$$
$$+((N+\gamma)h\sin((N+\gamma)h))(1-\cos 1) + d_1^+\,\sin((N+\gamma)h)$$
$$+d_2^+\,\cos((N+\gamma)h) - f((N+\gamma)h)\Big].$$

The series in the previous expressions are convergent, because $|\lambda_1| < 1$.

Theorem 4.3 *The coefficients of optimal quadrature formulas in the sense of Sard of the form* (1.1) *in the space* $K_2(P_2)$ *have the following representation*

$$C_\nu = \frac{4(1-\cos h)}{h+\sin h} + m\lambda_1^\nu + n\lambda_1^{N-\nu}, \quad \nu = 1, \dots, N-1, \tag{4.10}$$

where m *and* n *are defined above, and* λ_1 *is given in Theorem* 4.1.

Proof Let $\nu \in \{1, \dots, N-1\}$. Then from (4.7), using (4.5) and (4.9), we have

$$\begin{aligned}
C_\nu &= D(h\nu) * u(h\nu) = \sum_{\gamma=-\infty}^{\infty} D(h\nu - h\gamma)u(h\gamma) \\
&= \sum_{\gamma=-\infty}^{-1} D(h\nu - h\gamma)u(h\gamma) + \sum_{\gamma=0}^{N} D(h\nu - h\gamma)u(h\gamma) + \sum_{\gamma=N+1}^{\infty} D(h\nu - h\gamma)u(h\gamma) \\
&= D(h\nu) * f(h\nu) + \sum_{\gamma=1}^{\infty} \frac{A_1 p}{\lambda_1}\lambda_1^{\nu+\gamma}\Big[-\frac{1}{4}(\sin(-h\gamma) + h\gamma\cos(h\gamma))\sin 1 \\
&\quad + \frac{1}{4}(\cos(h\gamma) + h\gamma\sin(h\gamma))(1-\cos 1) - d_1^- \sin(h\gamma) + d_2^- \cos(h\gamma) - f(-h\gamma)\Big] \\
&\quad + \sum_{\gamma=1}^{\infty} \frac{A_1 p}{\lambda_1}\lambda_1^{N+\gamma-\nu}\Big[\frac{1}{4}(\sin((N+\gamma)h) - (N+\gamma)h\cos((N+\gamma)h))\sin 1 - \frac{1}{4}(\cos((N+\gamma)h) \\
&\quad + ((N+\gamma)h\sin((N+\gamma)h))(1-\cos 1) + d_1^+ \sin((N+\gamma)h) + d_2^+ \cos((N+\gamma)h) - f((N+\gamma)h)\Big].
\end{aligned}$$

Hence, taking into account the previous notations, we get

$$C_\nu = D(h\nu) * f(h\nu) + m\lambda_1^\nu + n\lambda_1^{N-\nu}. \tag{4.11}$$

Now, using Theorems 4.1 and 4.2 and equality (4.3), we calculate the convolution $D(h\nu) * f(h\nu)$. Namely,

$$D(h\nu)*f(h\nu) = D(h\nu)*1 = \sum_{\gamma=-\infty}^{\infty} D(h\gamma) = D(0)+2D(h)+2\sum_{\gamma=2}^{\infty} D(h\gamma) = \frac{4(1-\cos h)}{h+\sin h}.$$

Substituting this convolution into (4.11) we obtain (4.10), and Theorem 4.3 is proved. □

According Theorem 4.3 it is clear, that in order to obtain the exact expressions of the optimal coefficients C_ν we need only m and n. They can be found from an identity with respect to $(h\nu)$, which can be obtained by substituting the equality (4.10) into (4.1). Namely, equating the corresponding coefficients the left and the right hand sides of the equation (4.1) we find m and n. The coefficients C_0 and C_N follow directly from (4.2).

