FACTA UNIVERSITATIS (NIŠ)
SER. MATH. INFORM. 2 (1987), 7—28

VARIOUS EXTREMAL PROBLEMS OF MARKOV'S TYPE FOR ALGEBRAIC POLYNOMIALS

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Abstract. Extremal problems of Markov's type for algebraic polynomials in various norms and classes of polynomials are considered. Especially, the problems in L^2-norm on the set of all algebric polynomials of degree at most n or on some its subsets are investigated.

1. Introduction and preliminaries

We begin our investigation by considering the following extremal problem:

Let $\mathcal{P}_n$ be the set of all algebraic polynomials $P\,(\neq 0)$ of degree at most n on an interval (a, b) with a given norm $\|\,.\,\|$.

Determine the best constant in the inequality

$$\|P'\| \leqq A_n \|P\| \qquad (P \in \mathcal{P}_n), \tag{1.1}$$

i. e.,

$$A_n = \sup_{P \in \mathcal{P}_n} \frac{\|P'\|}{\|P\|}. \tag{1.2}$$

The first result at this area is well-known classical inequality of A. A. Markov [19].

Theorem 1.1. *Let* $(a, b) = (-1, 1)$ *and* $\|f\| = \|f\|_\infty = \max_{-1 \leqq t \leqq 1} |f(t)|$. *Then*

$$\|P'\|_\infty \leqq n^2 \|P\|_\infty \qquad (P \in \mathcal{P}_n), \tag{1.3}$$

with an equality case for $P(t) = T_n(t)$, *where* T_n *is Chebyshev polynomial of the first kind of degree* n.

An other type of these inequalities is Bernstein's inequality

$$\|P'\|_\infty \leqq n(1-t^2)^{-1/2} \|P\|_\infty \qquad (P \in \mathcal{P}_n). \tag{1.4}$$

Received May 15, 1987.

Markov's and Bernstein's inequalities are fundamental to the proofs of many inverse theorems in polynomial approximation theory [15], [21], [10].

Recently, these inequalities have been considered on disjoint intervals by P. S. Borwein [2].

In this paper, we will consider only inequalities of Markov's type.

A generalization of the inequality (1.2) for higher derivatives was given by V. A. Markov [20].

Theorem 1.2. *For each* $k=1, \ldots, n$, *the inequality*

$$\| P^{(k)} \|_\infty \leqq \frac{1}{(2k-1)!!} \prod_{i=0}^{k-1} (n^2-i^2) \| P \|_\infty \qquad (P \in \mathcal{P}_n) \tag{1.5}$$

holds. The extremal polynomial is T_n.

We note that the best constant in (1.5) is equal $\| T_n^{(k)} \|_\infty = T_n^{(k)}(1)$. So the inequality (1.5) can be written in the form

$$\| P^{(k)} \|_\infty \leqq T_n^{(k)}(1) \| P \|_\infty \qquad (P \in \mathcal{P}_n).$$

In 1964 G. Szegö [34] studied an extremal problem for the norm $\| f \| = \sup_{t \geqq 0} | f(t) e^{-t} |$ on $(0, +\infty)$. he proved the following:

Theorem 1.3. *Let* $(a, b) = (0, +\infty)$ *and* $\| f \| = \sup_{t \geqq 0} | f(t) e^{-t} |$. *There exists a positive constant* C *such that*

$$\| P' \| \leqq Cn \| P \|$$

for each $P \in \mathcal{P}_n$ $(n = 2, 3, \ldots)$.

If we put

$$\| f \|_{p, \mu} = \Big(\int_{-1}^{1} | f(t) (1-t^2)^\mu |^p \, dt \Big)^{1/p}, \qquad 1 \leqq p < \infty,$$

$$= \sup_{-1 \leqq t \leqq 1} | f(t) | (1-t^2)^\mu, \qquad p = +\infty,$$

where $p\mu > 1$ ($\mu \leqq 0$ if $p = +\infty$), we can consider the following general extremal problem (see [11])

$$A_{n,k}(p, \mu; q, \nu) = \sup_{P \in \mathcal{P}_n} \frac{\| P^{(k)} \|_{q, \nu}}{\| P \|_{p, \mu}}. \tag{1.6}$$

So the best constant in (1.5) is $A_{n,k}(+\infty, 0; +\infty, 0)$. We note that Bernstein's inequality (1.4) can be represented in the form

$$\| P' \|_{\infty, 1/2} \leqq n \| P \|_{\infty, 0} \qquad (P \in \mathcal{P}_n).$$

The case $k = n$ is especially interesting. Namely, then we have the following problem: Among all polynomials of degree n, with leading coefficient unity, find the polynomial which deviates least from zero in the norm $\| \cdot \|_{p, \mu}$.

Some more general results in the integral norms are given in [13], [27], [11]. When $p = q$ and $\mu = \nu$, there are several results.

E. Hille, G. Szegö, and J. D. Tamarkin [9] extended Markov's theorem in L^p norm ($p \geqq 1$) on $(-1,1)$ by proving the following result:

Theorem 1.4. *Let* $(a, b) = (-1, 1)$ *and* $\|f\| = \|f\|_{p,0}$ $(p \geqq 1)$. *Then*

$$\|P'\| \leq Cn^2 \|P\| \qquad (P \in \mathcal{P}_n), \tag{1.7}$$

where C is a positive constant which dependens only on p, but not on P or on n.

A. Markov's theorem (with a less precise value of the constant C) is obtained from (1.7) by allowing $p \to +\infty$. Another important case, namely, $p = 1$, was treated by N. K. Bari [1] and recently by S. V. Konyagin [12], who considered the extremal problem (1.6) for $p = q = 1$ and $\mu = \nu = 0$. He found an estimate for $A_{n,k} = A(1, 0;\ 1, 0)$.

Theorem 1.5. *There exist two constants* c_1 and c_2 $(0 < c_1 < c_2 < +\infty)$ *such that*

$$c_1 \frac{nT_n^{(k)}(1)}{(k+1)(n-k+1)} \leqq A_{n,k} \leqq c_2 \frac{nT_n^{(k)}(1)}{(k+1)(n-k+1)}$$

for each $n \in N$ *and* $k = 1, \ldots, n$.

Especially important cases are $p = q = 2$. In the following section we consider such cases. In Section 3 we give some classical results for the extremal problems on some restricted polynomial classes. In Section 4 we discuss the Varma's extremal problems in L^2-metric. A complete solution of Varma's one and related problems we give in Section 5. Finally, in Section 6 we consider some extremal problems in L^2-metric with Jacobi weight on $(-1, 1)$.

2. Extremal problems in L^2-norm

In the L^2-metric we give first the following result of E. Schmidt [30] and P. Turán [36]:

Theorem 2.1. (*a*) *Let* $(a, b) = (-\infty, +\infty)$ *and* $\|f\|^2 = \int_{-\infty}^{\infty} e^{-t^2} f(t)^2 \, dt$. *Then the best constant in* (1.2) *is* $A_n = \sqrt{2n}$. *An extremal polynomial is Hermite's polynomial* H_n;

(*b*) *Let* $(a, b) = (0, +\infty)$ *and* $\|f\|^2 = \int_0^{\infty} e^{-t} f(t)^2 \, dt$. *Then*

$$A_n = \left(2 \sin \frac{\pi}{4n+2}\right)^{-1}.$$

The extremal polynomial is

$$P(t) = \sum_{\nu=1}^{n} \sin \frac{\nu\pi}{2n+1} L_\nu(t),$$

where L_ν *is Laguerre polynomial.*

The Theorem 2.1, in this form, was formulated by P. Turán. E. Schmidt proved only

$$A_n = \frac{2n+1}{\pi}\left(\frac{\pi^2}{24\,(2n+1)^2} + \frac{R}{(2n+1)^4}\right)^{-1},$$

where $-8/3 < R < 4/3$.

