AN ESTIMATE FOR COEFFICIENTS
OF POLYNOMIALS IN $L^2$ NORM. II

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Dedicated to the memory of Professor S. Aljančić

Abstract. Let $\mathcal{P}_n$ be the class of algebraic polynomials $P(x) = \sum_{k=0}^{n} a_k x^k$ of degree at most $n$ and $\|P\|_{d\sigma} = \left( \int_{\mathbb{R}} |P(x)|^2 d\sigma(x) \right)^{1/2}$, where $d\sigma(x)$ is a nonnegative measure on $\mathbb{R}$. We determine the best constant in the inequality $|a_k| \leq C_{n,k} \|P\|_{d\sigma}$, for $k = 0, 1, \ldots, n$, when $P \in \mathcal{P}_n$ and such that $P(\xi_k) = 0$, $k = 1, \ldots, m$. The cases $C_{n,n}(d\sigma)$ and $C_{n,n-1}(d\sigma)$ were studied by Milovanović and Guessab [6]. In particular, we consider the case when the measure $d\sigma(x)$ corresponds to generalized Laguerre orthogonal polynomials on the real line.

1. Introduction

Let $\mathcal{P}_n$ be the class of algebraic polynomials $P(x) = \sum_{k=0}^{n} a_k x^k$ of degree at most $n$. The first inequality of the form $|a_k| \leq C_{n,k} \|P\|$ was given by Markov [3]. Namely, if $\|P\| = \|P\|_{\infty} = \max_{x \in [-1,1]} |P(x)|$ and $T_n(x) = \sum_{i=0}^{n} t_{n,i} x^i$ denotes the $n$-th Chebyshev polynomial of the first kind, then Markov proved that

$$|a_k| \leq \begin{cases} |t_{n,k}| \cdot \|P\|_{\infty} & \text{if } n-k \text{ is even,} \\ |t_{n-1,k}| \cdot \|P\|_{\infty} & \text{if } n-k \text{ is odd.} \end{cases} \quad (1.1)$$

For $k = n$ (1.1) reduces to the well-known Chebyshev inequality

$$|a_n| \leq 2^{n-1} \|P\|_{\infty}. \quad (1.2)$$

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Using a restriction on the polynomial class like \( P(1) = 0 \) or \( P(-1) = 0 \), Schur [8] found the following improvement of (1.2)

\[
|a_n| \leq 2^{n-1} \left( \cos \frac{\pi}{4n} \right)^{2n} \|P\|_\infty.
\]

This result was extended by Rahman and Schmeisser [7] for polynomials with real coefficients, which have at most \( n - 1 \) distinct zeros in \((-1, 1)\).

Similarly in \( L^2 \) norm,

\[
\|P\| = \|P\|_2 = \left( \int_{-1}^{1} |P(x)|^2 \, dx \right)^{1/2},
\]

Tariq [10] improved the following result of Labelle [2]

\[
|a_k| \leq \frac{1 \cdot 3 \cdot 5 \cdots (2k-1)}{k!} \left( k + \frac{1}{2} \right)^{1/2} \left( \left( \frac{(n-k)/2}{k+1/2} \right) \|P\|_2 \right),
\]

for \( P \in \mathcal{P}_n \) and \( 0 \leq k \leq n \), where the symbol \([x]\) denotes as usual the integral part of \( x \). Equality in this case is attained only for the constant multiplies of the polynomial

\[
\sum_{\nu=0}^{[\frac{n-k}{2}]} (-1)^\nu (4\nu + 2k + 1) \left( k + \frac{\nu - 1/2}{\nu} \right) P_{k+2\nu}(x),
\]

where \( P_n(x) \) denotes the Legendre polynomial of degree \( m \).