Finally, we can formulate and prove the following result:

Theorem 4.4 *The coefficients of the optimal quadrature formulas in the sense of Sard of the form* (1.1) *in the space* $K_2(P_2)$ *are*

$$C_\nu = \begin{cases} \dfrac{2\sin h-(h+\sin h)\cos h}{(h+\sin h)\sin h}+\dfrac{h-\sin h}{(h+\sin h)\sin h(1+\lambda_1^N)}\left(\lambda_1+\lambda_1^{N-1}\right), & \nu=0,N, \\ \dfrac{4(1-\cos h)}{h+\sin h}+\dfrac{2h(h-\sin h)\sin h}{(h+\sin h)(h\cos h-\sin h)(1+\lambda_1^N)}\left(\lambda_1^\nu+\lambda_1^{N-\nu}\right), & \nu=1,\dots,N-1, \end{cases}$$

where λ_1 *is given in Theorem* 4.1 *and* $|\lambda_1|<1$.

Proof First from equations (4.2) we have

$$C_0=\sin 1-\frac{\cos 1(1-\cos 1)}{\sin 1}-\sum_{\gamma=1}^{N-1}C_\gamma\cos(h\gamma)+\frac{\cos 1}{\sin 1}\sum_{\gamma=1}^{N-1}C_\gamma\sin(h\gamma),$$

$$C_N=\frac{1-\cos 1}{\sin 1}-\frac{1}{\sin 1}\sum_{\gamma=1}^{N-1}C_\gamma\sin(h\gamma).$$

Hence, using (4.10), after some simplifications we get

$$\begin{aligned} C_0 = {} & \frac{(h-\sin h)(1-\cos 1)+2\sin 1(1-\cos h)}{\sin 1(h+\sin h)} \\ & -m\,\frac{\lambda_1(\sin 1\cos h-\cos 1\sin h)+\lambda_1^{N+1}\sin h-\lambda_1^2\sin 1}{(\lambda_1^2+1-2\lambda_1\cos h)\sin 1} \\ & -n\,\frac{\lambda_1^{N+1}(\sin 1\cos h-\sin h\cos 1)+\lambda_1\sin h-\lambda_1^N\sin 1}{(\lambda_1^2+1-2\lambda_1\cos h)\sin 1}, \end{aligned} \tag{4.12}$$

$$\begin{aligned} C_N = {} & \frac{(h-\sin h)(1-\cos 1)+2\sin 1(1-\cos h)}{\sin 1(h+\sin h)} \\ & -m\,\frac{\lambda_1^{N+1}(\sin 1\cos h-\sin h\cos 1)+\lambda_1\sin h-\lambda_1^N\sin 1}{(\lambda_1^2+1-2\lambda_1\cos h)\sin 1} \\ & -n\,\frac{\lambda_1(\sin 1\cos h-\cos 1\sin h)+\lambda_1^{N+1}\sin h-\lambda_1^2\sin 1}{(\lambda_1^2+1-2\lambda_1\cos h)\sin 1}. \end{aligned} \tag{4.13}$$

Further, we consider the convolution $G(h\nu)*C_\nu$ in equation (4.1), i.e.,

$$\begin{aligned} G(h\nu)*C_\nu &= \sum_{\gamma=0}^{N}C_\gamma G(h\nu-h\gamma) \\ &= \sum_{\gamma=0}^{N}C_\gamma\frac{\operatorname{sign}(h\nu-h\gamma)}{4}\left[\sin(h\nu-h\gamma)-(h\nu-h\gamma)\cos(h\nu-h\gamma)\right] \\ &= S_1-S_2, \end{aligned} \tag{4.14}$$

where

$$S_1=\frac{1}{2}\sum_{\gamma=0}^{\nu}C_\gamma\left[\sin(h\nu-h\gamma)-(h\nu-h\gamma)\cos(h\nu-h\gamma)\right]$$

and

$$S_2 = \frac{1}{4} \sum_{\gamma=0}^{N} C_\gamma \left[\sin(h\nu - h\gamma) - (h\nu - h\gamma)\cos(h\nu - h\gamma)\right].$$

