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LEAST SQUARES APPROXIMATION WITH CONSTRAINT: GENERALIZED GEGENBAUER CASE

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Abstract. This paper consideres the least squares approximation of function $f \in L^2[-1, 1]$, $f(-1)=f(1)=0$. Using generalized Gegenbauer weight function $p(x)=|x|^{\mu}(1-x^2)^{\alpha}$ ($\mu, \alpha > -1$), some of the results from [8] are generalized. This approximation is compared with the least square approximation without constraint. The approximation is illustrated on two numerical examples.

1. Introduction

In [11] Wrigge and Fransén considered two families of functions and showed how these functions can be approximated on [0, 1] by the polynomials of the form $\sum_{n=1}^{k} c_{n,k}(x(1-x))^n$ and $(1-2x)\sum_{n=1}^{k} C_{n,k}(x(1-x))^n$. They used the L^2-norm with respect to the weight function $p(x)=(x(1-x))^q$, where $q \in \{0, 1, \ldots\}$. In [8] Milovanović and Wrigge presented a better and more natural way of approximation using Gegenbauer polynomials $C_{k,\lambda}(x)$ orthogonal with respect to the weight function $p(x)=(1-x^2)^{\lambda-1/2}$, $x \in [-1, 1]$, $\lambda > -1/2$. In this way they generalized the results from [11] and also avoided complicated manipulations with matrices.

Further generalization of these results can be obtained by using generalized Gegenbauer monic polynomials $W_k^{(\alpha,\beta)}(x)$, orthogonal on $[-1, 1]$ with respect to the weight function $p(x)=|x|^{\mu}(1-x^2)^{\alpha}$, $\mu, \alpha > -1$, $\beta=(\mu-1)/2$, which was introduced by Lascenov in [7] (see, also, [2, pp. 155—156]). It is interesting to say that these polynomials have been again „discovered" by J. Radecki ([9]).

2. Preliminaries

The relation between generalized Gegenbauer (monic) polynomials $W_k^{(\alpha,\beta)}(x)$ and Jacobi polynomials $P_k^{(\alpha,\beta)}(x)$ is given by

$$W_{2k}^{(\alpha,\beta)}(x)=\frac{k!}{(k+\alpha+\beta+1)_k}P_k^{(\alpha,\beta)}(2x^2-1), \tag{2.1}$$

$$W_{2k+1}^{(\alpha,\beta)}(x)=\frac{k!}{(k+\alpha+\beta+2)_k}xP_k^{(\alpha,\beta+1)}(2x^2-1). \tag{2.2}$$

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Notice that

$$W_{2k+1}^{(\alpha,\beta)}(x) = xW_{2k}^{(\alpha,\beta+1)}(x). \tag{2.3}$$

Using relations (2.1), (2.2) and Jacobi polynomials theory, it is possible to obtain a set of relations for generalized Gegenbauer polynomials. For example, the recurrence relation is

$$\begin{aligned} &W_{k+1}^{(\alpha,\beta)}(x) = xW_k^{(\alpha,\beta)}(x) - \Lambda_k W_{k-1}^{(\alpha,\beta)}(x), \qquad k=0,1,\ldots, \\ &W_{-1}^{(\alpha,\beta)}(x)=0, \quad W_0^{(\alpha,\beta)}(x)=1, \end{aligned} \tag{2.4}$$

where

$$\Lambda_{2k} = \frac{k(k+\alpha)}{(2k+\alpha+\beta)(2k+\alpha+\beta+1)}, \quad \Lambda_{2k-1} = \frac{(k+\beta)(k+\alpha+\beta)}{(2k+\alpha+\beta-1)(2k+\alpha+\beta)},$$

for $k=1, 2, \ldots$, except when $\alpha+\beta=-1$; then $\Lambda_1=(\beta+1)/(\alpha+\beta+2)$.

