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On an Inequality of Bogar and Gustafson

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In the year 1978 Bogar and Gustafson [3] have shown that the homogeneous boundary value problem

$$x^{(6)} - \sum_{i=0}^{2} p_i(t)\, x^{(i)} = 0, \tag{1}$$

$$x(a) = x'(a) = x''(a) = x'''(a) = x(b) = x'(b) = 0, \tag{2}$$

where $p_i \in C[a, b]$, $0 \leqslant i \leqslant 2$ has only the trivial solution provided the inequality

$$\frac{377.4}{10^6}(b-a)^6 \,\|p_0\| + \frac{156.91}{10^5}(b-a)^5 \,\|p_1\| + \frac{9}{2048}(b-a)^4 \,\|p_2\| < 1, \tag{3}$$

is satisfied. Their method of "partial inversion" of linear operators however does not seem to work for the complete differential equation

$$x^{(6)} - \sum_{i=0}^{5} p_i(t)\, x^{(i)} = 0, \tag{4}$$

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where $p_i \in C[a, b]$, together with the boundary conditions (2). The purpose of this paper is to prove the following:

THEOREM. *The boundary value problem* (4), (2) *has only the trivial solution provided*

$$\frac{1}{32805}(b-a)^6 \| p_0 \| + \left(\frac{25+34\sqrt{10}}{911250}\right)(b-a)^5 \| p_1 \| + \frac{1}{360}(b-a)^4 \| p_2 \|$$

$$+\frac{1}{30}(b-a)^3 \| p_3 \| + \frac{1}{5}(b-a)^2 \| p_4 \| + \frac{2}{3}(b-a) \| p_5 \| = \theta \leqslant 1. \tag{5}$$

Obviously, for the particular boundary value problem (1), (2) the inequality (5) provides a sharper upper estimate on the length of the interval (b–a) compared to Bogar and Gustafson's inequality (3).

For the proof, we need the following:

LEMMA 1. *Any function* $x \in C^{(6)}[0, 1]$ *satisfying the conditions*

$$x(0) = x'(0) = x''(0) = x'''(0) = x(1) = x'(1) = 0 \tag{6}$$

can be written as

$$x(t) = t^3(1-t)^2 F(t), \tag{7}$$

where

$$\begin{aligned} F(t) &= \int_0^t t_1^{\ 2} F_1(t_1)\, dt_1 \\ F_1(t) &= \int_0^t t_2^{-2} F_2(t_2)\, dt_2 \\ F_2(t) &= \int_0^t t_3^{-2} F_3(t_3)\, dt_3 \\ F_3(t) &= \int_0^t t_4^3 (1-t_4)^{-5} F_4(t_4)\, dt_4 \\ F_4(t) &= \int_1^t (1-t_5)^{-2} F_5(t_5)\, dt_5 \\ F_5(t) &= \int_1^t (1-t_6)^5 x^{(6)}(t_6)\, dt_6. \end{aligned} \tag{8}$$

Proof. Let $\phi(t)$ be the right side of (7), then it follows that

$$\phi'(t) = [3t^2(1-t)^2 - 2t^3(1-t)]\, F(t) + (1-t)^2\, tF_1(t) \tag{9}$$

$$\begin{aligned}\phi''(t) &= [6t(1-t)^2 - 12t^2(1-t) + 2t^3]\, F(t) \\ &\quad + [4(1-t)^2 - 4t(1-t)]\, F_1(t) + \frac{(1-t)^2}{t} F_2(t)\end{aligned} \tag{10}$$

$$\begin{aligned}\phi'''(t) &= [18t^2 - 36t(1-t) + 6(1-t)^2]\, F(t) \\ &\quad + \left[6t - 24(1-t) + \frac{6(1-t)^2}{t}\right] F_1(t) \\ &\quad + \left[\frac{3(1-t)^2}{t^2} - \frac{6(1-t)}{t}\right] F_2(t) + \frac{(1-t)^2}{t^3} F_3(t)\end{aligned} \tag{11}$$

$$\begin{aligned}\phi''''(t) &= [72t - 48(1-t)]\, F(t) \\ &\quad + \left[48 - \frac{48(1-t)}{t}\right] F_1(t) + \left[\frac{12}{t} - \frac{24(1-t)}{t^2}\right] F_2(t) \\ &\quad - \frac{8(1-t)}{t^3} F_3(t) + \frac{1}{(1-t)^3} F_4(t)\end{aligned} \tag{12}$$