Using (4.10) we obtain

$$\begin{aligned} S_1 &= \frac{1}{2} C_0[\sin(h\nu) - h\nu\cos(h\nu)] \\ &\quad + \frac{1}{2}\sum_{\gamma=1}^{\nu}(k + m\lambda_1^\gamma + n\lambda_1^{N+\gamma-\nu})[\sin(h\nu - h\gamma) - (h\nu - h\gamma)\cos(h\nu - h\gamma)] \\ &= \frac{1}{2} C_0[\sin(h\nu) - h\nu\cos(h\nu)] + \frac{1}{2}\sum_{\gamma=0}^{\nu-1}(k + m\lambda_1^{\nu-\gamma} + n\lambda_1^{N+\gamma-\nu})[\sin(h\gamma) - h\gamma\cos(h\gamma)], \end{aligned}$$

where $k = 4(1-\cos h)/(h+\sin h)$. After some calculations and simplifications S_1 can be reduced to the following form

$$\begin{aligned} S_1 &= 1 + \left[\frac{(h-\sin h)(1-\cos 1)}{2\sin 1(h+\sin h)} + m\left(\frac{(\lambda_1\cos 1 - \lambda_1^{N+1})\sin h}{2\sin 1(\lambda_1^2+1-2\lambda_1\cos h)} + \frac{h\sin h(\lambda_1 - \lambda_1^3)}{2(\lambda_1^2+1-2\lambda_1\cos h)^2}\right)\right. \\ &\quad \left. + n\left(\frac{(\lambda_1^{N+1}\cos 1 - \lambda_1)\sin h}{2\sin 1(\lambda_1^2+1-2\lambda_1\cos h)} + \frac{h\sin h(\lambda_1^{N+3} - \lambda_1^{N+1})}{2(\lambda_1^2+1-2\lambda_1\cos h)^2}\right)\right]\sin(h\nu) - \cos(h\nu) \\ &\quad - \left[\frac{\sin h}{h+\sin h} + m\frac{\lambda_1\sin h}{2(\lambda_1^2+1-2\lambda_1\cos h)} + n\frac{\lambda_1^{N+1}\sin h}{2(\lambda_1^2+1-2\lambda_1\cos h)}\right](h\nu)\sin(h\nu) \\ &\quad + \left[\frac{(h-\sin h)(\cos 1 - 1)}{h+\sin h} - m\frac{(\lambda_1\cos 1 - \lambda_1^{N+1})\sin h}{\lambda_1^2+1-2\lambda_1\cos h} - n\frac{(\lambda_1^{N+1}\cos 1 - \lambda_1)\sin h}{\lambda_1^2+1-2\lambda_1\cos h)}\right]\frac{h\nu\cos(h\nu)}{2\sin 1}, \end{aligned}$$

where we used the fact that λ_1 is a zero of the polynomial $Q_2(\lambda)$ defined by (4.6).

Also, keeping in mind (4.2), for S_2 we get following expression

$$\begin{aligned} S_2 &= \frac{1}{4}\Bigg[\sin 1\sin(h\nu) - (1-\cos 1)\cos(h\nu) - \sin 1(h\nu)\cos(h\nu) - (1-\cos 1)(h\nu)\sin(h\nu) \\ &\quad + \cos(h\nu)\sum_{\gamma=1}^{N} C_\gamma(h\gamma)\cos(h\gamma) + \sin(h\nu)\sum_{\gamma=1}^{N} C_\gamma(h\gamma)\sin(h\gamma)\Bigg]. \end{aligned}$$

Now, substituting (4.14) into equation (4.1) we get the following identity with respect to $(h\nu)$

$$S_1 - S_2 + d_1\sin(h\nu) + d_2\cos(h\nu) = f(h\nu), \tag{4.15}$$

where $f(h\nu)$ is defined by (4.3).