Under restriction \( P(1) = 0 \), Tariq [10] proved that

\[
|a_n| \leq \frac{n}{n+1} \left( \frac{2n+1}{2n} \right)^{1/2} \left( \frac{2n+1}{2} \right)^{1/2} \|P\|_2,
\]

with equality case \( P(x) = P_n(x) - \frac{1}{n^2} \sum_{\nu=0}^{n-1} (2\nu + 1)P_{\nu}(x) \). Also, he obtained that

\[
|a_{n-1}| \leq \frac{n^2 + 2}{n+1} \left( \frac{2n-2}{n} \right) \left( \frac{2n-1}{2} \right)^{1/2} \|P\|_2,
\]

with equality case

\[
P(x) = \frac{2n+1}{n^2+2} P_n(x) - P_{n-1}(x) + \frac{1}{n^2+2} \sum_{\nu=0}^{n-2} (2\nu + 1)P_{\nu}(x).
\]
In the absence of the hypothesis $P(1) = 0$ the factor $(n^2 + 2)^{1/2} / (n + 1)$ appearing on the right-hand side of (1.5) is to be dropped.

This result was extended by Milovanović and Guessab [4] for polynomials with real coefficients, which have $m$ zeros on real line.

In this paper we consider more general problem including $L^2$ norm of polynomials with respect to a nonnegative measure on the real line $\mathbb{R}$. The generalized Laguerre measure is also included.

2. Main results

Let $d\sigma(x)$ be a given nonnegative measure on the real line $\mathbb{R}$, with compact or infinite support, for which all moments $\mu_k = \int_{\mathbb{R}} x^k d\sigma(x)$, $k = 0, 1, \ldots$, exist and are finite, and $\mu_0 > 0$. In that case, there exist a unique set of orthonormal polynomials $\pi_n(\cdot) = \pi_n(\cdot ; d\sigma)$, $n = 0, 1, \ldots$, defined by

$$
\pi_n(x) = b^{(n)}_n(d\sigma)x^n + b^{(n)}_{n-1}(d\sigma)x^{n-1} + \cdots + b^{(n)}_0(d\sigma), \quad b^{(n)}_n(d\sigma) > 0,$$

$$
(\pi_n, \pi_m) = \delta_{nm}, \quad n, m \geq 0,
$$

where

$$
(f, g) = \int_{\mathbb{R}} f(x)\overline{g(x)} \, d\sigma(x) \quad (f, g \in L^2(\mathbb{R})).
$$

(2.1)

For $P \in \mathcal{P}_n$, we define

$$
||P||_{d\sigma} = \sqrt{\langle P, P \rangle} = \left( \int_{\mathbb{R}} |P(x)|^2 \, d\sigma(x) \right)^{1/2}.
$$

(2.2)

Also, for $\xi_k \in \mathbb{C}$, $k = 1, \ldots, m$, we define a restricted polynomial class

$$
\mathcal{P}_n(\xi_1, \ldots, \xi_m) = \{ P \in \mathcal{P}_n | P(\xi_k) = 0, \ k = 1, \ldots, m \} \quad (0 \leq m \leq n).
$$

In the case $m = 0$ this class of polynomials reduces to $\mathcal{P}_n$. The case $m = n$ is trivial. If $\xi_1 = \cdots = \xi_k = \xi$ ($1 \leq k \leq m$) then the restriction on polynomials at the point $x = \xi$ becomes $P(\xi) = P'(\xi) = \cdots = P^{(k-1)}(\xi) = 0$.

Let

$$
\prod_{i=1}^{m} (x - \xi_i) = x^n - s_1 x^{n-1} + \cdots + (-1)^{m-1} s_{m-1} x + (-1)^m s_m
$$

where $s_k$ denotes elementary symmetric functions of $\xi_1, \ldots, \xi_m$, i.e.,

$$
s_k = \sum \xi_1 \cdots \xi_k \quad \text{for} \quad k = 1, \ldots, m.
$$

(2.3)