$$\begin{aligned}\phi^{(5)}(t) &= 120F(t) + \frac{120}{t} F_1(t) + \frac{60}{t^2} F_2(t) \\ &\quad + \frac{20}{t^3} F_3(t) - \frac{5}{(1-t)^4} F_4(t) + \frac{1}{(1-t)^5} F_5(t)\end{aligned} \tag{13}$$

$$\begin{aligned}\phi^{(6)}(t) &= \frac{120}{t^2} F_1(t) - \frac{120}{t^2} F_1(t) + \frac{120}{t^3} F_2(t) - \frac{120}{t^3} F_2(t) \\ &\quad + \frac{60}{t^4} F_3(t) - \frac{60}{t^4} F_3(t) + \frac{20}{(1-t)^5} F_4(t) - \frac{20}{(1-t)^5} F_4(t) \\ &\quad - \frac{5}{(1-t)^6} F_5(t) + \frac{5}{(1-t)^6} F_5(t) + x^{(6)}(t),\end{aligned}$$

i.e., $\phi^{(6)}(t) = x^{(6)}(t)$, and since $\phi(t)$ obviously satisfies (6), we find that $x(t) \equiv \phi(t)$.

LEMMA 2. *Let $x(t)$ be as in Lemma 1, then*

$$x(1-t) = (1-t)^4\, tG(t), \tag{14}$$

where

$$\begin{aligned}
G(t) &= \int_0^t t_1^{-2} G_1(t_1)\, dt_1 \\
G_1(t) &= \int_0^t t_2(1-t_2)^{-3} G_2(t_2)\, dt_2 \\
G_2(t) &= \int_1^t (1-t_3)^{-2} G_3(t_3)\, dt_3 \\
G_3(t) &= \int_1^t (1-t_4)^{-2} G_4(t_4)\, dt_4 \\
G_4(t) &= \int_1^t (1-t_5)^{-2} G_5(t_5)\, dt_5 \\
G_5(t) &= \int_1^t (1-t_6)^5\, x^{(6)}(1-t_6)\, dt_6.
\end{aligned} \tag{15}$$

Proof. We note that $x(1-t)$ satisfies the conditions

$$x(1) = x'(1) = x''(1) = x'''(1) = x(0) = x'(0) = 0. \tag{16}$$

Let $\psi(t)$ be the right side of (14), then it follows that

$$\psi'(t) = [(1-t)^4 - 4(1-t)^3 t]\, G(t) + \frac{(1-t)^4}{t} G_1(t) \tag{17}$$

$$\psi''(t) = [-8(1-t)^3 + 12(1-t)^2 t]\, G(t) - \frac{8(1-t)^3}{t} G_1(t) + (1-t)\, G_2(t) \tag{18}$$

$$\psi'''(t) = [36(1-t)^2 - 24(1-t)\, t]\, G(t) + \frac{36(1-t)^2}{t} G_1(t) - 9G_2(t) + \frac{1}{(1-t)} G_3(t) \tag{19}$$

$$\psi''''(t) = [-96(1-t) + 24t]\, G(t) - \frac{96(1-t)}{t} G_1(t) + \frac{36}{(1-t)} G_2(t) - \frac{8}{(1-t)^2} G_3(t) + \frac{1}{(1-t)^3} G_4(t) \tag{20}$$

$$\psi^{(5)}(t)=120G(t)+\frac{120}{t}G_1(t)-\frac{60}{(1-t)^2}G_2(t)+\frac{20}{(1-t)^3}G_3(t)$$
$$-\frac{5}{(1-t)^4}G_4(t)+\frac{1}{(1-t)^5}G_5(t) \tag{21}$$

$$\psi^{(6)}(t)=\frac{120}{t^2}G_1(t)-\frac{120}{t^2}G_1(t)+\frac{120}{(1-t)^2}G_2(t)-\frac{120}{(1-t)^2}G_2(t)$$
$$-\frac{60}{(1-t)^4}G_3(t)+\frac{60}{(1-t)^4}G_3(t)+\frac{20}{(1-t)^5}G_4(t)$$
$$-\frac{20}{(1-t)^5}G_4(t)-\frac{5}{(1-t)^6}G_5(t)$$
$$+\frac{5}{(1-t)^6}G_5(t)+x^{(6)}(1-t),$$

i.e., $\psi^{(6)}(t)=x^{(6)}(1-t)$, and since $\psi(t)$ obviously satisfies (16), we find that $x(1-t)=\psi(t)$.