Starting from the relation ([1, p. 782])

$$(2k+\alpha+\beta+3)P_{k+1}^{(\alpha,\beta)}(x) = (k+\alpha+\beta+2)P_{k+1}^{(\alpha+1,\beta)}(x) - (k+\beta+1)P_k^{(\alpha+1,\beta)}(x)$$

and using (2.1), we obtain

$$W_{2k+2}^{(\alpha+1,\beta)}(x) = W_{2k+2}^{(\alpha,\beta)}(x) + \frac{(k+1)(k+\beta+1)}{(2k+\alpha+\beta+3)(2k+\alpha+\beta+2)} W_{2k}^{(\alpha+1,\beta)}(x). \tag{2.5}$$

In the sequel, the following formulas will be necessary

$$\begin{aligned} &\| W_{2k}^{(\alpha,\beta)} \|^2 = \int_{-1}^{1} W_{2k}^{(\alpha,\beta)}(x)^2 p(x)\, dx = \frac{k!}{(k+\alpha+\beta+1)_k} B(k+\alpha+1,\ k+\beta+1), \\ &\| W_{2k+1}^{(\alpha,\beta)} \|^2 = \| W_{2k}^{(\alpha,\beta+1)} \|^2 = \frac{k!}{(k+\alpha+\beta+2)_k} B(k+\alpha+1,\ k+\beta+2), \\ &W_{2k}^{(\alpha,\beta)}(1) = \frac{(\alpha+1)_k}{(k+\alpha+\beta+1)_k}, \quad W_{2k+1}^{(\alpha,\beta)}(1) = W_{2k}^{(\alpha,\beta+1)}(1) = \frac{(\alpha+1)_k}{(k+\alpha+\beta+2)_k} \end{aligned} \tag{2.6}$$

and, also

$$\begin{aligned} &W_{2k}^{(\alpha,\beta)}(x) = W_{2k}^{(\alpha,\beta)}(1)\, {}_2F_1(-k,\ k+\alpha+\beta+1;\ \alpha+1;\ 1-x^2), \\ &W_{2k+1}^{(\alpha,\beta)}(x) = xW_{2k+1}^{(\alpha,\beta)}(1)\, {}_2F_1(-k,\ k+\alpha+\beta+2;\ \alpha+1;\ 1-x^2), \end{aligned} \tag{2.7}$$

where ${}_2F_1$ is the hipergeometric function.

Lemma 2.1. *Let* $h_k = \| W_k^{(\alpha,\beta)} \|^2$. *Then the identities*

$$S_0^{(n)}(x) = \sum_{k=0}^{n} \frac{W_{2k}^{(\alpha,\beta)}(x)\, W_{2k}^{(\alpha,\beta)}(1)}{h_{2k}} = A_n^{(\alpha,\beta)} W_{2n}^{(\alpha+1,\beta)}(x) \tag{2.8}$$

and

$$S_1^{(n)}(x) = \sum_{k=0}^{n} \frac{W_{2k+1}^{(\alpha,\beta)}(x)\, W_{2k+1}^{(\alpha,\beta)}(1)}{h_{2k+1}} = A_n^{(\alpha,\beta+1)} W_{2n+1}^{(\alpha+1,\beta)}(x) \tag{2.9}$$

hold, where

$$A_n^{(\alpha,\beta)}=\frac{W_{2n}^{(\alpha,\beta)}(1)}{h_{2n}}=\frac{\Gamma(2n+\alpha+\beta+2)}{n!\,\Gamma(\alpha+1)\,\Gamma(n+\beta+1)}. \tag{2.10}$$

Proof. Our proof is based on induction. Note that (2.8) is correct for $n=0$. Now, let we suppose that (2.8) holds true for some n. Then we get

$$S_0^{(n+1)}(x)=A_n^{(\alpha,\beta)}W_{2n}^{(\alpha+1,\beta)}(x)+\frac{W_{2n+2}^{(\alpha,\beta)}(x)\,W_{2n+2}^{(\alpha,\beta)}(1)}{h_{2n+2}}.$$

On the basis of (2.10), we obtain

$$S_0^{(n+1)}(x)=A_{n+1}^{(\alpha,\beta)}\left(\frac{A_n^{(\alpha,\beta)}}{A_{n+1}^{(\alpha,\beta)}}W_{2n}^{(\alpha+1,\beta)}(x)+W_{2n+2}^{(\alpha,\beta)}(x)\right).$$

$$=A_{n+1}^{(\alpha,\beta)}\left(\frac{(n+1)(n+\beta+1)}{(2n+\alpha+\beta+3)(2n+\alpha+\beta+2)}W_{2n}^{(\alpha+1,\beta)}(x)+W_{2n+2}^{(\alpha,\beta)}(x)\right).$$