For $k = 0$ we have $s_0 = 1$, and $s_k = 0$ for $k > m$ or $k < 0$. 
Theorem 2.1. Let \( P \in \mathcal{P}_n(\xi_1, \ldots, \xi_m) \) and \( s_1, \ldots, s_m \) be given by (2.3). If the measure \( d\sigma(x) \) is given by

\[
d\sigma(x) = \prod_{k=1}^{m} |x - \xi_k|^2 \, d\sigma(x)
\]

and \( ||P||_{d\sigma} \) is defined by (2.2), then

\[
|a_{n-k}| \leq \left( \sum_{j=k}^{k} \left( \sum_{i=j}^{k} (-1)^{k-i} s_{k-i} \tilde{b}_{n-k}^{(n-m-j)} \right)^2 \right)^{1/2} ||P||_{d\sigma},
\]

for \( k = 0, 1, \ldots, n \), where \( \tilde{b}_{n-k}^{(n-m-j)}(d\sigma) = 0, 1, \ldots, \mu \), are the coefficients in the orthonormal polynomial \( \tilde{\pi}_\mu(\cdot) = \pi_\mu(\cdot; d\sigma) \).

Inequality (2.5) is sharp and becomes an equality if and only if \( P(x) \) is a constant multiple of the polynomial

\[
\left( \sum_{j=0}^{k} \tilde{\pi}_n-m-j(x) \sum_{i=j}^{k} (-1)^{k-i} s_{k-i} \tilde{b}_{n-k}^{(n-m-j)} \right) \prod_{k=1}^{m} (x - \xi_k).
\]

Proof. At first we consider the inner product (2.1). Then the polynomial \( P(x) = \sum_{\nu=0}^{n} a_{\nu} x^\nu \in \mathcal{P}_n \) can be represented in the form \( P(x) = \sum_{\nu=0}^{n} a_{\nu} \pi_\nu(x; d\sigma) \), where \( a_{\nu} = \langle P, \pi_\nu \rangle \), \( \nu = 0, 1, \ldots, n \). Then we have

\[
a_{n-k} = \sum_{i=0}^{k} a_{n-i} \tilde{b}_{n-k}^{(n-i)}(d\sigma) = \left( \sum_{i=0}^{k} b_{n-k}^{(n-i)}(d\sigma) \tilde{\pi}_n(\cdot) \right), \quad k = 0, 1, \ldots, n,
\]

where \( \pi_\nu(\cdot) = \pi_\nu(\cdot; d\sigma) \).

Suppose now that \( P \in \mathcal{P}_n(\xi_1, \ldots, \xi_m) \). Then we can write

\[
P(x) = Q(x) \prod_{k=1}^{m} (x - \xi_k),
\]

where \( Q(x) = a_{n-m} x^{n-m} + a_{n-m-1} x^{n-m-1} + \cdots + a_0 \in \mathcal{P}_{n-m} \). Also, we have

\[
\prod_{i=1}^{m} (x - \xi_i) = x^m - s_1 x^{m-1} + \cdots + (-1)^{m-1} s_{m-1} x + (-1)^m s_m
\]

where \( s_k, k = 0, 1, \ldots, m \), denotes elementary symmetric functions (2.3). Now, putting this in (2.7), we obtain

\[
P(x) = \sum_{i=0}^{n-m} \sum_{\nu=0}^{m} a_{\nu} (-1)^\nu s_\nu x^{m+i-\nu} = \sum_{k=0}^{n} a_{n-k} x^{n-k},
\]
where
\[ a_{n-k} = \sum_{i=0}^{k} a'_{n-m-i}(-1)^{k-i}s_{k-i}, \quad k = 0, 1, \ldots, n, \]  
(2.8)
and \( a'_{k} = 0 \) for \( k < 0 \) and \( k > n - m \).

Now, the corresponding equalities (2.6) for polynomial \( Q \) in the measure \( d\tilde{\sigma}(x) \), given by (2.4), become

\[ a'_{n-m-i} = \left( Q, \sum_{j=0}^{i} \tilde{b}_{n-m-i}^{(n-m-j)} \tilde{\sigma}_{n-m-j} \right), \quad i = 0, 1, \ldots, n - m, \]  
(2.9)
where \( \tilde{\sigma}_{v} = \pi_{v}(:d\tilde{\sigma}) \).