Unknowns in (4.15) are m, n, d_1 and d_2. Equating the corresponding coefficients of $(h\nu)\sin(h\nu)$ and $(h\nu)\cos(h\nu)$ of both sides of the identity (4.15), for unknowns m and n we get the following system of linear equations

$$\begin{cases} m + \lambda_1^N n = \dfrac{2h\sin h(h-\sin h)}{(h+\sin h)(h\cos h - \sin h)}, \\ \lambda_1^n m + n \ = \dfrac{2h\sin h(h-\sin h)}{(h+\sin h)(h\cos h - \sin h)}, \end{cases}$$

from which

$$m = n = \frac{2h\sin h(h-\sin h)}{(h+\sin h)(h\cos h-\sin h)(1+\lambda_1^N)}.$$

The coefficients d_1 and d_2 can be found also from (4.15) by equating the corresponding coefficients of $\sin(h\nu)$ and $\cos(h\nu)$. In this way the assertion of Theorem 4.4 is proved. □

Proving Theorem 4.4 we have just solved PROBLEM A, which is equivalent to PROBLEM 2. Thus, PROBLEM 2 is solved, i.e., the coefficients of the optimal quadrature formula (1.1) in the sense of Sard in the space $K_2(P_2)$ for equal spaced nodes are found.

Remark 4.1 Theorem 4.4 for $N = 2$ gives the result of the example (h) of the paper [13] when $[a, b] = [0, 1]$.

5 The norm of the error functional of the optimal quadrature formula in the sense of Sard

In this section we calculate square of the norm of the error functional (1.2) of the optimal quadrature formula (1.1). Furthermore, we give an asymptotic analysis of this norm.

The following result holds:

Theorem 5.1 *The square of the norm of the error functional* (1.2) *of the optimal quadrature formula* (1.1) *on the space* $K_2(P_2)$ *has the form*

$$\|\overset{\circ}{\ell}\|^2 = \frac{h(h-\sin h)(\sin h\ (\sin 1-1)+h\sin(h-1)+4\cos h)+(3h^2+h\sin h+8(\cos h-1))\sin h}{2h(h+\sin h)\sin h}$$
$$+\frac{h(h-\sin h)}{2(h+\sin h)(1+\lambda_1^N)\sin h}\Bigg(\frac{(h\sin 1-4)(\lambda_1+\lambda_1^N)(1-\lambda_1)-4(1-\lambda_1^{N-1})(\lambda_1^2+1-2\lambda_1\cos h)}{h(1-\lambda_1)}$$
$$+\frac{(\lambda_1^2-1)\Big((2-\cos 1)(\lambda_1^N-1)-(\lambda_1-\lambda_1^{N-1})\cos(h-1)\Big)\sin h}{\lambda_1^2+1-2\lambda_1\cos h}$$
$$+\frac{(\lambda_1^2\cos h-2\lambda_1+\cos h)(\lambda_1+\lambda_1^{N-1})\sin(h-1)}{\lambda_1^2+1-2\lambda_1\cos h}\Bigg),$$

where λ_1 *is given in Theorem 4.1 and* $|\lambda_1| < 1$.

Proof In the equal spaced case of the nodes, the expression (2.15), using (2.10), we can rewrite in the following form

$$\|\ell\|^2 = \sum_{\nu=0}^{N} C_\nu\left(\sum_{\gamma=0}^{N} C_\gamma G(h\nu-h\gamma)-f(h\nu)\right)-\sum_{\nu=0}^{N} C_\nu f(h\nu)+1-\frac{3}{2}\sin 1+\frac{1}{2}\cos 1,$$

where $f(h\nu)$ is defined by (4.3).