Finally, according to (2.5), we find $S_0^{(n+1)}(x)=A_{n+1}^{(\alpha,\beta)}W_{2n+2}^{(\alpha+1,\beta)}(x)$, showing that identity (2.8) holds true also for $n:=n+1$. The identity (2.9) is simply preved if we multiply identity (2.8), for $\beta:=\beta+1$, by x and using relations (2.3) and (2.6).

3. Approximations with constraint

Following Milovanović and Wrigge [8], we introduce two families of real functions, viz.

$$F_e=\{f: f(-x)=f(x),\ f(1)=0,\ f\in L^2[-1,\ 1]\}$$

and

$$F_o=\{f: f(-x)=-f(x),\ f(1)=0,\ f\in L^2[-1,\ 1]\},$$

where $L^2[-1,\ 1]=L_p^2[-1,\ 1]$, $p(x)=|x|^{\mu}(1-x^2)^{\alpha}$, $a,\ \mu>-1$, and

$$(f,g)=\int_{-1}^{1}f(x)\,g(x)\,p(x)\,dx\quad (f,\ g\in L^2[-1,\ 1]). \tag{3.1}$$

Let further $\mathcal{P}_m$ be the set of all real polynomials of degree at most m and such that the polynomials belong to the set F_e if m is even and to the set F_o if m is odd.

In this section, we will give the least squares approximation Φ_{2n} (or Φ_{2n+1}) for the function $f\in F_e$ (or F_o) in the class $\mathcal{P}_{2n}$ (or $\mathcal{P}_{2n+1}$), with respect to the norm $||f||=((f,f))^{1/2}$, where the inner product $(\cdot,\cdot)$ is defined by (3.1). For this approximation we have

$$\min_{\Phi\in\mathcal{P}_{2n}}||f-\Phi||=||f-\Phi_{2n}||\quad \text{when } f\in F_e, \tag{3.2}$$

or

$$\min_{\Phi\in\mathcal{P}_{2n+1}}||f-\Phi||=||f-\Phi_{2n+1}||\quad \text{when } f\in F_o. \tag{3.3}$$

Applying the same method as in the paper [8] and using the relations (2.6) — (2.10), we can prove two following theorems:

Theorem. 3.1. *If $f \in F_e$ then the least squares approximation in the class $\mathcal{P}_{2n}$ is given by*

$$\Phi_{2n}(x) = \sum_{i=1}^{n} d_{n,i}(1-x^2)^i, \tag{3.4}$$

where

$$d_{n,i} = \frac{(-1)^i}{\Gamma(\alpha+i+2)} \sum_{k=0}^{n} (f, W_{2k}^{(\alpha,\beta)}) \frac{\Gamma(2k+\alpha+\beta+2)}{k!\,\Gamma(k+\beta+1)} s_{k,i}^{(\alpha,\beta)}$$

and

$$s_{k,i}^{(\alpha,\beta)} = \begin{cases} -\binom{n}{i}(\alpha+1)\,(n+\alpha+\beta+2)_i, & k<i, \\ \binom{k}{i}(\alpha+i+1)\,(k+\alpha+\beta+1)_i - \binom{n}{i}(\alpha+1)(n+\alpha+\beta+2)_i, & k \geqq i. \end{cases}$$

Theorem 3.2. *If $f \in F_0$, then the least squares approximation in the class $\mathcal{P}_{2n+1}$ is given by*

$$\Phi_{2n+1}(x) = x \sum_{i=1}^{n} b_{n,i}(1-x^2)^i, \tag{3.5}$$

where

$$b_{n,i} = \frac{(-1)^i}{\Gamma(\alpha+i+2)} \sum_{k=0}^{n} (f, W_{2k+1}^{(\alpha,\beta)}) \frac{\Gamma(2k+\alpha+\beta+3)}{k!\,\Gamma(k+\beta+2)} g_{k,i}^{(\alpha,\beta)}$$

and

$$g_{k,i}^{(\alpha,\beta)} = \begin{cases} -\binom{n}{i}(n+\alpha+\beta+3)_i\,(\alpha+1), & k<i, \\ \binom{k}{i}(k+\alpha+\beta+2)_i(\alpha+i+1) - \binom{n}{i}(n+\alpha+\beta+3)_i(\alpha+1), & k \geqq i. \end{cases}$$

