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Gauss–Laguerre interval quadrature rule☆

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Abstract

In this paper we prove the existence and uniqueness of the Gaussian interval quadrature formula with respect to the generalized Laguerre weight function. An algorithm for numerical construction has also investigated and some suitable solutions are proposed. A few numerical examples are included.

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1. Introduction

By the Gaussian interval quadrature formula with respect to the positive weight function w, we assume a quadrature formula of the following form:

$$\int_a^b f w \,\mathrm{d}x \approx \sum_{k=1}^{n} \frac{\mu_k}{2h_k} \int_{x_k-h_k}^{x_k+h_k} f w \,\mathrm{d}x, \tag{1.1}$$

which integrates exactly all polynomials of degree less than $2n$.

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There are different results for the questions of existence of such quadrature rules (for example, see [1,4,7,9,11]).

The question of the existence for bounded a, b is proved in [2] in much wider context. Suppose that w is a weight function on $[-1, 1]$, i.e., a nonnegative Lebesgue integrable function, such as that for $I = (\alpha, \beta) \subset [-1, 1]$, $\alpha \neq \beta$, we have $\int_I w(x)\,\mathrm{d}x \neq 0$. In [2], Bojanov and Petrov proved the following statement: *Given the ordered set of odd integers* $\{v_1, \dots, v_n\}$, *with the property* $n + \sum_{k=1}^n v_k = N + 1$, *the Chebyshev system of functions* $\{u_0, \dots, u_N\}$ *on* $[-1, 1]$, *the Markov system of functions* $v_0, \dots, v_{m-1}$, *on* $[-1, 1]$, *where* $m = \max\{v_1, \dots, v_n\}$, *and a set of the lengths* $h_1 \geqslant 0, \dots, h_n \geqslant 0$, *with* $\sum h_k < 1$, *there exists an interpolatory quadrature formula of the form*

$$\int_{-1}^{1} f(x)w(x)\,\mathrm{d}x \approx \sum_{k=1}^{n} \sum_{v=0}^{v_k-1} \frac{\mu_{k,v}}{2h_k} \int_{I_k} f(x)v_v(x)w(x)\,\mathrm{d}x,$$

where intervals $I_k \subset [-1, 1]$, $k = 1, \dots, n$, *are non-overlapping, with the length of* I_k *equals* $2h_k$, *which integrates exactly every element of the linear span* $\{u_0, \dots, u_N\}$.

Also they proved that Gaussian interval quadrature formula for the Legendre weight $w(x) = 1$ on $[-1, 1]$ is unique (see [3]). The uniqueness of Gaussian interval quadrature formula for the Jacobi weight and its numerical construction was given in [6].

In this paper we present the existence and uniqueness results of the Gaussian interval quadrature formula for the generalized Laguerre weight function $w(x) = x^\alpha \mathrm{e}^{-x}$, $\alpha > -1$, on $(a, b) = (0, +\infty)$. The paper is organized as follows. In the remainder of this section we give some notation and state the main result. Preliminary and auxiliary results are given in Section 2 and the main result is proved in Section 3. Finally, a numerical algorithm and numerical results are presented in Section 4.

Denote by $\mathbf{H}_n^H$ the following set of the admissible lengths

$$\mathbf{H}_n^H = \left\{ \mathbf{h} \in \mathbb{R}^n \;\middle|\; h_i \geqslant 0, \sum_{k=1}^{n} h_k \leqslant H \right\}$$

and the corresponding set of the admissible nodes by

$$\mathbf{X}_n(\mathbf{h}) = \{\mathbf{x} \in \mathbb{R}^n \mid 0 < x_1 - h_1 \leqslant x_1 + h_1 < \cdots < x_n - h_n \leqslant x_n + h_n < +\infty\}.$$

Also, we introduce the set of the formal nodes

$$\widetilde{\mathbf{X}}_n(\mathbf{h}) = \{\mathbf{x} \in \mathbb{R}^n \mid 0 < x_1 - h_1 \leqslant x_1 + h_1 \leqslant \cdots \leqslant x_n - h_n \leqslant x_n + h_n < +\infty\}$$

and the set $\mathbf{X}_n^{L,\varepsilon_0,M}$ by

$$\begin{aligned}\mathbf{X}_n^{L,\varepsilon_0,M}(\mathbf{h}) = \{&\mathbf{x} \in \mathbb{R}^n \mid 0 < L < x_1 - h_1,\ x_{k+1} - h_{k+1} - x_k - h_k > \varepsilon_0 > 0,\\ &k = 1, \dots, n-1,\ x_n + h_n < M\}.\end{aligned}$$

Our main result can be stated in the following form.

Theorem 1.1. *For every* $\mathbf{h} \in \mathbf{H}_n^H$, *the Gaussian interval quadrature rule* (1.1) *with respect to the generalized Laguerre weight* $w(x) = x^\alpha \mathrm{e}^{-x}$, $\alpha > -1$, *on* $(a, b) = (0, +\infty)$, *with nodes* $\mathbf{x} \in \widetilde{\mathbf{X}}_n(\mathbf{h})$ *and positive weights* μ_k, $k = 1, \dots, n$, *exists uniquely. Moreover, there exist the positive constants L,* ε_0, *and M, depending on n and H, such that* $\mathbf{x} \in \mathbf{X}_n^{L,\varepsilon_0,M}(\mathbf{h})$.

2. Preliminary and auxiliary results

Let $\mathscr{P}_n$, $n \in \mathbb{N}_0$, be the set of all algebraic polynomials of degree at most n and $\mathscr{P}$ be the set of all algebraic polynomials.