Recently, L. Mirsky [26] considered the case of L^2-metric with an arbitrary weight function $w : (a, b) \to R_+$ $(-\infty \leqq a < b \leqq +\infty)$ for which all moments are finite.

Theorem 2.2. *There exists a number $A_n = A_n(a, b\,;\, w)$ such that, for every polynomial P with complex coefficients and of degree not exceeding n, the inequality* (1.1) *holds. Furthermore, we have*

$$A_n \leqq \left(\sum_{k=1}^{n} k \,\|\, \pi_k' \,\|^2\right)^{1/2}, \tag{2.1}$$

where (π_k) is a system of polynomials orthonormal with respect to the weight function w.

The main interest of this result is, however, qualitative, for the bound specified by (2.1) can be very crude. For example, when $w(t) = e^{-t^2}$ on $(-\infty, +\infty)$, the estimate (2.1) becomes

$$A_n \leqq \left(\sum_{k=1}^{n} 2k^2\right)^{1/2} \quad O(n^{3/2}).$$

The contrast between this estimate and $A_n = \sqrt{2n}$ (see (*a*) in Theorem 2.1) is evident.

In [6] P. Dörfler considered the analogous inequality for derivatives of higher order and compute the best possible constant:

Theorem 2.2. *Let P be any polynomial with complex coefficients of degree at most n. Then the best possible constant $A_{n,m}$ such that*

$$\| P^{(m)} \| \leqq A_{n,m} \| P \|,$$

is the largest singular value of the matrix $A_n^{(m)}$, where

$$A_n^{(m)} = \begin{bmatrix} e_{0,0}^{(m)} & \cdots & e_{n,0}^{(m)} \\ \vdots & & \\ e_{0,n-m}^{(m)} & & e_{n,n-m}^{(m)} \end{bmatrix}, \qquad e_{k,j}^{(m)} = \int_a^b \pi_k^{(m)}(t)\, \pi_j(t)\, w(t)\, dt.$$

Moreover, the estimation

$$\max_{0 \leqq k \leqq n} \| \pi_k^{(m)} \| \leqq A_{n,m} \leqq \left(\sum_{k=0}^{n} \| \pi_k^{(m)} \|^2\right)^{1/2}$$

holds.

The exact constant in (1.1) can be found as a maximal eigenvalue of a matrix of Gram's type. Now, we consider a more general case with a given nonnegative measure $d\lambda(t)$ on the real line R, with compact or infinite support, for which all moments

$$\mu_k = \int_R t^k \, d\lambda(t), \qquad k=0, 1, \dots ,$$

exist and are finite, and $\mu_0 > 0$. There exists, then, a unique set of orthonormal polynomials $\pi_k(\,.\,) = \pi_k(\,.\,; d\lambda)$, $k = 0, 1, \dots,$ defined by

$$\pi_k(t) = a_k t^k + \text{lower degree terms}, \quad a_k > 0,$$

(2.2)
$$\int_R \pi_k(t)\,\pi_m(t)\,d\lambda(t) = \delta_{km}, \qquad k, m \geq 0,$$

(For any polynomial $P \in \mathcal{P}_n$, with complex coefficients, we take

$$\|P\| = \Big(\int_R |P(t)|^2 \, d\lambda(t)\Big)^{1/2}$$

and consider the extremal problem

(2.3)
$$A_{n,m} = A_{n,m}(d\lambda) = \sup_{P \in \mathcal{P}_n} \frac{\|P^{(m)}\|}{\|P\|} \qquad (1 \leq m \leq n).$$

Theorem 2.4. *The best constant* $A_{n,m}$ *defined in* (2.3) *is given by*

2.4)
$$A_{n,m} = (\lambda_{\max}(B_{n,m}))^{1/2},$$

where $\lambda_{\max}(B_{n,m})$ *is the maximal eigenvalue of the matrix* $B_{n,m} = [b_{ij}^{(m)}]_{m \leq i,\, j \leq n}$, *which the elements are given by*

(2.5)
$$b_{i,j}^{(m)} = \int_R \pi_i^{(m)}(t)\,\pi_j^{(m)}(t)\,d\lambda(t), \qquad m \leq i, j \leq n.$$

An extremal polynomial is

$$P^*(t) = \sum_{k=m}^{n} c_k \pi_k(t),$$

where $[c_m, c_{m+1}, \dots, c_n]$ T *is an eigenvector of the matrix* $B_{n,m}$ *corresponding to the eigenvalue* $\lambda_{\max}(B_{n,m})$.

Proof. Let $P \in \mathcal{P}_n$. Then we can write $P(t) = \sum_{k=0}^{n} c_k \pi_k(t)$ and $P^{(m)}(t) = \sum_{k=m}^{n} c_k \pi_k^{(m)}(t)$, $m \leq n$, where the coefficients c_k are uniquely determined. Hence, by (2.2) and (2.5),

$$\|P\|^2 = \sum_{k=0}^{n} |c_k|^2 \quad \text{and} \quad \|P^{(m)}\|^2 = \sum_{i,j=m}^{n} c_i \bar{c}_j b_{ij}^{(m)}.$$

Now we have

$$\frac{\| P^{(m)} \|^2}{\| P \|^2} \leq \frac{\sum_{i,j=m}^{n} c_i \bar{c}_j b_{ij}^{(m)}}{\sum_{i=m}^{n} | c_i |^2} = \frac{\langle B_{n,m} \mathbf{c}, \mathbf{c} \rangle}{\langle \mathbf{c}, \mathbf{c} \rangle}. \tag{2.6}$$

with equality case $c_0 = \ldots = c_{m-1} = 0$, where $\langle .,. \rangle$ is the standard inner product in an $(n-m+1)$-dimensional space.

The matrix $B_{n,m}$ is evidently positive definite. Since the right side in (2.6) is not greater than the maximal eigenvalue of this matrix we obtain

$$\| P^{(m)} \|^2 \leq \lambda_{\max} (B_{n,m}) \| P \|^2. \tag{2.7}$$

In order to show that $A_{n,m}$, given by (2.4), is best possible, we note that (2.7) reduces to an equality if we put $P(t) = P^*(t) = \sum_{k=m}^{n} c_k^* \pi_k(t)$, where $[c_m^*, c_{m+1}^*, \ldots, c_n^*]^T$ is an eigenvector of the matrix $B_{n,m}$ corresponding to $\lambda_{\max}(B_{n,m})$.

An alternative result like Theorem 2.3 is the following theorem:

Theorem 2.5. *The best constant $A_{n,m}$ defined in* (2.3), *is equal to the spectral norm of one triangular matrix $Q_{n,m}^T$, $Q_{n,m} = [q_{ij}^{m}]_{m \leq i, j \leq n}$ $(q_{ij}^{(m)} = 0 \Leftarrow i > j)$, i.e.*

$$A_{n,m} = \sigma(Q_{n,m}^T) = (\lambda_{\max}(Q_{n,m} Q_{n,m}^T))^{1/2}, \tag{2.8}$$

where the elements $q_{ij}^{(m)}$ are given by the following inner products

$$q_{ij}^{(m)} = (\pi_j^{(m)}, \pi_{i-m}) \qquad (m \leq i, j \leq n).$$

Alternatively, (2.8) *can be expresed in the form*

$$A_{n,m} = (\lambda_{\min}(C_{n,m}))^{-1/2}, \tag{2.9}$$

where $C_{n,m} = (Q_{n,m} Q_{n,m}^T)^{-1}$.