Let $\tilde{\Phi}_{2n+q}(x)$ be the least squares approximation without constraint given by (see, e. g., [10, pp. 50—51])

$$\tilde{\Phi}_{2n+q}(x) = \sum_{k=0}^{n} \frac{(f, W_{2k+q}^{(\alpha,\beta)})}{h_{2k+q}} W_{2k+q}^{(\alpha,\beta)}(x),$$

where $q=0$ or $q=1$. It is easy to see that the approximation with constraint $\Phi_{2n+q}(x)$ turns out to be the truncated expansion in generalized Gegenbauer polynomials with a multiple of $S_q^{(n)}(x) \left(= A_n^{(\alpha,\beta+q)} W_{2n+q}^{(\alpha+1,\beta)}(x)\right)$ added to satisfy the constraint at $x=1$, i. e.,

$$\Phi_{2n+q}(x) = \tilde{\Phi}_{2n+q}(x) - \tilde{\Phi}_{2n+q}(1) \frac{S_q^{(n)}(x)}{S_q^{(n)}(1)}, \tag{3.7}$$

where $q=0$ or $q=1$.

4. General case

In the general case, when f is neither an even nor an odd function, but $f\in L^2[-1, 1]$ and $f(-1)=f(1)=0$, then the least squares approximation ψ_m (in the class of real polynomials of degree $\leq m$), which satisfies the conditions $\psi_m(-1)=\psi_m(1)=0$, is given by

(4.1) $$\psi_m(x)=\Phi_{2n}(x)+\Phi_{2n+1}(x) \qquad \text{when } m=2n+1$$

and

$$\psi_m(x)=\Phi_{2n}(x)+\Phi_{2n-1}(x) \qquad \text{when } m=2n,$$

where Φ_{2n} and Φ_{2n+1} are the solutions of (3.2) and (3.3). This can be seen by writing

(4.2) $$f(x)=f^{(e)}(x)+f^{(o)}(x)=\frac{1}{2}(f(x)+f(-x))+\frac{1}{2}(f(x)-f(-x)).$$

It is of some interest to compare approximations without constraint with our approximations with constraint.

In the set of all real polynomials Π_{2n+1} of degree at most $m=2n+1$, the least squares approximation without constraint for the function f, given by (4.2), can be represented in the form

(4.3) $$\tilde{\psi}_{2n+1}(x)=\tilde{\Phi}_{2n}(x)+\tilde{\Phi}_{2n+1}(x),$$

where the even and odd parts, $\tilde{\Phi}_{2n}$ and $\tilde{\Phi}_{2n+1}$, are given by (3.6) for $q=0$ and $q=1$ respectively.

According to (4.1), (3.7) and (4.3) we have that the corresponding least squares approximation with constraint is given by

(4.4) $$\psi_{2n+1}(x)=\tilde{\psi}_{2n+1}(x)-h(x),$$

where

$$h(x)=\tilde{\Phi}_{2n}(1)\frac{S_0^{(n)}(x)}{S_0^{(n)}(1)}+\tilde{\Phi}_{2n+1}(1)\frac{S_1^{(n)}(x)}{S_1^{(n)}(1)}.$$

Assuming $f\in L^2[-1, 1]$ and $f(-1)=f(1)=0$, define

(4.5) $$D^*=\min_{\psi\in\mathcal{P}_{2n}\cup\mathcal{P}_{2n+1}}\|f-\psi\|^2=\|f-\psi_{2n+1}\|^2$$

and

(4.6) $$\tilde{D}^*=\min_{\psi\in\Pi_{2n+1}}\|f-\psi\|^2=\|f-\tilde{\psi}_{2n+1}\|^2.$$

We note that $\mathcal{P}_{2n}\cup\mathcal{P}_{2n+1}\subset\Pi_{2n+1}$.