First, we give some preliminary definitions and results. Denote $\phi = x$

$$d\mu = w\,dx = x^{\alpha} e^{-x}\,dx \quad \text{on } [0, +\infty),$$

where $\alpha > -1$, and $\psi = 1 + \alpha - x$, such that we have the following Pearson's equation $(\phi w)' = \psi w$ holds (see [5]).

Lemma 2.1. *For any polynomial $p \in \mathscr{P}_n$, there exists $q \in \mathscr{P}_{n-1}$ and $\gamma \in \mathbb{C}$, such that*

$$\int p w\,dx = q\phi w + \gamma \Gamma[1+\alpha, x] + A,$$

where A is an integration constant and Γ is the incomplete gamma function defined by

$$\Gamma[a, x] = \int_x^{+\infty} t^{a-1} e^{-t}\,dt, \quad a > 0, \quad x \geqslant 0.$$

Proof. Let $p \in \mathscr{P}_n$. For every polynomial $p = r'\phi + r\psi$, $r \in \mathscr{P}_{n-1}$, we have

$$\int p w\,dx = \int (r'\phi + r\psi) x^{\alpha} e^{-x}\,dx = \int (r\phi w)'\,dx = r\phi w + A$$

so that we can identify $q = r$ and $\gamma = 0$.

Now, we consider the linear space

$$\mathscr{L}_n = \{r'\phi + r\psi \mid r \in \mathscr{P}_{n-1}\}.$$

Obviously its basis is

$$\ell_\nu = (x^{\nu-1})' x + x^{\nu-1}(1+\alpha-x) = -x^\nu + (\nu+\alpha)x^{\nu-1}, \quad \nu = 1, \ldots, n.$$

Adding $\ell_0 = 1$ to this basis we get a complete basis for $\mathscr{P}_n (= \mathscr{L}_n \dot{+} \mathscr{P}_0)$. Taking any polynomial $p \in \mathscr{P}_n$, we have

$$p = \sum_{k=0}^{n} \gamma_k \ell_k$$

and therefore,

$$\int p w\,dx = \gamma_0 \int \ell_0 w\,dx + \sum_{k=1}^{n} \gamma_k \int \ell_k w\,dx = \gamma_0 \Gamma[1+\alpha, x] + \phi w \sum_{k=1}^{n} \gamma_k x^{k-1} + A$$

from where we can identify uniquely q as $\sum_{k=1}^{n} \gamma_k x^{k-1}$ and $\gamma = \gamma_0$. $\square$

Now, we define the following sets:

$$I_k = (x_k - h_k, x_k + h_k), \quad \overline{I}_k = [x_k - h_k, x_k + h_k], \quad k = 1, \ldots, n,$$

$$I = \bigcup_{k=1}^{n} I_k, \quad \overline{I} = \bigcup_{k=1}^{n} \overline{I}_k, \quad O = [0, +\infty) \backslash \overline{I}$$

as well as $O_1 = [0, x_1 + h_1)$, $O_{k+1} = (x_k + h_k, x_{k+1} - h_{k+1})$, $k = 1, \ldots, n-1$, $O_{n+1} = (x_n + h_n, +\infty)$. These definitions enables us to express results in a shorter form.

The next definition gives precisely what we mean by the Gauss–Laguerre interval quadrature rule.

Definition 2.2. The Gaussian interval quadrature rule with respect to the generalized Laguerre measure $\mathrm{d}\mu = x^{\alpha}\mathrm{e}^{-x}\,\mathrm{d}x$, $\alpha > -1$, for $\mathbf{h} \in \mathbf{H}_n^H$ is an interpolatory quadrature rule of the form

$$\int p \,\mathrm{d}\mu = \sum_{k=1}^{n} \frac{\mu_k}{2h_k} \int_{I_k} p \,\mathrm{d}\mu, \quad p \in \mathscr{P}_{2n-1} \tag{2.1}$$

provided $\mathbf{x} \in \widetilde{\mathbf{X}}_n(\mathbf{h})$.

The following statement is very important, since it enables us to prove almost all of our results. Similar results for finite intervals can be found in [2,3,6].

Lemma 2.3. (i) *Assume* $1 \leqslant j_k \leqslant 2$, $k = 1, \ldots, n$, *with* $\sum_{k=1}^{n} j_k = N + 1$, $\mathbf{h} \in \mathbf{H}_n^H$, $\mathbf{x} \in \widetilde{\mathbf{X}}_n(\mathbf{h})$, *and let* $f_{m,k}$, $m = 1, j_k$, $k = 1, \ldots, n$, *be arbitrary numbers. Then the interpolation problem*

$$\frac{1}{2h_k} \int_{I_k} p^{(m-1)} \,\mathrm{d}\mu = f_{m,k}, \quad m = 1, j_k, \quad k = 1, \ldots, n \tag{2.2}$$

has the unique solution in $\mathscr{P}_N$.

(ii) *Assume that* $1 \leqslant j_k \leqslant 2$, $k = 1, \ldots, n$, *with* $\sum_{k=1}^{n} j_k = N + 1$, $\mathbf{h} \in \mathbf{H}_n^H$, $\mathbf{x} \in \widetilde{\mathbf{X}}_n(\mathbf{h})$, *then for every* $c \in \mathbb{C}$ *there exists the unique* $q_c \in \mathscr{P}_N$, *such that* $p = cx^{N+1} + q_c$, *solves the following interpolation problem*:

$$\frac{1}{2h_k} \int_{I_k} p^{(m-1)} \,\mathrm{d}\mu = 0, \quad m = 1, j_k, \quad k = 1, \ldots, n$$

and there holds $q_c = cq_1$. *In every* $\overline{I}_k$, *the polynomial p has exactly* j_k *zeros and those are all its zeros*.