Proof. It is enough to consider only real polynomial set $\mathcal{P}_n$. Let $P \in \mathcal{P}_n$ and $\pi_j^{(m)}(t) = \sum_{i=m}^{n} q_{ij}^{(m)} \pi_{i-m}(t)$, where $q_{ij}^{(m)} = (\pi_j^{(m)}, \pi_{i-m})$. Then we have

$$P^{(m)}(t) = \sum_{j=m}^{n} c_j \sum_{i=m}^{j} q_{ij}^{(m)} \pi_{i-m}(t) = \sum_{i=m}^{n} \left(\sum_{j=m}^{n} c_j q_{ij}^{(m)} \right) \pi_{i-m}(t)$$

and

$$\| P^{(m)} \|^2 = \sum_{i=m}^{n} \left(\sum_{j=i}^{n} c_j q_{ij}^{(m)} \right)^2 = \sum_{i=m}^{n} Y_i^2,$$

where we put

$$Y_i = \sum_{j=i}^{n} c_j q_{ij}^{(m)}, \qquad i = m, \ldots, n. \tag{2.10}$$

Let $c=[c_m, \ldots, c_n]^T$, $Y=[Y_m, \ldots, Y_n]^T$, and $Q_{n,m}=[q_{ij}^{(m)}]_{m\le i,j\le n}$. Since $Y=Q_{n,m}c$ we have

$$\frac{\|P^{(m)}\|^2}{\|P\|}\le\frac{\langle Y, Y\rangle}{\langle c, c\rangle}=\frac{\langle Y, Y\rangle}{\langle (Q_{n,m}Q_{n,m}^T)^{-1}Y, Y\rangle},$$

wherefrom we conclude that (2.8) and (2.9) hold.

Example 2.1. $d\lambda(t)=\exp(-t^2)\,dt$, $-\infty<t<+\infty$. Here we have $\pi_k(t)=\hat{H}_k(t)=(\sqrt{\pi}\,2^k k!)^{-1/2}H_k(t)$, where H_k is Hermite polynomial of degree k. Since $H_k'(t)=2kH_{k-1}(t)$, i.e., $\hat{H}_k'(t)=\sqrt{2k}\,\hat{H}_{k-1}(t)$, we have

$$\hat{H}_k^{(m)}(t)=\sqrt{2k}\sqrt{2(k-1)}\cdots\sqrt{2(k-m+1)}\,\hat{H}_{k-m}(t)=\sqrt{2^m m!\binom{k}{m}}\,\hat{H}_{k-m}(t)$$

and

$$b_{i,j}^{(m)}=2^m m!\binom{i}{m}\delta_{i,j},\quad m\le i,j\le n,$$

So, we find $\lambda_{\max}(B_{n,m})=2^m m!\binom{n}{m}$ and $A_{n,m}=2^{m/2}\sqrt{n!/(n-m)!}$.

Also, this result can be found in unpublished Ph. D. Thesis of L. F. Shampine [31] and [6].

Example 2.2. $d\lambda(t)=t^\alpha e^{-t}\,dt$, $0<t<\infty$. Here we have the generalized Laguerre case with $\pi_k(t)=\hat{L}_k^\alpha(t)=\sqrt{k!/\Gamma(k+\alpha+1)}\sum_{i=0}^{k}(-1)^{k-i}\binom{k+\alpha}{k-i}\frac{x^i}{i!}$, where Γ is the gamma function.

First, we consider the case $m=1$. Since

$$\frac{d}{dt}\hat{L}_j^\alpha(t)=\sum_{i=1}^{j}q_{ij}^{(1)}\hat{L}_{i-1}^\alpha(t),\quad q_{ij}^{(1)}=-\sqrt{\frac{j!}{\Gamma(j+\alpha+1)}}\cdot\sqrt{\frac{\Gamma(i+\alpha)}{(i-1)!}},$$

from the equalities (2.10) it follows that

$$c_i=Y_{i+1}-\sqrt{\frac{i+\alpha}{i}}\,Y_i,\quad i=1,\ldots,n,$$

where we put $Y_{n+1}=0$. The elements $p_{ij}^{(1)}$ of the matrix $P_{n,1}=Q_{n,1}^{-1}$ are

$$p_{ii}^{(1)}=-\sqrt{1+\frac{\alpha}{i}},\quad i=1,\ldots,n;\quad p_{i,i+1}^{(1)}=1,\quad i=1,\ldots,n-1;$$

$$p_{ij}^{(1)}=0,\ \text{otherwise},$$

so that

$$C_{n,1}=P_{n,1}^T P_{n,1}=-\begin{bmatrix}\alpha_0 & \sqrt{\beta_1} & & & \mathbf{0}\\ \sqrt{\beta_1} & \alpha_1 & \sqrt{\beta_2} & & \\ & \cdot & \cdot & \cdot & \\ & & \cdot & \cdot & \cdot\\ & & & \cdot & \sqrt{\beta_{n-1}}\\ \mathbf{0} & & & \sqrt{\beta_{n-1}} & \alpha_{n-1}\end{bmatrix}=-J_n,$$

where

$$\alpha_0 = -(1+\alpha), \quad \alpha_k = -\left(2+\frac{\alpha}{k+1}\right), \quad \beta_k = 1+\frac{\alpha}{k}, \quad k=1, \ldots, n-1.$$

We see that J_n is the Jacobi matrix for monic polynomials (Q_k), which satisfy the following three-term recurrence relation

$$Q_{k+1}(t) = (t-\alpha_k) Q_k(t) - \beta_k Q_{k-1}(t), \qquad k=0, 1, \ldots,$$
$$Q_{-1}(t)=0, \quad Q_0(t)=1.$$

The eigenvalues of $C_{n,1}$ are $\lambda_\nu = -t_\nu$, where $Q_n(t_\nu)=0$, $\nu=1, \ldots, n$.

The standard Laguerre case ($\alpha=0$) can be exactly solved. Namely, then for $t=2(z-1)$ and $-1 \leqq z \leqq 1$, we have

$$Q_k(t) = \cos(2k+1)\frac{\theta}{2} / \cos\frac{\theta}{2}, \quad z=\cos\theta.$$

The eigenvalues of the matrix $C_{n,1}$ are

$$\lambda_\nu = -t_\nu = 4\sin^2\frac{(2\nu-1)\pi}{2(2n+1)}, \quad \nu=1, \ldots, n.$$

Since $\lambda_{\min}(C_{n,1}) = \lambda_1$, we obtain $A_{n,1} = \left(2\sin\frac{\pi}{2(2n+1)}\right)^{-1}$. This is Turán's result (Theorem 2.1 (*b*)).

Now, we consider the case when $m=2$ and $\alpha=0$. First, we note that

$$\frac{d^m}{dt^m}\hat{L}_j(t) = (-1)^m \sum_{i=m}^{j} \binom{j-i+m-1}{m-1} \hat{L}_{i-m}(t).$$

The formulas (2.10), for $m=2$, become

$$Y_i = \sum_{j=i}^{n} (j-i+1)c_j, \quad i=2, \ldots, n.$$

Since $\Delta^2 Y_i = c_i \; (Y_{n+1}=Y_{n+2}=0)$, we find a five-diagonal symmetric matrix of the order $n-1$

$$C_{n,2} = \begin{bmatrix} 1 & -2 & 1 & & & & & \mathbf{0} \\ -2 & 5 & -4 & 1 & & & & \\ 1 & -4 & 6 & -4 & 1 & & & \\ & 1 & -4 & 6 & -4 & 1 & & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & & 1 & -4 & 6 & -4 & 1 \\ & & & & 1 & -4 & 6 & -4 \\ \mathbf{0} & & & & & 1 & -4 & 6 \end{bmatrix}.$$

So, using the minimal eigenvalue of this matrix we obtain the best constant $A_{n,2}=(\lambda_{\min}(C_{n,2}))^{-1/2}$. These constants, for $n = 4\,(1)10$ are presented in Table 2.1 (with seven decimal digits). Numbers in parentheses indicate decimal exponents. For $n=2$ and $n=3$ we have $A_{2,2}=1$ and $A_{3,2}=(3+2\sqrt{2})^{1/2}$ respectively.