Hence taking into account equality (3.1) we get

$$\|\ell\|^2 = \sum_{\nu=0}^{N} C_\nu \left(-d_1 \sin(h\nu) - d_2 \cos(h\nu)\right) - \sum_{\nu=0}^{N} C_\nu f(h\nu) + 1 - \frac{3}{2}\sin 1 + \frac{1}{2}\cos 1.$$

Using equalities (4.3) and (3.2), after some simplifications, we obtain

$$\|\ell\|^2 = -d_1(1-\cos 1) - d_2 \sin 1 - \frac{1}{4}\Bigg(4\sum_{\nu=0}^{N} C_\nu + \sin 1 \sum_{\nu=0}^{N} C_\nu (h\nu)\cos(h\nu)$$
$$-(1+\cos 1)\sum_{\nu=0}^{N} C_\nu (h\nu)\sin(h\nu)\Bigg) - \frac{1}{2}\sin 1 + \frac{1}{4}\cos 1 + \frac{5}{4}. \tag{5.1}$$

Now from (4.15) equating the corresponding coefficients of $\sin(h\nu)$ and $\cos(h\nu)$, for d_1 and d_2 we get the following expressions

$$d_1 = \frac{1}{4}\cos 1 - \frac{1}{4}\sin 1 + \frac{1}{4}\sum_{\gamma=0}^{N} C_\gamma (h\gamma)\sin(h\gamma) - \frac{h(h-\sin h)(\lambda_1^2-1)(\lambda_1^N-1)}{2(h+\sin h)(1+\lambda_1^N)(\lambda_1^2+1-2\lambda_1\cos h)},$$

$$d_2 = \frac{1}{4} - \frac{1}{4}\cos 1 - \frac{1}{4}\sin 1 + \frac{1}{4}\sum_{\gamma=0}^{N} C_\gamma (h\gamma)\cos(h\gamma).$$

Substituting these expressions in (5.1) we find

$$\|\ell\|^2 = \frac{3}{2} - \frac{1}{2}\sin 1 + \frac{h(1-\cos 1)(h-\sin h)(\lambda_1^2-1)(\lambda_1^N-1)}{2(h+\sin h)(1+\lambda_1^N)(\lambda_1^2+1-2\lambda_1\cos h)}$$
$$+\frac{\cos 1}{2}\sum_{\gamma=1}^{N} C_\gamma (h\gamma)\sin(h\gamma) - \frac{\sin 1}{2}\sum_{\gamma=1}^{N} C_\gamma (h\gamma)\cos(h\gamma) - \sum_{\gamma=1}^{N} C_\gamma.$$

Finally, using the expression for optimal coefficients C_γ from Theorem 4.4, after some calculations and simplifications, we get the assertion of Theorem 5.1. □

Theorem 5.2 *For the norm of the error functional* (1.2) *of the optimal quadrature formula of the form* (1.1) *we have*

$$\|\overset{\circ}{\ell}\,|K_2^*(P_2)\|^2 = \frac{1}{720}h^4 + O(h^5) \quad as \quad N\to\infty. \tag{5.2}$$

Proof Since

$$\lambda_1 = \frac{2h - \sin 2h - 2\sin h\sqrt{h^2-\sin^2 h}}{2(h\cos h - \sin h)} = (\sqrt{3}-2) + O(h^2)$$

that is $|\lambda_1|<1$ and then $\lambda_1^N \to 0$ as $N\to\infty$. Thus, when $N\to\infty$ from Theorem 5.1 for $\|\ell\|^2$ we get the following asymptotic expression

$$\|\ell\|^2 = \frac{h(h-\sin h)(\sin h\ (\sin 1-1) + h\sin(h-1) + 4\cos h) + (3h^2 + h\sin h + 8(\cos h - 1))\sin h}{2h(h+\sin h)\sin h}$$
$$+\frac{h(h-\sin h)}{2(h+\sin h)\sin h}\Bigg(\frac{(h\sin 1-4)(\lambda_1-\lambda_1^2) - 4(\lambda_1^2+1-2\lambda_1\cos h)}{h(1-\lambda_1)}$$
$$+\frac{(\lambda_1^2-1)(\cos 1 - 2 - \lambda_1\cos(h-1))\sin h + (\lambda_1^2\cos h - 2\lambda_1 + \cos h)\lambda_1\sin(h-1)}{\lambda_1^2+1-2\lambda_1\cos h}\Bigg).$$

The expansion of the last expression in a series in powers of h gives the assertion of Theorem 5.2. □

The next theorem gives an asymptotic optimality for our optimal quadrature formula.