Proof. In order to prove this lemma, we show that the corresponding homogenous system of equations (2.2), with $f_{m,k} \equiv 0$, has only a trivial solution. Note that this system can be expressed as a system of linear equations for the coefficients of p.

The proof for the part (i) is already given in [3, Lemma 1]. Here, we give this proof for the sake of completeness. We can simply count zeros to see that in every subinterval I_k there are j_k zeros, so that in total we have $\sum_k j_k = N + 1$ zeros. This means that if the solution is not trivial it has a degree at least $N + 1$, and it is not a solution in $\mathscr{P}_N$.

For the part (ii), we can rewrite the interpolation problem in the following form:

$$\frac{1}{2h_k}\int_{I_k} q_c^{(m-1)}\,\mathrm{d}\mu = -\frac{c}{2h_k}\int_{I_k} (x^{N+1})^{(m-1)}\,\mathrm{d}\mu, \quad m=1, j_k, \quad k=1,\ldots,n. \tag{2.3}$$

Now, we can apply the first part of this lemma with

$$f_{m,k} = -\frac{c}{2h_k}\int_{I_k} (x^{N+1})^{(m-1)}\,\mathrm{d}\mu, \quad m=1, j_k, \quad k=1,\ldots,m$$

to the interpolation problem (2.3), and denote the unique solution by q_c. Obviously, the linear system of equations which defines q_c has a free vector from which c can be factorized, so that $q_c = cq_1$. For the last statement we refer to the proof of part (i). □

The next lemma shows that for every $\mathbf{h} \in \mathbf{H}_n^H$ the Gauss–Laguerre interval quadrature rule must have nodes in $\mathbf{X}_n(\mathbf{h})$.

Lemma 2.4. *Suppose* $\mathbf{h} \in \mathbf{H}_n^H$ *and there exists a Gauss–Laguerre interval quadrature rule with nodes* $\mathbf{x} \in \widetilde{\mathbf{X}}_n(\mathbf{h})$, *then* $\mathbf{x} \in \mathbf{X}_n(\mathbf{h})$.

Proof. Let $\mathbf{h} \in \mathbf{H}_n^H$ and $\mathbf{x} \in \widetilde{\mathbf{X}}_n(\mathbf{h})$, but $\mathbf{x} \notin \mathbf{X}_n(\mathbf{h})$. Then at least one of the equalities

$$x_k + h_k = x_{k+1} - h_{k+1}, \quad k=1,\ldots,n-1$$

holds. Suppose, it is the case for some $k \in \{1,\ldots,n-1\}$. According to the interpolation Lemma 2.3, part (ii), there exists a monic $p \in \mathscr{P}_{2n-2}$, with the properties

$$\frac{1}{2h_\nu}\int_{I_\nu} p^{(m-1)}\,\mathrm{d}\mu = 0, \quad \nu = 1,\ldots,k-1,k+2,\ldots,n$$

and

$$\frac{1}{2h_k}\int_{I_k} p\,\mathrm{d}\mu = \frac{1}{2h_{k+1}}\int_{I_{k+1}} p\,\mathrm{d}\mu = 0, \quad h_k \neq 0, \text{ or } h_{k+1} \neq 0,$$

$$p(x_k) = p'(x_k) = 0, \quad h_k = h_{k+1} = 0.$$

Obviously such p annihilates the Gauss–Laguerre interval quadrature sum and it is of a constant sign on O. This means that $\int p\,\mathrm{d}\mu = \int_O p\,\mathrm{d}\mu \neq 0$, which is a contradiction. □

An immediate consequence of the previous lemma is that all weights μ_k, are positive in the Gauss–Laguerre interval quadrature rule.

Lemma 2.5. *For the Gauss–Laguerre interval quadrature rule* (2.1), *we have* $\mu_k > 0$, $k=1,\ldots,n$.

Proof. Suppose there is some index $k \in \{1,\ldots,n\}$, such that $\mu_k \leqslant 0$. According to the interpolation Lemma 2.3, there exists a monic polynomial $p \in \mathscr{P}_{2n-2}$, such that

$$\frac{1}{2h_\nu}\int_{I_\nu} p^{(m-1)}\,\mathrm{d}\mu = 0, \quad m=1,2, \quad \nu=1,\ldots,k-1,k+1,\ldots,n.$$

For this p the Gauss–Laguerre quadrature formula (3.1) is exact. On the other side this polynomial p is positive on $A=O\cup I_k$, so that we conclude $\mu_k=2h_k\int p\,\mathrm{d}\mu/\int_{I_k} p\,\mathrm{d}\mu=2h_k\int_A p\,\mathrm{d}\mu/\int_{I_k} p\,\mathrm{d}\mu>0$, which is a contradiction. □

The following theorem shows that there exists a uniform bound for nodes in (2.1) regarding to $\mathbf{h}\in\mathbf{H}_n^H$. This is an important result, which enables us to think about Gauss–Laguerre interval quadrature rule as it is given with respect to some measure on the bounded supporting set.

Theorem 2.6. *Let $n\in\mathbb{N}$ and $H\geqslant 0$ be given. Then there exists a constant $M>0$ such that for every $\mathbf{h}\in\mathbf{H}_n^H$ and $\mathbf{x}\in\mathbf{X}_n(\mathbf{h})$, for which* (2.1) *is the Gauss–Laguerre interval quadrature rule, we have*

$$x_n<M. \tag{2.4}$$

Proof. Suppose, it is not the case for some $n\in\mathbb{N}$ and $H\geqslant 0$. Then for every $M>0$ there exists a quadrature rule of the form (2.1) for some $\mathbf{h}\in\mathbf{H}_n^H$, such that there exists an index $m\in\{1,\ldots,n\}$ for which $x_m\geqslant M$.