Table 2.1.

n	$\lambda_{\min}(C_{n,2})$	$A_{n,2}$
4	5.1590055 (—2)	4.4026788
5	2.0635581 (—2)	6.9613208
6	9.8237813 (—3)	10.0892912
7	5.2614253 (—3)	13.7863181
8	3.0685649 (—3)	18.0522919
9	1.9090449 (—3)	22.8871610
10	1.2494144 (—3)	28.2908989

Remark 2.1. The last problem could be interpreted as an extremal problem of Wirtinger's type

$$\sum_{i=2}^{n} Y_i^2 \leq A_{n,2}^2 \sum_{i=2}^{n} (\Delta^2 Y_i)^2, \qquad Y_{n+1}=Y_{n+2}=0.$$

Similar problems were given in [8] by K. Fan, O. Taussky, and J. Todd.

Remark 2.2. In 1965 L. F. Shampine [31] proved that

$$\frac{1}{n^4} A_{n,2}^2 = \frac{1}{k_0^4} - R, \qquad 0<R\leq \frac{1}{2n} - \frac{1}{6n^2},$$

where $k_0=1.8751041\ldots$ (k_0 is the smallest root of the equation $1+\cos k \cosh k=0$).

On the end of this section we consider a case with a special even weight function. Namely, let $d\lambda(t) = w(t)\,dt$ on $(-a,a)$, $0<a<\infty$, where $w(-t) = w(t)$. Then we have

$$\pi_i'(t) = \frac{1}{r_i} \sum_{j=1}^{\left[\frac{i+1}{2}\right]} q_{i,j}\, \pi_{i-2j+1}(t), \qquad r_i\neq 0.$$

Now, we consider a class of weight functions for which $q_{i,j}=q_{i+2,j+1}$ (for example, this property holds for Gegenbauer weight). In this case, for $P\in\mathcal{P}_n$, we have

$$P'(t) = \sum_{i=1}^{n} c_i \pi_i'(t) = \sum_{i=1}^{n} q_{i,1} \Big(\sum_{j\geq 0} c_{i+2j}\, r_{i+2j}^{-1}\Big)\, \pi_{i-1}(t)$$

and

$$\| P' \|^2 = \sum_{i=1}^{n} Y_i^2,$$

where

$$Y_i = q_{i,1} \sum_{j\geq 0} c_{i+2j}\, r_{i+2j}^{-1}, \qquad i=1,\ldots,n. \tag{2.11}$$

If we put $q_{i,1}=p_i$ and $Y_{n+1}=Y_{n+2}=0$, from (2.11) follows

$$c_i=r_i\left(\frac{Y_i}{p_i}-\frac{Y_{i+2}}{p_{i+2}}\right), \qquad i=1, \ldots, n.$$

Then

$$\|P\|^2=\sum_{i=1}^{n} c_i^2=\frac{r_1^2}{p_1^2}Y_1^2+\frac{r_2^2}{p_2^2}Y_2^2+\sum_{i=3}^{n}\frac{r_i^2+r_{i-2}^2}{p_i^2}Y_i^2-2\sum_{i=1}^{n-2}\frac{r_i^2}{p_i p_{i+2}}Y_i Y_{i+2}.$$

The corresponding matrix $C_{n,1}$ (see Theorem 2.5) is given by

$$C_{n,1}=\begin{bmatrix}
\alpha_1 & 0 & \beta_1 & & & & & \\
0 & \alpha_2 & 0 & \beta_2 & & & \mathbf{0} & \\
\beta_1 & 0 & \alpha_3 & 0 & \beta_3 & & & \\
 & \beta_2 & 0 & \alpha_4 & 0 & \beta_4 & & \\
 & & \ddots & \ddots & \ddots & \ddots & \ddots & \\
 & & & \beta_{n-4} & 0 & \alpha_{n-2} & 0 & \beta_{n-2} \\
 & \mathbf{0} & & & \beta_{n-3} & 0 & \alpha_{n-1} & 0 \\
 & & & & & \beta_{n-2} & 0 & \alpha_n
\end{bmatrix}, \tag{2.12}$$

where

$$\alpha_i=\frac{r_i^2+r_{i-2}^2}{p_i^2}, \qquad \beta_i=-\frac{r_i^2}{p_i p_{i+2}} \quad (r_{-1}=r_0=0).$$

Now, we define two sequences of polynomials (R_i) and (S_i) by the following three-term recurrence relations

$$tR_{i-1}(t)=\beta_{2i-1}R_i(t)+\alpha_{2i-1}R_{i-1}(t)+\beta_{2i-3}R_{i-2}(t), \quad i=1, \ldots, \left[\frac{n+1}{2}\right], \tag{2.13}$$
$$R_{-1}(t)=0, \quad R_0(t)=R_0=\text{const}$$

and

$$tS_{i-1}(t)=\beta_{2i}S_i(t)+\alpha_{2i}S_{i-1}(t)+\beta_{2i-2}S_{i-2}(t), \quad i=1, \ldots, \left[\frac{n}{2}\right], \tag{2.14}$$
$$S_{-1}(t)=0, \quad S_0(t)=S_0=\text{const}.$$

Theorem 2.6. *The eigenvalues of the matrix $C_{n,1}$, given by* (2.12), *are the zeros of polynomials*

(a) S_{k-1} *and* R_k, *when* $n=2k-1$,

(b) S_k *and* R_k, *when* $n=2k$,

so that

(c) $A_{2k-1,1}=\left(\min(s_1^{(k-1)}, r_1^{(k)})\right)^{-1/2}$

and

(d) $A_{2k,1}=\left(\min(s_1^{(k)}, r_1^{(k)})\right)^{-1/2}$,

where $s_1^{(m)}$ and $r_1^{(m)}$ are the minimal zeros of the polynomials S_m and R_m respectively.

Proof. Firstly, we put $v = v(t) = [R_0(t), S_0(t), R_1(t), S_1(t), \ldots]^T$, where the last coordinate of this $(n-1)$-dimensional vector is $R_{k-1}(t)$ (if $n=2k-1$) or $S_{k-1}(t)$ (if $n=2k$). Using the matrix notation, the relations (2.13) and (2.14) can be interpreted in the form

(2.15) $$t v = C_{n,1} v + w_n,$$

where w_n, in depending on n, is given by

$$w_n = \begin{cases} \beta_{n-1} S_{k-1}(t) e_{n-2} + \beta_n R_k(t) e_{n-1}, & \text{if } n = 2k-1, \\ \beta_{n-1} R_k(t) e_{n-2} + \beta_n S_k(t) e_{n-1}, & \text{if } n = 2k, \end{cases}$$

and e_s is an $(n-1)$-dimensional vector which s-th coordinate is equal one, and others are zero.

Putting firstly $R_0 = 0$ and $S_0 \neq 0$, and then $R_0 \neq 0$ and $S_0 = 0$, we conclude that w_n is a zero-vector if $S_{k-1}(t) = 0$ and $R_k(t) = 0$, when $n = 2k-1$. In the case $n = 2k$, we have the same situation if $S_k(t) = 0$ and $R_k(t) = 0$. Now, according to (2.14) we can conclude that (a) and (b) in Theorem 2.6 are valid. Finally, (c) and (d) follow from (2.9).

Example 2.3. The conditions $q_{i,j} = q_{i+2,j+1}$ are satisfied for Gegenbauer measure $d\lambda(t) = (1-t^2)^{\lambda-1/2}\,dt$, $-1 < t < 1$. Namely, we have

$$\frac{d}{dt} \hat{C}_i^\lambda(t) = \frac{2}{h_i^{1/2}} \sum_{j=1}^{\left[\frac{i+1}{2}\right]} (i + \lambda - 2j + 1)\, h_{i-2j+1}^{1/2}\, \hat{C}_{i-2j+1}^\lambda(t),$$

where $\hat{C}_k^\lambda$ is the normalized Gegenbauer polynomial of the order k, $h_i = \| C_i^\lambda \|^2 = \sqrt{\pi}\, \dfrac{(2\lambda)_i\, \Gamma\left(\lambda + \frac{1}{2}\right)}{(i+\lambda)\, i!\, \Gamma(\lambda)}$, and $(p)_i = p(p+1) \cdots (p+i-1)$. This formula follows immediately from [25, Lemma on the p. 552].