Theorem 5.3 *Optimal quadrature formula of the form* (1.1) *with the error functional* (1.2) *in the space* $K_2(P_2)$ *is asymptotic optimal in the Sobolev space* $L_2^{(2)}(0,1)$, *i.e.,*

$$\lim_{N\to\infty}\frac{\|\overset{\circ}{\ell}\,|K_2^*(P_2)\|^2}{\|\overset{\circ}{\ell}\,|L_2^{(2)*}(0,1)\|^2}=1. \tag{5.3}$$

Proof Using Corollary 5.2 from [22] (for $m=2$ and $\eta_0=0$), for square of the norm of the error functional (1.2) of the optimal quadrature formula of the form (1.1) on the Sobolev space $L_2^{(2)}(0,1)$ we get the following expression

$$\|\overset{\circ}{\ell}\,|L_2^{(2)*}(0,1)\|^2=\frac{1}{720}h^4-\frac{h^5}{12}d\sum_{i=1}^{4}\frac{q^{N+i}+(-1)^iq}{(1-q)^{i+1}}\Delta^i0^4=\frac{1}{720}h^4+O(h^5), \tag{5.4}$$

where d is known, $q=\sqrt{3}-2$, $\Delta^i\gamma^4$ is the finite difference of order i of γ^4, $\Delta^i0^4=\Delta^i\gamma^4|_{\gamma=0}$.

Using (5.2) and (5.4) we obtain (5.3). Thus, Theorem 5.3 is proved. □

As said in the introduction of this paper the error (1.4) of optimal quadrature formula of the form (1.1) in the space $K_2(P_2)$ is estimated by Cauchy-Schwarz inequality

$$|R_N(\varphi)|\le\|\varphi|K_2(P_2)\|\cdot\|\overset{\circ}{\ell}\,|K_2^*(P_2)\|. \tag{5.5}$$

Hence taking into account Theorem 5.2 we get

$$|R_N(\varphi)|\le\|\varphi|K_2(P_2)\|\cdot\left(\frac{\sqrt{5}}{60}h^2+O(h^{5/2})\right).$$

From here we conclude that order of convergence of our optimal quadrature formula is $O(h^2)$.

In the next section we give some numerical examples which confirm our theoretical results.

6 Numerical results

As numerical examples we consider the following functions

$$f_1(x)=e^x,\quad f_2(x)=\tan x,\quad f_3(x)=\frac{313x^4-6900x^2+15120}{13x^4+660x^2+15120}$$

and corresponding integrals

$$I=\int_0^1 \mathrm{e}^x\,\mathrm{d}x=\mathrm{e}-1=1.718281828459045\dots\ ,$$

$$J=\int_0^1\tan x\,\mathrm{d}x=-\log(\cos 1)=0.6156264703860142\dots\ ,$$

$$K=\int_0^1 \frac{313x^4-6900x^2+15120}{13x^4+660x^2+15120}\,\mathrm{d}x = 0.84147101789394123457\ldots\ .$$

Applying the optimal quadrature formula (1.1), with $N = 10, 100, 1000$, to the previous integrals we obtain their approximate values denoted by I_N, J_N, and K_N, respectively. Using Theorem 5.1 and taking into account inequality (5.5) we get upper bounds for absolute errors of these integrals. The corresponding absolute errors and upper bounds are displayed in Table 6.1. Numbers in parentheses indicate decimal exponents.

Table 6.1 Absolute errors of quadrature approximations I_N, J_N, K_N and corresponding upper bounds

N	$\lvert I_N-I\rvert$	$\lVert f_1\rVert\cdot\lVert \mathring{\ell}\rVert$	$\lvert J_N-J\rvert$	$\lVert f_2\rVert\cdot\lVert \mathring{\ell}\rVert$	$\lvert K_N-K\rvert$	$\lVert f_3\rVert\cdot\lVert \mathring{\ell}\rVert$
10	1.779(−4)	1.454(−3)	2.796(−4)	1.287(−3)	6.985(−10)	1.515(−9)
100	1.788(−7)	1.299(−5)	2.933(−7)	1.149(−5)	7.577(−13)	1.354(−11)
1000	1.789(−10)	1.282(−7)	2.941(−10)	1.136(−7)	7.612(−16)	1.334(−13)