Suppose that for all $M>0$ we have that all $x_\nu+h_\nu$, $\nu=1,\ldots,k$, are bounded by some M_k, and that $x_\nu>M$ for $\nu=k+1,\ldots,n$. For sufficiently large M there exists a fixed constant M' such that $M_k<M'<M$.

According to the interpolation Lemma 2.3, part (ii), there exists a monic polynomial p of degree $2n-1$, satisfying the conditions

$$\frac{1}{2h_\nu}\int_{I_\nu} p^{(m-1)}\,\mathrm{d}\mu=0,\quad m=1,2,\ \ \nu=1,\ldots,n-1$$

and

$$\frac{1}{2h_n}\int_{I_n} p\,\mathrm{d}\mu=0.$$

This polynomial p annihilates the Gauss–Laguerre interval quadrature sum. Choose $P=-p$. So that P is positive on $O\backslash O_{n+1}$ and negative on O_{n+1}. Since $\int_I P\,\mathrm{d}\mu=0$, we have

$$\int P\,\mathrm{d}\mu=\int_{O\backslash O_{n+1}} P\,\mathrm{d}\mu-\int_{O_{n+1}}(-P)\,\mathrm{d}\mu=0.$$

Then for chosen M', $M_k<M'<M$, we have

$$\int_{O\backslash O_{n+1}} P\,\mathrm{d}\mu>\int_{M_k}^{M'}(x-M_k)^{2k}(M'-x)^{2n-2k-1}\,\mathrm{d}\mu=J_1>0.$$

Note that, according to Lemma 2.1, there exist q and γ such that

$$J_1=(q\phi w)(M')-(q\phi w)(M_k)+\gamma(\Gamma(1+\alpha,M')-\Gamma(1+\alpha,M_k)).$$

Similarly, there exist q_1 and γ_1 such that

$$0<\int_{O_{n+1}}(-P)\,\mathrm{d}\mu<\int_M^{+\infty}x^{2k}(x-M')^{2n-2k-1}\,\mathrm{d}\mu=-(q_1\phi w)(M)-\gamma_1\Gamma(1+\alpha,M)=J_2.$$

We see that J_2 tends to zero as M increases, so that

$$\int P\,\mathrm{d}\mu = \int_{O\setminus O_{n+1}} P\,\mathrm{d}\mu - \int_{O_{n+1}} (-P)\,\mathrm{d}\mu > J_1 - J_2 > 0$$

for sufficiently large M. This is a contradiction, i.e., x_{k+1} must be bounded.

Repeating the same arguments we prove that $x_{k+2}, \ldots, x_n$ must be bounded, which is a contradiction. □

Remark 2.7. According to (2.4), $x_n + h_n$ is also bounded, i.e., $x_n + h_n < M + H$.

Almost with the same arguments, we can prove the following result.

Lemma 2.8. *Let $n \in \mathbb{N}$ and $H \geqslant 0$ be given. Then there exists a constant $L > 0$ such that for every $\mathbf{h} \in \mathbf{H}_n^H$ and $\mathbf{x} \in \mathbf{X}_n(\mathbf{h})$, for which* (2.1) *is the Gauss–Laguerre interval quadrature rule, we have*

$$0 < L < x_1 - h_1.$$

Proof. Suppose it is not the case for some $n \in \mathbb{N}$ and $H \geqslant 0$. Then for every $L > 0$, there exist $\mathbf{h} \in \mathbf{H}_n^H$ and respective nodes of the Gauss–Laguerre interval quadrature rule $\mathbf{x} \in \mathbf{X}_n(\mathbf{h})$, such that there exists an index $k \in \{1, \ldots, n\}$ such that $0 < x_\nu - h_\nu \leqslant x_\nu + h_\nu < L$, $\nu = 1, \ldots, k$.

According to the interpolation Lemma 2.3, part (ii), there exists a monic polynomial P of degree $2n-1$, such that

$$\frac{1}{2h_1}\int P\,\mathrm{d}\mu = \frac{1}{2h_\nu}\int_{I_\nu} P^{(m-1)}\,\mathrm{d}\mu = 0, \quad m = 1, 2, \quad \nu = 2, \ldots, n.$$

This polynomial P annihilates the Gauss–Laguerre interval quadrature sum. In order to prove that P cannot annihilate $\int P\,\mathrm{d}\mu$, for every $L > 0$, we consider

$$\int P\,\mathrm{d}\mu = \int_{O\setminus O_1} P\,\mathrm{d}\mu - \int_{O_1} (-P)\,\mathrm{d}\mu.$$

Note that P has a positive sign on $O\setminus O_1$ and negative on O_1. According to Theorem 2.6 and corresponding remark, there exists $M > 0$ such that $x_n + h_n < M' = M + H$. Then, using Lemma 2.1, with the corresponding q and γ, we have

$$\int_{O\setminus O_1} P\,\mathrm{d}\mu > \int_{M'}^{+\infty} (x - M')^{2n-1}\,\mathrm{d}\mu = -(q\phi w)(M') - \gamma(\Gamma(1+\alpha, M')) = J_1 > 0.$$

Also, with the corresponding q_1 and γ_1, we have

$$0 < \int_{O_1} (-P)\,\mathrm{d}\mu < \int_0^L (x - M')^{2n-1}\,\mathrm{d}\mu = (q_1\phi w)(L) + \gamma_1(\Gamma(1+\alpha, L) - \Gamma(1+\alpha)) = J_2,$$

where J_2 evidently tends to zero as $L \to 0^+$. Thus, for a sufficiently small L, we have

$$\int P\,\mathrm{d}\mu = \int_{O\setminus O_1} P\,\mathrm{d}\mu - \int_{O_1} (-P)\,\mathrm{d}\mu > J_1 - J_2 > 0$$

which is a contradiction. Therefore, $x_1 - h_1$ must be uniformly bounded from zero. □

In the sequel we use the following notation.