LEMMA 3. *Let $x(t)$ be as in Lemma 1, then*

$$|x(t)|\leqslant\frac{1}{32805}M \tag{22}$$

$$|x'(t)|\leqslant\left(\frac{25+34\sqrt{10}}{911250}\right)M \tag{23}$$

$$|x''(t)|\leqslant\frac{1}{360}M \tag{24}$$

$$|x'''(t)|\leqslant\frac{1}{30}M \tag{25}$$

$$|x''''(t)|\leqslant\frac{1}{5}M \tag{26}$$

$$|x^{(5)}(t)|\leqslant\frac{2}{3}M, \tag{27}$$

where $M=\max_{0\leqslant t\leqslant 1}|x^{(6)}(t)|$.

Proof. From (8), it is immediate that

$$\begin{aligned} &|F_5(t)|\leqslant\tfrac{1}{6}(1-t)^6M, && |F_4(t)|\leqslant\tfrac{1}{30}(1-t)^5M, && |F_3(t)|\leqslant\tfrac{1}{120}t^4M,\\ &|F_2(t)|\leqslant\tfrac{1}{360}t^3M, && |F_1(t)|\leqslant\tfrac{1}{720}t^2M, && |F(t)|\leqslant\tfrac{1}{720}tM. \end{aligned} \tag{28}$$

Thus, from (7) it follows that

$$|x(t)| \leqslant \tfrac{1}{720} t^4 (1-t)^2 M. \tag{29}$$

In (29) the right side attains its maximum at $t = \frac{2}{3}$, and we get

$$|x(t)| \leqslant \tfrac{1}{32805} M.$$

This completes the proof of (22).

To prove (23), from (15) we have

$$|G_5(t)| \leqslant \tfrac{1}{6}(1-t)^6 M, \qquad |G_4(t)| \leqslant \tfrac{1}{30}(1-t)^5 M, \qquad |G_3(t)| \leqslant \tfrac{1}{120}(1-t)^4 M,$$
$$|G_2(t)| \leqslant \tfrac{1}{360}(1-t)^3 M, \qquad |G_1(t)| \leqslant \tfrac{1}{720} t^2 M, \qquad |G(t)| \leqslant \tfrac{1}{720} t M. \tag{30}$$

Further, from (17) and (15) successively we get

$$\begin{aligned} -x'(1-t) &= [(1-t)^4 - 4(1-t)^3 t]\, G(t) + \frac{(1-t)^4}{t} G_1(t) \\ &= [(1-t)^4 - 4(1-t)^3 t] \\ &\quad \times \left[-\frac{1}{t_1} G_1(t_1)|_0^t + \int_0^t \frac{1}{t_1} G_1'(t_1)\, dt_1 \right] + \frac{(1-t)^4}{t} G_1(t) \\ &= 4(1-t)^3 \int_0^t \frac{t_1}{(1-t_1)^3} G_2(t_1)\, dt_1 \\ &\quad + [(1-t)^4 - 4(1-t)^3 t] \int_0^t \frac{1}{(1-t_1)^3} G_2(t_1)\, dt_1. \end{aligned} \tag{31}$$

Thus, from (30) it follows that

$$\begin{aligned} |x'(1-t)| &\leqslant \frac{M(1-t)^3}{360} \int_0^t \left| \frac{4t_1}{(1-t_1)^3} + \frac{(1-5t)}{(1-t_1)^3} \right| (1-t_1)^3\, dt_1 \\ &= \frac{M(1-t)^3}{360} \int_0^t |4t_1 + 1 - 5t|\, dt_1 \\ &= \frac{M(1-t)^3}{360} \begin{cases} t(1-3t), & 0 \leqslant t \leqslant \frac{1}{5} \\ \frac{1}{4}(1-6t+13t^2), & \frac{1}{5} \leqslant t \leqslant 1. \end{cases} \end{aligned} \tag{32}$$

In (32) the right side attains its maximum at $t = (5-\sqrt{10})/15$, and we get

$$|x'(1-t)| \leqslant \left(\frac{25 + 34\sqrt{10}}{911250} \right) M$$

which is same as (23).