Applying the corresponding trapezoidal rule

$$T_N(\varphi)=h\left(\frac{1}{2}\varphi(0)+\sum_{\nu=1}^{N-1}\varphi(\nu h)+\frac{1}{2}\varphi(1)\right),\quad h=\frac{1}{N},$$

for example, to the first integral we have

$$T_N=T_N(\exp(\,\cdot\,))=\frac{1}{N}\,\mathrm{e}^{1/N}\,\frac{\mathrm{e}-1}{\mathrm{e}^{1/N}-1}-\frac{\mathrm{e}-1}{2N},$$

which evidently tends to $\mathrm{e}-1$ as $N\to\infty$, with the relative errors

$$\frac{T_N-I}{I}=\frac{h^2}{12}-\frac{h^4}{720}+O(h^5).$$

For example, for $N = 10$, 100, and 1000, we get 8.332(−4), 8.333(−6), and 8.333(−8), respectively. Otherwise, it is well-known that the error of the trapezoidal rule for a function $\varphi\in C^2[0,1]$ can be expressed in the form $-(h^2/12)\varphi''(\xi)$ for some $\xi\in(0,1)$.

Notice that the optimal quadrature formula gives the best results for the last integral, because its integrand is a rational approximation for the function $\cos x$ (cf. [10, p. 66]).

At the end we consider numerical integration of the function

$$\varphi_m(x)=\sum_{k=0}^{m}(-1)^k\,\frac{x^{2k}}{(2k)!}\,,$$

i.e., a finite part of the Taylor series of the function $\cos x$, applying our optimal quadrature formula (OQF) and the trapezoidal rule (TR), taking a small number of nodes ($N = 5, 10$, and 15).

The relative errors for $m = 1(1)8$ are presented in Table 6.2. We can see that for a fixed N the optimal quadrature formula gives better results when degree of the polynomial φ_m increases (better approximation of $\cos x$!). On the other side, the

Table 6.2 Relative errors of quadrature approximations for the function $\varphi_m(x)$, $m = 1(1)8$, obtained by the optimal quadrature formula (OQF) and the trapezoidal rule (TR)

	$N=5$		$N=10$		$N=15$	
m	OQF	TR	OQF	TR	OQF	TR
1	1.11(−4)	4.00(−3)	1.42(−5)	1.00(−3)	4.23(−6)	4.44(−4)
2	8.77(−6)	3.30(−3)	1.15(−6)	8.25(−4)	3.46(−7)	3.67(−4)
3	2.72(−7)	3.34(−3)	3.74(−8)	8.34(−3)	1.14(−8)	3.70(−4)
4	4.44(−9)	3.34(−3)	6.48(−10)	8.33(−3)	1.20(−10)	3.70(−4)
5	4.47(−11)	3.34(−3)	6.96(−12)	8.33(−3)	2.18(−12)	3.70(−4)
6	3.06(−13)	3.34(−3)	5.07(−14)	8.33(−3)	1.62(−14)	3.70(−4)
7	1.51(−15)	3.34(−3)	2.67(−16)	8.33(−3)	8.67(−17)	3.70(−4)
8	5.66(−18)	3.34(−3)	1.06(−18)	8.33(−3)	3.53(−19)	3.70(−4)
$\cos x$	m.p.	3.34(−3)	m.p.	8.33(−3)	m.p.	3.70(−4)

trapezoidal rule has the same accuracy for each $m > 3$, as well as for the trigonometric function $\cos x$ (see the last row in Table 6.2).

All calculations were performed in MATHEMATICA with 34 decimal digits mantissa. The same results can be obtained using FORTRAN in quadruple precision arithmetic (with machine precision m.p. $\approx 1.93 \times 10^{-34}$).

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