Thus,

$$r_i = \frac{1}{2}\sqrt{h_i}, \qquad p_i = q_{i,1} = (i + \lambda - 1)\sqrt{h_{i-1}},$$

so, for $n=1$ and $n=2$, we have $A_{1,1} = \sqrt{2(\lambda+1)}$ and $A_{2,1} = \sqrt{\dfrac{8(\lambda+1)(\lambda+2)}{2\lambda+1}}$.

In a special case, when $\lambda = 1/2$ (Legendre case), we obtain

$$\alpha_1 = \frac{1}{3}, \quad \alpha_2 = \frac{1}{15}, \quad \alpha_i = \frac{2}{(2i+1)(2i-3)}, \quad i = 3, \ldots, n;$$

$$\beta_i = -\frac{1}{(2i+1)\sqrt{(2i-1)(3i+3)}}, \quad i = 1, \ldots, n-2.$$

Similarly, in Chebyshev case $(\lambda = 0)$ we have

$$\alpha_1 = \frac{1}{2}, \quad \alpha_2 = \frac{1}{16}, \quad \alpha_i = \frac{1}{4}\left(\frac{1}{i^2} + \frac{1}{(i-2)^2}\right), \quad i = 3, \ldots, n;$$

$$\beta_1 = -\frac{\sqrt{2}}{4}, \quad \beta_i = -\frac{1}{4i^2}, \quad i = 2, \ldots, n-2.$$

Some numerical results are presented in Table 2.2.

Table 2.2.

n	$A_{n,1}$ ($\lambda=1/2$)	$A_{n,1}$ ($\lambda=0$)
1	1.7320508	1.4142136
2	3.8729833	4.0000000
3	6.5215962	7.3948177
4	9.7498094	11.6832385
5	13.5914030	16.8974115
6	18.0596447	23.0482821
7	23.1597543	30.1399752
8	28.8939700	38.1742735
9	35.2633549	47.1520448
10	42.2684628	57.0737521

3. Restricted polynomials

In this section and further we will consider extremal problems on some restricted polynomial classes, i. e.,

(3.1) $$A_n = \sup_{P\in W_n} \frac{\|P'\|}{\|P\|},$$

where W_n is some subset of $\mathcal{P}_n$. We can restrict (see [17]): (a) zeros of P_n; (b) coefficients of the polynomials. In this way, the inequality (1.1) can be improved.

So for the uniform norm on $[-1, 1]$ P. Erdös [7] proved the following result:

Theorem 3.1. *If $P \in \mathcal{P}_n$ has only real roots, none of which are in $(-1,1)$, then $A_n = \frac{1}{2}\, en$.*

For the same norm, Q. I. Rahman and G. Schmeisser [28] proved:

Theorem 3.2. *If $P\in\mathcal{P}_n$ has at most $n-1$ distanct zeros in $(-1, 1)$, then $A_n = \left(n \cos\frac{\pi}{4n}\right)^2$. The extremal polynomial is*

$$T_n\left(\pm\left(\cos\frac{\pi}{4n}\right)^2 t + \left(\sin\frac{\pi}{4n}\right)^2\right).$$

There have been several related results (e. g. [4], [32], [33], [3]).

In 1963 G. G. Lorentz [14] introduced polynomials with positive coefficients in t, $1-t$ on $(0,1)$, i. e., the polynomials of the form

(3.2) $$P(t) = \sum_{k=0}^{n} b_k t^k (1-t)^{n-k}, \quad b_k \geqq 0.$$

Also, these polynomials were studied extensively by J. T. Scheick [29].

The Lorentz theorem can be stated in the following form:

Theorem 3.3. *There exists a constant $C>0$ such that for each polynomial P of the form* (3.2),

$$(3.3)\qquad \|P'\|_\infty \leqq Cn \|P\|_\infty \quad (n=1, 2, \ldots),$$

for the uniform norm on [0,1].

The inequality (3.3) is much better than basic Markov's inequality (1.3). Namely, the exponent 2 is replaced by 1.

In 1968 G. G. Lorentz [16] considered the problem of G. Szegö (see Theorem 1.3) for the special polynomials with nonnegative coefficients in t,

$$(3.4)\qquad P(t) = \sum_{k=0}^{n} a_k t^k, \quad a_k \geqq 0,$$

and the norm of a function on $(0, +\infty)$ is given by $\|f\| = \sup\limits_{t\geqq 0} |f(t)\, e^{-\omega(t)}|$. Here ω is a positive differentiable function which, together with $t \mapsto t\,\omega'(t)$, is strictly increasing to $+\infty$.

Theorem 3.4. *Let* ω *satisfy the inequalities*

$$\omega(t) - \omega(0) \leqq At\,\omega'(t), \quad t \geqq 0,$$

and

$$\omega'(\tau) \leqq A\,\omega'(t), \quad \tau < t,$$

for some positive constant A. Then for some constant $C>0$, the inequality

$$\|P'\| \leqq C \frac{\|p_n'\|}{\|p_n\|} \|P\|, \quad p_n(t) = t^n,$$

is valid, for each polynomial P of the form (3.4).

M. A. Malik [18] studied an extremal problem in the L^p-norm $(p>1)$ on $(-1,1)$. Namely, he found the following improvement of Theorem 1.4 under only a little restriction on the location of the zeros of P:

Theorem 3.5. *Let $p>1$ and $P \in \mathcal{P}_n$ have no zeros in the two cicular regions $|z \pm a| < 1-a$ $(0 \leqq a < 1)$. Then $\|P'\| \leqq Bn^{1+1/p} \|P\|$, where B is a constant which depends only on p and a, but not on P or n.*

Note that a can be taken as close to 1 as we like, except that $1-a$ has to be positive. Thus, we have the interesting conclusion that

$$\frac{\|P'\|}{\|P\|} = O(n^{1+1/p})$$

howsoever small the two exceptional circles of the theorem may be.

Similarly, S. Zhou [43] showed in $L^p(-1, 1)$, $1 \leqq p \leqq +\infty$:

Theorem 3.6. *If $P \in \mathcal{P}_n$ has at most k roots in $(-1, 1)$ then*

$$\|P'\| \leqq C(k)\, n \|P\|,$$

where $C(k)$ is a positive constant depending only on k.

The following result was given by V. I. Buslaev [5].

Theorem 3.7. *Let the polynomial P be represented in the form* $P(t)=Q(t)\,R(t)$, *where*

$$Q(t)=\prod_{i=1}^{m}(t-\tau_i),\qquad |\tau_i|\geq 1\ (i=1,\ldots,m)$$

and R is an arbitrary polynomial of degree r.

Then for every segment $[a, b]$ *lying strictly in the interior if the interval* $[-1,1]$

$$\|P'\|_{L^p(a,b)}\leq C(a,b)\,\mu\,\|P\|_{L^p(-1,1)},$$

where

$$\mu=r+1+\sum_{i=1}^{m}|\tau_i|^{-2},$$

and $C(a,b)$ *depends only on a and b.*

The extremal problems in L^2-norm on the restricted polynomial classes are especially interesting. In the next sections we will investigate such problems. Several results at this area were given by A. K. Varma [37], [38], [39], [40], [41].

4. Extremal problems of A. K. Varma

In several papers A. K. Varma studied the extremal problems of the form (3.1) in L^2-norm on $(-1,1)$ and $(0,+\infty)$ for some restricted classes of polynomials. So he got several inequalities of the form

$$\|P'\|^2\leq C_n\|P\|^2\quad (P\in W_n).$$

Beside that, he considered some oposite inequalities.

Let W_n be the set of all algebraic polynomials whose degree is n and whose zeros are all real and lie inside $[-1,1]$.