According to (4.5), (4.4) and (4.6) we have

$$D^*=(f-\psi_{2n+1}, f-\psi_{2n+1})=\tilde{D}^*+2(f-\tilde{\psi}_{2n+1}, h)+(h, h).$$

It is easy to show that $(f-\tilde{\psi}_{2n+1}, h)=0$ and

$$(h, h)=\frac{(\tilde{\Phi}_{2n}(1))^2}{S_0^{(n)}(1)}+\frac{(\tilde{\Phi}_{2n+1}(1))^2}{S_1^{(n)}(1)}.$$

So, we obtain

$$D^*-\tilde{D}^*=\frac{(\tilde{\Phi}_{2n}(1))^2}{A_n^{(\alpha,\beta)}W_{2n}^{(\alpha+1,\beta)}(1)}+\frac{(\tilde{\Phi}_{2n+1}(1))^2}{A_n^{(\alpha,\beta+1)}W_{2n+1}^{(\alpha+1,\beta)}(1)}.$$

Thus we see that the difference $D^*-\tilde{D}^*$ is a linear combination of the squares of errors $\tilde{\Phi}_{2n}(1)-f^{(e)}(1)=\tilde{\Phi}_{2n}(1)$ and $\tilde{\Phi}_{2n+1}(1)-f^{(o)}(1)=\tilde{\Phi}_{2n+1}(1)$, where $f^{(e)}(1)=f^{(o)}(1)=f(1)=0$.

A similar result can be obtained if we consider our approximations in the case $m=2n$.

5. Examples

As may be seen from Theorems 3.1 and 3.2, a main difficulty when calculating the least squares approximations is to achieve high — precision values of the inner products $(f, W_k^{(\alpha,\beta)})$.

An appropriate numerical method for the determination of these inner products is the application of Gauss — Christoffel quadrature with the generalized Gegenbauer weight. The parameters of these quadratures can be calculated from corresponding Jacobi matrix by using QR — algorithm ([4], [5]). The elements of Jacobi matrix are determinated by three — term recurrence relation (2.4).

A better approach is use of Gauss — Lobatto quadratures with the same weight ([6]). Namely, we can achieve higher accurace than with the above one, using the same number of knots because of the conditions $f(-1)=f(1)=0$.

Example 5.1. $f(x)=\cos(\pi x/2)$, $x\in[-1, 1]$.

In this case the approximating polynomial is given by (3.4), where the coefficients $d_{n,i}$ $(n=1, 2, 3, 4)$ are displayed in Table 5.1. The corresponding absolute errors

$$e_n=\max_{-1\leq x\leq 1}|f(x)-\Phi_{2n}(x)| \qquad (n=1,\ldots,4)$$

are given, too. Numbers in parenthesis indicate decimal exponents.

Table 5.1

		$\mu=0$.	$\alpha=-0.5$	$\mu=-0.5$	$\alpha=-0.5$
n	i	$d_{n,i}$	e_n	$d_{n,i}$	e_n
1	1	0.962270459871	3.84 (—2)	0.979346973677	4.60 (—2)
2	1	0.777230028062	7.47 (—4)	0.776199638179	9.00 (—4)
	2	0.222048518171		0.223462069048	
3	1	0.785557128489	8.05 (—6)	0.785579574340	9.69 (—6)
	2	0.195401796805		0.195322260565	
	3	0.019033372405		0.019094870042	
4	1	0.785396470018	5.46 (—8)	0.785396215336	6.56 (—8)
	2	0.196365747628		0.196367406887	
	3	0.017380885279		0.017377843941	
	4	0.000856845176		0.000858513050	

We can notice that for $\beta=-0.5$ ($\mu=0$) and $\alpha=\lambda-1/2$, the problem is reduced to the Gegenbauer's case which is considered in [8].