Definition 2.9. We denote

$$\Omega = \prod_{\nu=1}^{n} (x - x_\nu - h_\nu)(x - x_\nu + h_\nu),$$

$$\Omega_k = \frac{\Omega}{(x - x_k - h_k)(x - x_k + h_k)}, \quad k = 1, \dots, n$$

and

$$\Delta_k(\Omega_k \phi w) = \begin{cases} \dfrac{(\Omega_k \phi w)(x_k + h_k) - (\Omega_k \phi w)(x_k - h_k)}{2h_k}, & h_k \neq 0, \\ \partial_{x_k}[(\Omega_k \phi w)(x_k)], & h_k = 0 \end{cases}$$

for $k = 1, \dots, n$, where $\partial_{x_k} = \partial/\partial x_k$.

Theorem 2.10. *For every* $\mathbf{h} \in \mathbf{H}_n^H$, *the nodes* $\mathbf{x} \in \mathbf{X}_n(\mathbf{h})$ *of the quadrature rule* (2.1) *satisfy the system of equations*

$$\Delta_k(\Omega_k \phi w) = 0, \quad k = 1, \dots, n. \tag{2.5}$$

For $\mathbf{h} \in \mathbf{H}_n^H$, *every solution* $\mathbf{x} \in \mathbf{X}_n(\mathbf{h})$ *of system* (2.5) *defines the nodes for the Gauss–Laguerre interval quadrature rule* (2.1).

Proof. Applying the Gauss–Laguerre interval quadrature rule (2.1) to the polynomial $(\Omega_\nu \phi w)'/w$ of degree $2n - 1$, we have

$$0 = \int \frac{(\Omega_\nu \phi w)'}{w} \, \mathrm{d}\mu = \sum_{k=1}^{n} \frac{\mu_k}{2h_k} \int_{I_k} \frac{(\Omega_\nu \phi w)'}{w} \, \mathrm{d}\mu = \mu_\nu \Delta_\nu(\Omega_\nu \phi w),$$

i.e., if $\mathbf{x}$ are nodes of the Gauss–Laguerre interval quadrature rule, they must satisfy (2.5), since according to Lemma 2.5, we have $\mu_\nu > 0$, $\nu = 1, \dots, n$.

For any $p \in \mathscr{P}_{2n-2}$ we have

$$\sum_{k=1}^{n} \left(\frac{p(x_k + h_k)}{\Omega'(x_k + h_k)} + \frac{p(x_k - h_k)}{\Omega'(x_k - h_k)} \right) = 0. \tag{2.6}$$

This can be proved by applying Cauchy Residue Theorem to the rational function p/Ω over the contour $\Lambda_R = \{x \mid |x - M/2| = R\}$, $R > M/2$, and letting $R \to +\infty$.

Now, suppose that for $\mathbf{x} \in \mathbf{X}_n(\mathbf{h})$ we have

$$\frac{1}{(\Omega' \phi w)(x_k + h_k)} + \frac{1}{(\Omega' \phi w)(x_k - h_k)} = 0. \tag{2.7}$$

Then obviously, according to (2.6), we have for any $p \in \mathscr{P}_{2n-2}$

$$
\begin{aligned}
0 &= \sum_{k=1}^{n} \frac{1}{(\Omega'\phi w)(x_k+h_k)}[(p\phi w)(x_k+h_k)-(p\phi w)(x_k-h_k)] \\
&= \sum_{k=1}^{n} \frac{1}{(\Omega'\phi w)(x_k+h_k)} \int_{I_k} (p\phi w)'\,\mathrm{d}x.
\end{aligned}
$$

But also

$$
\int \frac{(p\phi w)'}{w}\,\mathrm{d}\mu = (p\phi w)\Big|_0^{+\infty} = 0,
$$

so that for every $r \in \mathscr{P}_{2n-1}$ of the form $r = p'\phi + p\psi$, $p \in \mathscr{P}_{2n-2}$, we have

$$
\int r\,\mathrm{d}\mu = C \sum_{k=1}^{n} \frac{1}{(\Omega'\phi w)(x_k+h_k)} \int_{I_k} r\,\mathrm{d}\mu
$$

for a constant C. Now we can choose C such that the previous formula is exact for all $r \in \mathscr{P}_{2n-1}$. According to the proof of Lemma 2.1, it is enough to adjust this formula to be exact for $r = \ell_0 = 1$, which gives

$$
C = \frac{m_0}{\sum_{k=1}^{n} \dfrac{2h_k m_{0,k}}{(\Omega'\phi w)(x_k+h_k)}},
$$

where

$$
m_0 = \int \mathrm{d}\mu, \quad m_{0,k} = \frac{1}{2h_k}\int_{I_k} \mathrm{d}\mu, \quad k = 1,\ldots,n.
$$

The system of equations (2.7) defines the Gauss–Laguerre interval quadrature rule. However, it is equivalent to (2.5), because of

$$
\Omega'(x_k \pm h_k) = \pm 2h_k \Omega_k(x_k \pm h_k), \quad k = 1,\ldots,n.
$$

Using these equations, by definition of Ω_k, we can conclude that

$$
(\Omega'\phi w)(x_k+h_k) = 2h_k(\Omega_k\phi w)(x_k+h_k) > 0
$$

for $\mathbf{h} \in \mathbf{H}_n^H$ and $\mathbf{x} \in \mathbf{X}_n(\mathbf{h})$, which gives $C > 0$. Thus, all weights in the constructed quadrature rule are positive, as we know also from Lemma 2.5.