Theorem 4.1. *Let* $\|f\|^2=\int_{-1}^{1}f(t)^2\,dt$. *If* $P\in W_n$ *and* $n=2m$; *then*

$$\|P'\|^2\geq\left(\frac{n}{2}+\frac{3}{4}+\frac{3}{4(n-1)}\right)\|P\|^2,$$

where equality holds iff $P(t)=(1-t^2)^m$. *Moreover, if* $n=2m-1$, *then*

$$\|P'\|^2\geq\left(\frac{n}{2}+\frac{3}{4}+\frac{5}{4(n-2)}\right)\|P\|^2,\qquad n\geq 3,$$

where equality holds iff $P(t)=(1-t)^{m-1}(1+t)^m$ *or* $P(t)=(1-t)^m(1+t)^{m-1}$.

This theorem is proved by Varma in his paper [41] and it is an improvement of an earlier his result [38]. Similar results in uniform norm and L^p-norm $(p\geq 1)$ on $(-1,1)$ were given by P. Turán [35] and S. Zhou [44] respectively.

In 1979 A. K. Varma [39] proved the three following results:

Theorem 4.2. *Let* $\|f\|^2=\int_{-1}^{1}(1-t^2)f(t)^2\,dt$. *If* $P\in W_n$ *then*

$$\|P'\|^2\geqq\left(\frac{n}{2}+\frac{1}{4}-\frac{1}{4(n+1)}\right)\|P\|^2$$

with equality for $P(t)=(1-t^2)^m$, $n=2m$.

Theorem 4.3. *Let P be an algebraic polynomial of degree* $\leqq n$ *having all real roots and no root inside the interval* $[-1,1]$, *then we have*

$$\|P'\|^2\leqq\frac{n(n+1)(2n+3)}{4(2n+1)}\|P\|^2.$$

with equality for $P(t)=(1+t)^n$ *or* $P(t)=(1-t)^n$. *The norm is the same as in the above theorem.*

Theorem 4.4. *Let P be an algebraic polynomial of degree n having all zeros* τ_k $(k=1,\ldots,n)$ *inside* $[0,+\infty)$. *Let* $P(0)=0$ *or*

$$\sum_{k=1}^{n}\tau_k^{-1}\geqq\frac{1}{2};$$

then

$$\|P'\|^2\geqq\frac{n}{2(2n-1)}\|P\|^2$$

with equality for $P(t)=t^n$. *Here* $\|f\|^2=\int_0^\infty e^{-t}f(t)^2\,dt$.

In 1981 Varma has investigated the problem of determinating the best constant in the inequality

$$\|P'\|^2\leqq C_n(\alpha)\|P\|^2, \tag{4.1}$$

for polynomials with nonnegative coefficients, with respect to the generalized Laguerre weight function $t\to t^\alpha e^{-t}$ $(\alpha>-1)$.

Theorem 4.5. *Let P be an algebraic polynomial of exact degree n with nonnegative coefficients. Then for* $\alpha\geqq(\sqrt{5}-1)/2$,

$$\int_0^\infty(P'(t))^2t^\alpha e^{-t}\,dt\leqq\frac{n^2}{(2n+\alpha)(2n+\alpha-1)}\int_0^\infty P^2(t)\,t^\alpha e^{-t}\,dt,$$

equality holding for $P(t)=t^n$. *For* $0\leqq\alpha\leqq1/2$ *we have*

$$\int_0^\infty(P'(t))^2t^\alpha e^{-t}\,dt\leqq\frac{1}{(2+\alpha)(1+\alpha)}\int_0^\infty P^2(t)\,t^\alpha e^{-t}\,dt. \tag{4.2}$$

Moreover, (4.2) is also best possible in the sence that for $P(t) = t^n + \lambda t$ the expression on the left can be made arbitrarily close to the right by choosing λ positive and sufficiently large.

The case $\alpha = 1$ was considered in [39]. The cases $\alpha \in (-1,0)$ and $\alpha \in (1/2, (\sqrt{5}-1)/2)$ were not solved. D. Xie [42] tried to solve this problem in $(1/2, (\sqrt{5}-1)/2$. Namely, he proved the following complicated and crude result:

Theorem 4.6. *Let*

$$b_n = \frac{n^2}{(2n+\alpha)(2n-1+\alpha)}, \quad n = 1, 2, \ldots,$$

and

$$\alpha_n = \frac{1-2n-4n^2+\sqrt{16n^4+32n^3+20n^2+4n+1}}{2(2n+1)}, \quad n = 1, 2, \ldots.$$

Then, for each $P \in W_n$,

$$\|P'\|^2 \leq b_n(\alpha) \, \|P\|^2, \quad \textit{for } \alpha \geqq \alpha_1;$$

$$\|P'\|^2 \leq \begin{cases} b_1(\alpha) \, \|P\|^2, & \textit{for } \alpha_k \leqq \alpha < \alpha_{k-1} \textit{ and } n \leqq k, \\ [b_1(\alpha) + b_n(\alpha) - b_k(\alpha)] \, \|P\|^2, & \textit{for } \alpha_k \leqq \alpha < \alpha_{k-1} \textit{ and } n > k, \end{cases}$$

where $k = 2, 3, \ldots$.

In the next section we give a complete solution of the extremal problem (4.1.)

5. Extremal problems for polynomial with nonnegative coefficients in $L^2(0, \infty)$ norm

First, we consider the extremal problem (4.1).

Let W_n be the set of all algebraic polynomials of exact degree n, all coefficients of which are nonnegative, i. e.,

$$W_n = \left\{ P \mid P(t) = \sum_{k=0}^{n} a_k t^k, \; a_k \geqq 0 \; (k = 0, 1, \ldots, n-1), \; a_n > 0 \right\}.$$

We denote by W_n^0 the subset of W_n for which $a_0 = 0$ (i. e., $P(0) = 0$).

Let $w(t) = t^\alpha e^{-t}$ $(\alpha > -1)$ be a weight function on $[0, +\infty)$, and let $\|f\|^2 = (f, f)$, where

$$(f, g) = \int_0^\infty w(t) f(t) g(t) \, dt \qquad (f, g \in L^2[0, +\infty)).$$

In the paper [22] we gave a complete solution of Varma's problem (4.1), i. e. we determined

$$C_n(\alpha) = \sup_{P\in W_n} \frac{\|P'\|^2}{\|P\|^2} \tag{5.1}$$

for all $\alpha\in(-1,+\infty)$.

Theorem 5.1. *The best constant* $C_n(\alpha)$ *defined in* (5.1) *is*

$$C_n(\alpha)=\begin{cases} 1/(2+\alpha)(1+\alpha) & (-1<\alpha\leqq\alpha_n),\\ n^2/(2n+\alpha)(2n+\alpha-1) & (\alpha_n\leqq\alpha<+\infty),\end{cases} \tag{5.2}$$

where

$$\alpha_n=\frac{1}{2}(n+1)^{-1}\left((17n^2+2n+1)^{1/2}-3n+1\right). \tag{5.3}$$

Note that the supremum in (5.1) is attained for some $P\in W_n^0$. Indeed

$$\sup_{P\in W_n}\frac{\|P'\|}{\|P\|}=\sup_{\substack{P\in W^0\\ a_0\geqq 0}}\frac{\|P'\|}{\|P+a_0\|}=\sup_{P\in W_n^0}\frac{\|P'\|}{\|P\|}.$$

We can see that $P(t)=t^n$ is an extremal polynomial for $\alpha\geqq\alpha_n$. Furthermore, if $-1<\alpha\leqq\alpha_n$, there exists a sequence of polynomials, for example, $p_k(t)=$ $=t^n+kt$, $k=1,2,\ldots$, for which

$$\lim_{k\to\infty}\frac{\|p_k'\|^2}{\|p_k\|^2}=C_n(\alpha).$$

From Theorem 5.1 we can see:

(a) $C_n(\alpha_n-0)=C_n(\alpha_n+0)$;

(b) $C_{n+1}(\alpha)\leqq C_n(\alpha)$;

(c) The sequence (α_k) is decreasing, i. e., $\alpha_1>\alpha_2>\alpha_2>\ldots>\alpha_\infty$, where

$$\alpha_1=(\sqrt{5}-1)/2,\quad \alpha_2=(\sqrt{73}-5)/6,\quad \alpha_3=(\sqrt{10}-2)/2,\ \text{etc.},$$

and

$$\alpha_\infty=\lim_{n\to\infty}\alpha_n=(\sqrt{17}-3)/2=0.561552812\ldots.$$

Remark. The statement of Theorem 5.1 holds if W_n is the set of all algebraic polynomials $P(\not\equiv 0)$ of degree at most n (not only of exact degree n), with nonnegative coefficients. In this case, if $-1<\alpha\leqq\alpha_n$, we can see that $\tilde{P}(t)=$ $=\lambda t$ $(\lambda>0)$ is an extremal polynomial.