According to (2.7), we have

\[ a_{n-k} = \sum_{i=0}^{k} (-1)^{k-i}s_{k-i} \left( Q, \sum_{j=0}^{i} \tilde{b}_{n-m-i}^{(n-m-j)} \tilde{\sigma}_{n-m-j} \right) = (Q, W_{n-m}) \]  
(2.10)
where

\[ W_{n-m}(x) = \sum_{i=0}^{k} (-1)^{k-i}s_{k-i} \sum_{j=0}^{i} \tilde{b}_{n-m-i}^{(n-m-j)} \tilde{\sigma}_{n-m-j}(x) \]
\[ = \sum_{j=0}^{k} \tilde{\sigma}_{n-m-j}(x) \sum_{i=j}^{k} (-1)^{k-i}s_{k-i} \tilde{b}_{n-m-i}^{(n-m-j)} \]

and \( \tilde{b}_{\nu} = 0 \) for \( \nu < 0 \). Now, using Cauchy inequality we get

\[ |a_{n-k}| \leq C_{n,n-k}\|Q\|_{d\tilde{\sigma}} \]

where \( C_{n,n-k} = \|W_{n-m}\|_{d\tilde{\sigma}} = \left( \sum_{j=0}^{k} \left( \sum_{i=j}^{k} (-1)^{k-i}s_{k-i} \tilde{b}_{n-m-i}^{(n-m-j)} \right)^{2} \right)^{1/2} \). Since

\[ \|Q\|_{d\tilde{\sigma}}^{2} = \int_{\mathbb{R}} |Q(x)|^{2}d\tilde{\sigma}(x) = \int_{\mathbb{R}} |P(x)|^{2}d\tilde{\sigma}(x) = \|P\|_{d\tilde{\sigma}}^{2} \]

we obtain inequality (2.5).

The extremal polynomial is \( x \mapsto W_{n-m}(x) \prod_{k=1}^{m} (x - \xi_{k}) \). \( \square \)

**Remark 2.1.** For \( k = 0 \) and \( k = 1 \) Theorem 2.1 gives the results obtained by Milovanović and Guessab [4] (see also [6, pp. 432–439]).

Consider now the generalized Laguerre measure \( d\tilde{\sigma}(x) = x^{\alpha}e^{-x}dx, \alpha > -1 \), on \((0, +\infty)\). With \( \tilde{L}_{n}^{(\alpha)}(x) \) we denote the generalized orthonormal Laguerre polynomial. The coefficient \( \tilde{b}_{k}^{(n)} \) of \( x^{k} \) in \( \tilde{L}_{n}^{(\alpha)}(x) \) is given by

\[ \tilde{b}_{k}^{(n)} = (-1)^{n-k} \binom{n}{k} \frac{(\alpha + k + 1)_{n-k}}{\sqrt{n! \Gamma(n + \alpha + 1)}} \]

As a direct corollary of Theorem 2.1, we have:
Corollary 2.2. Under restriction \( P^{(i)}(0) = 0, i = 0, 1, \ldots, m - 1 \), we have that

\[
|a_{n-k}| \leq \sqrt{A_{n,k} \| P \|_2},
\]

where

\[
A_{n,k} = \frac{1}{(n - m - k)!\Gamma(n + m - k + \alpha + 1)} \sum_{j=0}^{k} \binom{n + m - j + \alpha}{k - j} \binom{n - m - j}{k - j}
\]

for \( n - k \geq m \), and \( A_{n,k} = 0 \) for \( n - k < m \). The equality is attained if and only if \( P(x) \) is a constant multiple of the polynomial

\[
x^m \sum_{j=0}^{k} b^{(n-m-j)}_{n-m-k} L_{n-m-j}(x).
\]

References


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