To be completely fair, we need to give an explanation for the case $h_k = 0$ for some k. Since the corresponding term of (2.6), in that case is given by

$$
\frac{p'\Omega_k - p\Omega_k'}{\Omega_k^2}(x_k),
$$

we can transform it to the form

$$\frac{p'\Omega_k - p\Omega_k'}{\Omega_k^2} = \frac{p'\Omega_k\phi w + p\Omega_k(\phi w)' - p\Omega_k(\phi w)' - p\Omega_k'\phi w}{\Omega_k^2\phi w} = \frac{(p\phi w)'\Omega_k - p(\Omega_k\phi w)'}{\Omega_k^2\phi w}$$

and we require that term with p vanish so that we have

$$(\Omega_k\phi w)'(x_k) = \partial_{x_k}[(\Omega_k\phi w)(x_k)] = 0.$$

This is exactly what equation of system (2.5) becomes for $h_k = 0$. □

Remark 2.11. According to the proof Theorem 2.10, we have the following formulas:

$$\mu_k = \frac{m_0}{\sum_{v=1}^n \dfrac{m_{0,v}}{(\Omega_v\phi w)(x_v + h_v)}} \frac{1}{(\Omega_k\phi w)(x_k + h_k)}, \quad k = 1, \ldots, n \tag{2.8}$$

for the weights in the Gauss–Lagurre quadrature formula (2.1).

Lemma 2.12. *Suppose $n \in \mathbb{N}$ and $H \geqslant 0$ are given. There exist $\varepsilon_0 > 0$, $L > 0$ and $M > 0$, such that for all $\mathbf{h} \in \mathbf{H}_n^H$ and all nodes $\mathbf{x} \in \mathbf{X}_n(\mathbf{h})$ of the Gauss–Laguerre interval quadrature rule* (2.1), *we have $\mathbf{x} \in \mathbf{X}_n^{L,\varepsilon_0,M}$.*

Proof. The existences of L and M is already proved, so we prove now the existence of ε_0. Assume contrary, then for every $\varepsilon_0 > 0$, there exists $\mathbf{h}^{\varepsilon_0} \in \mathbf{H}_n^H$ and the respective set of nodes $\mathbf{x}^{\varepsilon_0} \in \mathbf{X}_n(\mathbf{h}^{\varepsilon_0})$, for which (2.1) is Gauss–Laguerre quadrature formula, with the property that at least one of the following equalities:

$$x_k^{\varepsilon_0} + h_k^{\varepsilon_0} + \varepsilon_0 = x_{k+1}^{\varepsilon_0} - h_{k+1}^{\varepsilon_0}, \quad k = 1, \ldots, n-1$$

holds. Since the sets $\mathbf{h}^{\varepsilon_0}$ and $\mathbf{x}^{\varepsilon_0}$ are bounded, there are the convergent sequences $\mathbf{h}^k$, $\mathbf{x}^k$, $k \in \mathbb{N}$, with the limits $\mathbf{h}^0$ and $\mathbf{x}^0$, such that at least one of the equalities

$$x_k^0 + h_k^0 = x_{k+1}^0 - h_{k+1}^0, \quad k = 1, \ldots, n-1$$

holds. Since the weights μ_v, $v = 1, \ldots, n$, are continuous functions of $\mathbf{h}$ and $\mathbf{x}$, according to (2.8), for $\mathbf{h}^0$ and the respective set of nodes $\mathbf{x}^0$, we have that the rule

$$\int p \, \mathrm{d}\mu = \sum_{k=1}^n \frac{\mu_k^0}{2h_k^0} \int_{I_k^0} p \, \mathrm{d}\mu$$

constructed from the nodes $\mathbf{x}^0$ and lengths $\mathbf{h}^0$, is exact for $p \in \mathscr{P}_{2n-1}$, because of continuity. Since for this Gauss–Laguerre interval quadrature rule we have at least two intervals which have the boundary point in common, we can apply the same arguments as in the proof of Lemma 2.4 to produce a contradiction. □

3. Proof of the main result

To prove the main result, we are going to need the following topological result, which can be found in [8,10].

Assume D is a bounded open set in $\mathbb{R}^n$, with the closure $\overline{D}$ and the boundary ∂D, and $\Phi\colon \overline{D} \to \mathbb{R}^n$ is a continuous mapping. By $\deg(\Phi, D, \mathbf{c})$ we denote the topological degree of Φ with respect to D and $\mathbf{c} \notin \Phi(\partial D)$.

Lemma 3.1. (i) *If* $\deg(\Phi, D, \mathbf{c}) \neq 0$, *the equation* $\Phi(\mathbf{x}) = \mathbf{c}$ *has a solution in* D.

(ii) *Let* $\Phi(\mathbf{x}, \lambda)$ *be a continuous map* $\Phi : \overline{D} \times [0,1] \to \mathbb{R}^n$, *such that* $\mathbf{c} \notin \Phi(\partial D, [0,1])$, *then* $\deg(\Phi(\mathbf{x}, \lambda), D, \mathbf{c})$ *is a constant independent of* λ.