Using the same method as in [22], we can solve the following general extremal problem for higher derivatives

$$C_{n,m}(\alpha)=\sup_{P\in W_n}\frac{\|P^{(m)}\|^2}{\|P\|^2}\qquad(1\leqq m\leqq n).$$

Theorem 5.2. *The best constant* $C_{n,m}(\alpha)$ *is given by*

$$C_{n,m}(\alpha)=\begin{cases}\dfrac{(m!)^2}{(\alpha+1)_{2m}}, & -1<\alpha\leqq\alpha_{n,m},\\[2ex] \dfrac{n^2(n-1)^2\cdots(n-m+1)^2}{(2n+\alpha)^{(2m)}}, & \alpha\geqq\alpha_{n,m},\end{cases}$$

where $\alpha_{n,m}$ *is the unique positive root of the equation*

$$\frac{(2n+\alpha)^{(2m)}}{(2m+\alpha)^{(2m)}}=\binom{n}{m}^2.$$

Here $(p)_k=p(p+1)\ldots(p+k-1)$ *and* $p^{(k)}=p(p-1)\ldots(p-k+1)$.

In the special case, when $n\to+\infty$, we have

$$\lim_{n\to\infty}C_{n,m}(\alpha)=\begin{cases}\dfrac{(m!)^2}{(\alpha+1)_{2m}}, & -1<\alpha\leqq\alpha_m^*,\\[2ex] \dfrac{1}{4^m} & \alpha_m^*\leqq\alpha<+\infty,\end{cases}$$

where α_m^* is the unique positive root of the equation

$$(2m+\alpha)^{(2m)}=2^{2m}(m!)^2.$$

We note that $\alpha_1^*=\alpha_\infty=(\sqrt{17}-3)/2$. These roots α_m^* for $m=2,\ldots,6$ are presented in Table 5.1 (with seven decimal digits).

Table 5.1

m	2	3	4	5	6
α_m^*	0.5515992	0.5461112	0.5425236	0.5399438	0.5379725

The extremal problem for the polynomials with nonnegative coefficients can be investigated with other weight functions on $(0,+\infty)$, for example, $w(t)=$ $=t^\alpha\exp(-t^s)$, $\alpha>-1$, $s>0$. The corresponding best constant we will denote by $C_n(\alpha;s)$.

In this case, using the same method we can prove that for $P\in W_n^0$

$$\|P\|^2=(P,P)=\frac{1}{s}\sum_{k=2}^{2n}b_k\Gamma\left(\frac{\alpha+k+1}{s}\right)$$

and

$$\|P'\|^2=(P',P')\leqq\frac{1}{s}\sum_{k=2}^{2n}H_k(\alpha;s)\,b_k\Gamma\left(\frac{\alpha+k+1}{s}\right),$$

where $(f, g) = \int\limits_0^\infty w(t) f(t) g(t)\, \mathrm{d}t$ and

$$H_k(\alpha; s) = \frac{k^2}{2} \cdot \frac{\Gamma\left(\frac{\alpha+k+1}{s}\right)}{\Gamma\left(\frac{\alpha+k+1}{s}\right)}.$$

For $s = 2$ we get a simple result:

Theorem 5.3. *The best constant $C_n(\alpha; 2)$ is given by*

$$C_n(\alpha; 2) = \begin{cases} \dfrac{2}{\alpha+1}, & -1 < \alpha \leqq -\dfrac{n-1}{n+1}, \\ \dfrac{2n^2}{2n+\alpha-1}, & -\dfrac{n-1}{n+1} \leqq \alpha < +\infty. \end{cases}$$

If we take, e. g. $\alpha = 0$, we have the following inequality

$$\int\limits_0^\infty e^{-t^2} P'(t)^2\, \mathrm{d}t \leqq \frac{2n^2}{2n-1} \int\limits_0^\infty e^{-t^2} P(t)^2\, \mathrm{d}t$$

for each $P \in W_n$.

In connection with these results see the paper [23].

6. Extremal problems for Lorentz classes of polynomials

Let L_n be the set of algebraic polynomials of the form

$$(6.1) \qquad P(t) = \sum_{k=0}^{n} b_k (1-t)^k (1+t)^{n-k}, \quad b_k \geqq 0 \ (k = 0, 1, \ldots, n).$$

These polynomials (transformed to [0, 1]) were introduced by G. G. Lorentz [14] (see Section 3). A subset of Lorentz's class L_n for which $P^{(i-1)}(\pm 1) = 0$ $(i=1, \ldots, m)$ we denote by $L_n^{(m)}$. Notice that $L_n^{(0)} \supset L_n^{(1)} \supset \cdots$, where $L_n^{(0)} \equiv L_n$. The corresponding representation of a polynomial P from $L_n^{(m)}$ is

$$(6.2) \qquad P(t) = \sum_{k=m}^{n-m} b_k (1-t)^k (1+t)^{n-k}, \quad b_k \geqq 0 \ (k = m, \ldots, n-m).$$

If $\|f\|^2 = \int\limits_{-1}^{1} (1-t)^\alpha (1+t)^\beta f(t)^2\, \mathrm{d}t$ $(\alpha, \beta > -1)$, we can consider the following extremal problem

$$(6.3) \qquad C_n^{(m)}(\alpha, \beta) = \sup_{P \in L_n^{(m)} \setminus \{0\}} \frac{\|P'\|^2}{\|P\|^2},$$

where $m=0,1,\ldots,\left[\frac{n}{2}\right]$. The corresponding problem in the class L_n for the uniform norm was considered by G. G. Lorentz (see Theorem 3.3).

Here, we mention only some special cases of general results obtained by G. V. Milovanović and M. S. Petković [24].

Theorem 6.1. *If* $P \in L_n$ *and* $\alpha, \beta \geqq 1$, *then the best constant* $C_n^{(0)}(\alpha, \beta)$ *defined in* (6.3) *is*

$$C_n^{(0)}(\alpha, \beta)=\frac{n^2(2n+\alpha+\beta)(2n+\alpha+\beta+1)}{4(2n+\lambda)(2n+\lambda-1)},$$

where $\lambda=\min(\alpha, \beta)$.

Theorem 6.2. *If* $P \in L_n^{(m)}$, $m \geqq 1$, $\alpha=\beta>-1$, *then*

$$C_n^{(m)}(\alpha, \alpha)=\frac{(n+\alpha)(2n+2\alpha+1)\left(\alpha(\alpha-1)n^2+2m(n-m)(n-1+3\alpha-2\alpha^2)\right)}{2(2m+\alpha-1)(2m+\alpha)(2n-2m+\alpha-1)(2n-2m+\alpha)}.$$

In the special case when $\alpha=1$ we obtain:

Corollary 6.3. *If* $P \in L_n^{(m)}$, $m \geqq 1$, *we have*

$$C_n^{(m)}(1,1)=\frac{n(n+1)(2n+3)}{4(2m+1)(2n-2m+1)}. \tag{6.4}$$

From Theorem 6.1 we see that (6.4) holds for $m=0$ (see also Theorem 4.3).

In the proofs of these theorems we use the representations of Lorenz polynomials (6.1) and (6.2) and an analogue of the Lemma 1 from [22]. Another interesting results on this topic can be found in the mentioned paper [27].