Example 5.2. The error function

$$\mathrm{erf}(x)=\frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt$$

is an odd one. In order to approximate this function by means of Theorem 3.2, let us define a new function f by

$$f(x)=\mathrm{erf}(ax)-\mathrm{erf}(a)\,x \quad (|x|\leqq 1),$$

where a is a positive constant. For the function $f\in F_0$, according to Theorem 3.2, the approximating polynomial is given by (3.5). Taking into account the above, we obtain the approximation of the form

$$\mathrm{erf}(ax)\cong x\left(\mathrm{erf}(a)+\sum_{i=1}^{n} b_{n,i}(1-x^2)^i\right). \tag{5.1}$$

The coefficients $b_{n,i}$ $(i=1,\dots,n)$ and $b_{n,0}=\mathrm{erf}(a)$, for $a=0.5$ and $\alpha=\beta=-0.5$ and 0.5, are given in Table 5.2.

Table 5.2

		$\alpha=\beta=-0.5$	$\alpha=\beta=0.5$
n	i	$d_{n,i}$	$d_{n,i}$
6	0	0.52049987781305	0.52049987781305
	1	0.04055429417069	0.04055429417017
	2	0.00295376508012	0.00295376508471
	3	0.00017297426446	0.00017297425369
	4	0.00000832270120	0.00000832270347
	5	0.00000033642194	0.00000033643767
	6	0.00000001309590	0.00000001308425

Table 5.3 contains the maximum values of the absolute error, i.e.

$$\max_{|x|\leqq 1} |f(x)-\Phi_{2n+1}(x)|,$$

for $a=0.5\,(0.5)\,2$, when $\alpha=\beta=-0.5$ and $\alpha=\beta=0.5$.

It can be seen that the increase of the parameter a produces an increase of the error.

The obtained approximation for the error function given by (5.1) is very efficient fot usage because it requires a small number of arithmetic operations, i.e. n additions, one substraction, and $n+2$ multiplications, assuming that the Horner's sheme is used.

Table 5.3

	$a=0.5$		$a=1.0$	
n	$\alpha=\beta=-0.5$	$\alpha=\beta=0.5$	$\alpha=\beta=-0.5$	$\alpha=\beta=0.5$
1	2.6 (—4)	2.9 (—4)	6.1 (—3)	7.1 (—3)
2	3.6 (—6)	4.7 (—6)	3.4 (—4)	4.5 (—4)
3	4.3 (—8)	6.4 (—8)	1.6 (—5)	2.4 (—5)
4	4.3 (—10)	7.2 (—10)	6.1 (—7)	1.1 (—6)
5	3.7 (—12)	7.0 (—12)	2.1 (—8)	4.2 (—8)
6	2.9 (—14)	5.9 (—14)	6.4 (—10)	1.4 (—9)
	$a=1.5$		$a=2.0$	
n	$\alpha=\beta=-0.5$	$\alpha=\beta=0.5$	$\alpha=\beta=-0.5$	$\alpha=\beta=0.5$
1	3.1 (—2)	3.6 (—2)	7.7 (—2)	8.9 (—2)
2	3.6 (—3)	4.9 (—3)	1.6 (—2)	2.1 (—2)
3	3.7 (—4)	5.7 (—4)	2.7 (—3)	4.1 (—3)
4	3.2 (—5)	5.7 (—5)	4.0 (—4)	6.9 (—4)
5	2.5 (—6)	4.8 (—6)	5.2 (—5)	1.1 (—4)
6	1.7 (—7)	3.6 (—7)	6.1 (—6)	1.4 (—5)

All computations, which include the error function, have been performed using rational approximation to this function given in [3].

All calculations were performed in double precision arithmetic on a HONEYWELL DPS 6/92 computer.

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SREDNJE-KVADRATNA APROKSIMACIJA SA OGRANIČENJEM: GENERALISANI GEGENBAUEROV SLUČAJ

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U radu se razmatra srednje-kvadratna aproksimacija funkcije $f \in L^2[-1, 1]$ pod uslovom da je $f(-1)=f(1)=0$. Korisćenjem generalisane Gegenbauerove težinske funkcije dobijena su uopštenja nekih rezultata Milovanovića i Wriggea [8]. Rezultati su upoređeni sa odgovarajućom srednje-kvadratnom aproksimacijom bez ograničenja. U poslednjem odeljku rada data su dva numerička primera.