(iii) *Suppose* $\Phi \in C^1(D)$, $\mathbf{c} \notin \Phi(\partial D)$ *and* $\det(\Phi'(\mathbf{x})) \neq 0$ *for any* $\mathbf{x} \in D$ *such that* $\Phi(\mathbf{x}) = \mathbf{c}$. *Then, the equation* $\Phi(\mathbf{x}) = \mathbf{c}$ *has only finitely many solutions* $\mathbf{x}^\nu$ *in* D *and there holds*

$$\deg(\Phi, D, \mathbf{c}) = \sum_{\mathbf{x}^\nu} \operatorname{sgn}(\det(\Phi'(\mathbf{x}^\nu))).$$

Now we are ready to prove the main result given by Theorem 1.1.

Proof of Theorem 1.1. Let

$$\Psi_k = -\Delta_k(\Omega_k \phi w) = 0, \quad k = 1, \ldots, n. \tag{3.1}$$

Suppose $\mathbf{x} \in \mathbf{X}_n^{L,\varepsilon_0,M}(\mathbf{h})$ is solution of (3.1). Then we have

$$\begin{aligned}\partial_{x_k}\Psi_k = (\Omega_k\phi w)(x_k+h_k)\Bigg(&\sum_{\nu\neq k}\frac{1}{(x_k+h_k-x_\nu-h_\nu)(x_k-h_k-x_\nu-h_\nu)}\\ &+\frac{1}{(x_k+h_k-x_\nu+h_\nu)(x_k-h_k-x_\nu+h_\nu)}+\frac{1+\alpha}{x_k^2-h_k^2}\Bigg)>0\end{aligned}$$

and inequality is obvious. Now, we have

$$\begin{aligned}\partial_{x_m}\Psi_k = -(\Omega_k\phi w)(x_k+h_k)\Bigg(&\frac{1}{(x_k+h_k-x_m-h_m)(x_k+h_k-x_m+h_m)}\\ &+\frac{1}{(x_k-h_k-x_m-h_m)(x_k-h_k-x_m+h_m)}\Bigg)<0\end{aligned}$$

inequality is obvious.

Also it is clear that

$$\partial_{x_k}\Psi_k + \sum_{m\neq k}\partial_{x_m}\Psi_k = (\Omega_k\phi w)(x_k+h_k)\frac{1+\alpha}{x_k^2-h_k^2} > 0,$$

which gives

$$\partial_{x_k}\Psi_k > -\sum_{m\neq k}\partial_{x_m}\Psi_k = \sum_{m\neq k}|\partial_{x_m}\Psi_k|.$$

This means that Jacobian is diagonally dominant, with positive elements on the main diagonal and negative elsewhere, so that

$$\operatorname{sgn}(|\partial_{x_m}\Psi_k|_{m,k=1,\ldots,n}) = 1.$$

The rest of the proof goes exactly as it is given in [3] or [6]. Choose $\delta = \min\{L, \varepsilon_0\}$. The proof has N steps, where N is defined by $h = (N+\eta)\delta/4$, $0 < \eta \leqslant 1$, with $h = \max\{h_1, \ldots, h_n\}$. At the jth step, the uniqueness is proved for the set of lengths $\mathbf{h}^{(j)} = (j+\eta)\frac{\delta}{4h}\mathbf{h}$, $j = 0, \ldots, N$.

In the first step, the mappings

$$\Phi^{(0)}(\mathbf{x}, \lambda) = (\Psi_1(\mathbf{x}, \lambda\mathbf{h}^{(0)}), \ldots, \Psi_n(\mathbf{x}, \lambda\mathbf{h}^{(0)}))$$

are considered on $\mathbf{X}_n^{L,\varepsilon_0,M}(\mathbf{0})$ for each $0 \leqslant \lambda \leqslant 1$. It is obvious $\Phi^{(0)}(\mathbf{x}, 0) = 0$ has solution for $\lambda = 0$ and that solution is unique. That solution is, really, the classical Gauss–Laguerre quadrature rule. Since the sign of the determinant of the Jacobian is positive, using Lemma 3.1, we know that

$$\deg(\Phi^{(0)}(\mathbf{x}, 0), \mathbf{X}_n^{L,\varepsilon_0,M}(\mathbf{0}), \mathbf{0}) = 1.$$

For $\mathbf{x} \in \mathbf{X}_n^{L,\varepsilon_0,M}(\mathbf{0})$ and $0 \leqslant \lambda \leqslant 1$, we have

$$0 < x_1 - \lambda h_1^0, \quad x_k + \lambda h_k^0 < x_{k+1} - \lambda h_{k+1}^0, \quad k = 1, \ldots, n-1.$$

Then, for any solution $\mathbf{x}$ of the system $\Phi^{(0)}(\mathbf{x}, \lambda) = 0$, we have that sign of $\det(J(\mathbf{x}, \lambda\mathbf{h}^{(0)}))$ is positive. Hence, according to Lemma 3.1, part (ii), we have

$$\deg(\Phi^{(0)}(\mathbf{x}, \lambda), \quad \mathbf{X}_n^{L,\varepsilon_0,M}(\mathbf{0}), \mathbf{0}) = 1$$

for all $\lambda \in [0, 1]$, and in particular for $\lambda = 1$. This means that the system $\Phi^{(0)}(\mathbf{x}, 1) = 0$ has a unique solution in $\mathbf{X}_n^{L,\varepsilon_0,M}(\mathbf{0})$. It is also the unique solution on the smaller set $\mathbf{X}_n^{L,\varepsilon_0,M}(\mathbf{h}^{(1)})$, according to Lemma 2.12.