REFERENCES

1. N. K. Bari: *Generalization of inequalities of S. N. Bernstein and A. A. Markov.* Izv. Akad. Nauk SSSR Ser. Mat. 18 (1954), 159—176 (Russian).
2. P. B. Borwein: *Markov's and Bernstein's inequalities on disjoint intervals.* Can. Math. 33 (1981), 201—209.
3. P. B. Borwein: *Markov's inequality for polynomials with real zeros.* Proc. Amer. Math. Soc. 93 (1985), 43—47.
4. V. I. Buslaev: *Estimation of the derivative of a polynomial with real coefficients.* Izv. Akad. Nauk SSSR Ser. Mat. 39 (1975), 413—417 (Russian).
5. V. I. Buslaev: *Some inequalities for polynomials with real coefficients.* Soviet Math. Dokl. 16 (1975), 803—805.
6. P. Dörfler: *New inequalities of Markov typs.* SIAM J. Math. Anal. 18 (1987), 490—494.
7. P. Erdös: *On extremal properties of the derivatives of polynomials.* Ann. of Math. (2) 41 (1940), 310—313.
8. K. Fan, O. Taussky, and J. Todd: *Discrete analogs of inequalities of Wirtinger.* Monatsh. Math. 59 (1955), 73—79.
9. E. Hille, G. Szegö and J. D. Tamarkin: *On some generalisations of a theorem of A. Markoff.* Duke Math. J. 3 (1937), 729—739.
10. V. I. Ivanov: *Some extremal properties of polynomials and inverse inequalities in approximation theory.* Trudy Mat. Inst. Steklov, 145 (1979), 79—110 (Russian).

11. S. V. KONJAGIN: *Estimations of the derivatives of polynomials.* Dokl. Akad. Nauk SSSR 243 (1978), 1116—1118 (Russian).
12. S. V. KONJAGIN: *On Markov's inequality for polynomials in L metric.* Trudy Mat. Inst. Steklov, 145 (1979), 117–125 (Russian).
13. G. K. LEBED': *Inequalities for polynomials and their derivatives.* Dokl. Akad. Nauk SSSR 117 (1957), 570—572 (Russian).
14. G. G. LORENTZ: *The degree of approximation by polynomials with positive coefficients* Math. Ann. 151 (1963), 239—251.
15. G. G. LORENTZ: *Approximation of functions.* Holt, Rinehart and Winston, New York, 1966.
16. G. G. LORENTZ: *Derivatives of polynomials with positive coefficients.* J. Approx. Theory 1 (1968), 1—4.
17. G. G. LORENTZ, K. JETTER and S. D. RIEMENSCHNEIDER: *Birkhoff interpolation.* Encyclopedia of Math. and Appl., Vol. XIX, Addison—Wesley, Reading, Mass., 1983.
18. M. A. MALIK: *On a theorem of Hille, Szegö, and Tamarkin.* J. Math. Anal. Appl. 30 (1970), 503—509.
19. A. A. MARKOV: *On a problem of D. I. Mendeleev,* Zap. Imp. Akad. Nauk, St. Petersburg 62 (1889), 1—24 (Russian).
20. V. A. MARKOV: *On functions deviating least from zero in a given interval.* Izdat. Imp. Akad. Nauk, St. Petersburg, 1892 (Russian); German transl. Math. Ann. 77 (1916), 218—258.
21. G. MEINARDUS: *Approximation of functions: Theory and numerical methods.* Springer--Verlag, New York, 1967.
22. G. V. MILOVANOVIĆ: *An extremal problem for polynomials with nonnegative coefficients* Proc. Amer. Math. Soc. 94 (1985), 423—426.
23. G. V. MILOVANOVIĆ and R. Ž. DJORDJEVIĆ: *An extremal problem for polynomials with nonnegative coefficients. II.* Facta Univ. Ser. Math. Inform. 1 (1986), 7—11.
24. G. V. MILOVANOVIĆ and M. S. PETKOVIĆ: *Extremal problems for Lorentz classes of nonnegative polynomials in L^2-metric with Jacobi weight.* Proc. Amer. Math. Soc. (to appear).
25. G. V. MILOVANOVIĆ and S. WRIGGE: *Least squares approximation with constraints.* Math. Comp. 46 (1986), 551—565.
26. L. MIRSKY: *An inequality of the Markov-Bernstein type for polynomials.* SIAM J. Math. Anal. 14 (1983), 1004—1008.
27. M. K. POTAPOV: *Some inequalities for polynomials and their derivatives.* Vestnik Moskov. Univ. Ser. I Mat. Meh. No 2 (1960), 10—20 (Russian).
28. Q. I. RAHMAN and G. SCHMEISSER: *Inequalities for polynomials on the unit interval.* Trans. Amer. Math. Soc. 231 (1977), 93—100.
29. J. T. SCHEICK: *Inequalities for derivatives of polynomials of special type.* J. Aprox.Theory 6 (1972), 354—358.
30. E. SCHMIDT: *Über die nebst ihren Ableitungen orthogonalen Polynomensysteme und das zugehörige Extremum.* Math. Ann. 119 (1944), 165—204.
31. L. F. SHAMPINE: *Asymptotic L_2 inequalities of Markoff type.* Ph. D. Thesis, California Institute of Technology, Pasadena, 1964.
32. J. SZABADOS: *Bernstein and Markov type estimates for the derivative of a polynomial with real zeros.* Functional Analysis and Approximation (P. L. Butzer, B. Sz.-Nagy and E. Görlick, eds.), Birkhauser, Basel 1981, pp. 177—188.
33. J. SZABADOS and A. K. VARMA: *Inequalities for derivatives of polynomial having real zeros.* Approximation Theory III. (E. W. Cheney, ed.), Academic Press, New York 1980, pp. 881—887.
34. G. SZEGÖ: *On some problems of approximations.* Magyar Tud. Akad. Mat. Kutato Int. Dozl. 2 (1964), 3—9.
35. P. TURÁN: *Über die Ableitung von Polynomen.* Compositio Math. 7 (1939), 85—95.
36. P. TURÁN: *Remark on a theorem of Erhard Schmidt.* Mathematica 2 (25) (1960), 373—378.
37. A. K. VARMA: *An analogue of some inequalities of P. Turán concerning algebraic polynomials having all zeros inside* [—1,1]. Proc. Amer. Math. Soc. 55 (1976), 305—309.

38. A. K. VARMA: *An analogue of some inequalities of P. Turán concerning algebraic polynomials having all zeros inside* [—1,1]. *II.* Proc. Amer. Math. Soc. 69 (1978), 25—33.
39. A. K. VARMA: *Some inequalities of algebraic polynomials having real zeros.* Proc. Amer. Math. Soc. 75 (1979), 243—250.
40. A. K. VARMA: *Derivatives of polynomials with positive coefficients.* Proc. Amer. Math. Soc. 83 (1981), 107—112.
41. A. K. VARMA: *Some inequalities of algebraic polynomials having all zeros inside* [—1,1]. Proc. Amer. Math. Soc. 88 (1983), 227—233.
42. D. XIE: *A replenishment for the theorem of Varma, A..* J. Hangzhou Univ. 10 (1983), 266—270 (Chinese, English summary).
43. S. ZHOU: *Inequalities for derivatives of polynomials having real zeros.* J. Hangzhou Univ. 10 (1983), 448—457 (Chinese, English summary).
44. S. ZHOU: *On the Turán inequality in L_p-norm.* J. Hangzhou Univ. 11 (1984), 28—33 (Chinese, English summary).

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RAZLIČITI EKSTREMALNI PROBLEMI MARKOVLJEVOG TIPA ZA ALGEBARSKE POLINOME

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U radu se razmatraju ekstremalni problemi Markovljevog tipa za algebarske polinome korišćenjem različitih normi i više polinomialnih klasa. Posebna pažnja je posvećena ekstremalnim problemima u L^2-normi na skupu svih algebarskih polinoma ne višeg stepena od n ili na nekim njegovim podskupovima.