In the case $N \neq 0$, we proceed with the same arguments to the mappings

$$\Phi^{(1)}(\mathbf{x}, \lambda) = (\Psi_1(\mathbf{x}, \lambda\mathbf{h}^{(1)} + (1-\lambda)\mathbf{h}^{(0)}), \ldots, \Psi_n(\mathbf{x}, \lambda\mathbf{h}^{(1)} + (1-\lambda)\mathbf{h}^{(0)}))$$

to prove that there is a unique solution in $\mathbf{X}_n^{L,\varepsilon_0,M}(\mathbf{h}^{(0)})$, which is also unique in the set $\mathbf{X}_n^{L,\varepsilon_0,M}(\mathbf{h}^{(1)})$, according to Lemma 2.12.

After that, the same arguments are iterated to the mappings

$$\Phi^{(j)}(\mathbf{x}, \lambda) = (\Psi_1(\mathbf{x}, \lambda\mathbf{h}^{(j)} + (1-\lambda)\mathbf{h}^{(j-1)}), \ldots, \Psi_n(\mathbf{x}, \lambda\mathbf{h}^{(j)} + (1-\lambda)\mathbf{h}^{(j-1)}))$$

until j reaches N. $\square$

Note that we have proved the existence and uniqueness.

Table 1
Nodes and weights in (2.1) for $\mathbf{h} = (2^{-11}, \ldots, 2^{-11})$ and $w(x) = \mathrm{e}^{-x}$

k	x_k	μ_k
1	1.377940166284629(−1)	3.540104794852169(−1)
2	7.294553294941193(−1)	8.319023631683221(−1)
3	1.808343714231451	1.330288577914199
4	3.401434520637697	1.863063909509826
5	5.552496967372349	2.450255561241085
6	8.330153576459341	3.122764156923402
7	1.184378666901005(1)	3.934152696675963
8	1.627925866340221(1)	4.992414872941587
9	2.199658664463931(1)	6.572202485670547
10	2.992069784541086(1)	9.784695840808932

4. Numerical results

For numerical construction of the weights μ_k, once nodes are constructed there is nothing better than relations (2.8), since all the terms included are positive. However, it is obvious there can be some cancellation in the calculation of Ω, provided $x_k + h_k$ and $x_{k+1} - h_{k+1}$, are close enough for some k. We did not encounter any such problems, since in the examples we are presenting we keep relatively small number of nodes.

For the construction of nodes in the Gauss–Laguerre interval quadrature rule, we propose an algorithm on the system of equations (2.5). Since the system of equations (3.1) defines x_k, $k = 1, \ldots, n$, as implicit functions of $\mathbf{h}$, according to the proof of the main theorem, we know that these functions are continuous. We can start with the classical Gauss–Laguerre quadrature rule and increase $\mathbf{h}$ for small amount from $\mathbf{0}$ and solve (2.5) using the Newton–Kantorovich method. If during iterations, some of the intervals I_k interlace or if $x_1 - h_1 < 0$, we should start again with a smaller increment in $\mathbf{h}$. We iterate the procedure until we reach the desired $\mathbf{h}$. We point-out that according to the proof of main Theorem 1.1, we know that the Jacobian of the system of equations (2.5) is diagonally dominant, so that it is always invertible.

We can summarize the previous facts in the following procedure:

1° *Using QR-algorithm, construct the classical Gauss–Laguerre quadrature rule* ($\mathbf{h} = \mathbf{0}$).
2° *Increase the vector of lengths* $\mathbf{h}$ *for some small amounts and solve* (2.5) *for such a* $\mathbf{h}$. *If during computations some solution goes out of* $[-1, 1]$ *or if there is overlapping between the intervals* $\overline{I}_\nu$, $\nu = 1, \ldots, n$, *the process should start again with a smaller increment in* $\mathbf{h}$.
3° *If a desired* $\mathbf{h}$ *is reached, go to the next step*; *if it is not go back to the step* 2°.
4° *Use Eq.* (2.8) *for the construction of weights* μ_k, $k = 1, \ldots, n$.

Three examples are given. In two of them we take $n = 10$ nodes (Tables 1 and 2), and in the third example we take only four nodes (Table 3). All calculations are performed in double precision arithmetic with machine precision $\approx 2.22 \times 10^{-16}$. Numbers in parentheses indicate decimal exponents.

Table 2
Nodes and weights in (2.1) for $\mathbf{h} = (2^{-7}, 2^{-11}, \ldots, 2^{-11})$ and $w(x) = x^{-1/2}e^{-x}$

k	x_k	μ_k
1	6.052967278273032(−2)	2.412481462579694(−1)
2	5.442474084979854(−1)	7.283299804186028(−1)
3	1.523295984637797	1.233869234227906
4	3.022858246758531	1.772188329401966
5	5.085249814290907	2.363573236867304
6	7.777779889478171	3.038970646692715
7	1.120847005470487(1)	3.851789365809746
8	1.556150267238522(1)	4.909484833536406
9	2.119423108827917(1)	6.485050289965512
10	2.902528907312939(1)	9.682293806768922

Table 3
Nodes and weights in (2.1) for $\mathbf{h} = (\frac{1}{4}, \frac{1}{20}, \frac{1}{20}, \frac{1}{6})$ and $w(x) = x^{-1/2}e^{-x}$

k	x_k	μ_k
1	2.666576691057568(−1)	7.265863111854613(−1)
2	1.484028741362700	1.808913281049689
3	4.059621417868565	3.428967366512284
4	8.720111988907440	6.285437046077216

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