Next, we shall prove (24). For this, from (31) we have

$$x''(1-t) = (1-t)\,G_2(t) + \int_0^t \frac{[-12(1-t)^2 t_1 - 8(1-t)^3 + 12(1-t)^2 t]}{(1-t_1)^3}\, G_2(t_1)\, dt_1$$

and hence (30) gives that

$$\begin{aligned} |x''(1-t)| &\leqslant \frac{(1-t)^4 M}{360} \\ &\quad + \int_0^t \frac{|-12(1-t)^2 t_1 - 8(1-t)^3 + 12(1-t)^2 t|}{(1-t_1)^3}\, \frac{M(1-t_1)^3}{360}\, dt_1 \\ &= \frac{M(1-t)^2}{360}\left[(1-t)^2 + \int_0^t |-12t_1 - 8 + 20t|\, dt_1\right] \\ &= \frac{M(1-t)^2}{360}\left[(1-t)^2 + \frac{2}{3}(29t^2 - 28t + 8)\right], \qquad \frac{2}{5} \leqslant t \leqslant 1 \\ &= \frac{M}{1080}(1-t)^2(61t^2 - 62t + 19), \qquad \frac{2}{5} \leqslant t \leqslant 1 \\ &\leqslant \frac{M}{360}\left(\frac{297}{625}\right), \qquad \frac{2}{5} \leqslant t \leqslant 1 \\ &< \frac{M}{360}, \qquad \frac{2}{5} \leqslant t \leqslant 1. \end{aligned}$$

Thus, we find that

$$|x''(t)| \leqslant \frac{M}{360}, \qquad 0 \leqslant t \leqslant \frac{3}{5}. \tag{33}$$

Also, from (10) and (28) we have

$$\begin{aligned} |x''(t)| &\leqslant |6t(1-t)^2 - 12t^2(1-t) + 2t^3|\, \frac{1}{720}\, tM \\ &\quad + |4(1-t)^2 - 4t(1-t)|\, \frac{1}{720}\, t^2 M + \frac{(1-t)^2}{t} \times \frac{1}{360}\, t^3 M \\ &= \frac{Mt^2}{360}\left[|10t^2 - 12t + 3| + (1-t)|2-4t| + (1-t)^2\right] \end{aligned}$$

$$= \frac{Mt^2}{360}\begin{cases}(-13t^2+16t-4), & \frac{3}{5} \leqslant t \leqslant \frac{6+\sqrt{6}}{10} \\ (7t^2-8t+2), & \frac{6+\sqrt{6}}{10} \leqslant t \leqslant 1\end{cases}$$

$$\leqslant \frac{M}{360}, \qquad \frac{3}{5} \leqslant t \leqslant 1. \tag{34}$$

Combining (33) and (34), we get (24).

To prove (25), from (19) and (30), we have

$$|x'''(1-t)| \leqslant |36(1-t)^2 - 24(1-t)\,t| \frac{1}{720} tM + \frac{36(1-t)^2}{t} \frac{1}{720} t^2 M$$

$$+ 9 \frac{1}{360} (1-t)^3 M + \frac{1}{(1-t)} \frac{1}{120} (1-t)^4 M$$

$$= M\left[\frac{1}{30}(1-t)^3 + \frac{1}{20} t(1-t)^2 + \frac{1}{60} t(1-t)|3(1-t)-2t|\right]$$

$$= \frac{M}{30}\begin{cases}(1-t)^3 + 3t(1-t)^2 - t^2(1-t), & 0 \leqslant t \leqslant \frac{3}{5} \\ (1-t)^3 + t^2(1-t), & \frac{3}{5} \leqslant t \leqslant 1\end{cases}$$

$$\leqslant \frac{M}{30}, \qquad 0 \leqslant t \leqslant 1.$$

Thus, we have

$$|x'''(t)| \leqslant \frac{M}{30}, \qquad 0 \leqslant t \leqslant 1.$$

Now we shall prove (26). For this, from (12) and (28) we have

$$|x''''(t)| \leqslant |72t - 48(1-t)| \frac{1}{720} tM + \left|48 - \frac{48(1-t)}{t}\right| \frac{1}{720} t^2 M$$

$$+ \left|\frac{12}{t} - \frac{24(1-t)}{t^2}\right| \frac{1}{360} t^3 M + \frac{8(1-t)}{t^3} \frac{1}{120} t^4 M$$

$$+ \frac{1}{(1-t)^3} \frac{1}{30} (1-t)^5 M$$

$$= \frac{M}{30}\left[|5t-2|\,t + |4t-2|\,t + |3t-2|\,t + (1-t^2)\right]$$

$$=\frac{M}{30}\begin{cases}1+6t-13t^2, & 0\leqslant t\leqslant\frac{2}{5}\\ 1+2t-3t^2, & \frac{2}{5}\leqslant t\leqslant\frac{1}{2}\\ 1-6t+11t^2, & \frac{1}{2}\leqslant t\leqslant 1.\end{cases} \tag{35}$$

In (35) the right side attains its maximum at $t=1$, and we get

$$|x''''(t)|\leqslant\frac{M}{5}, \qquad 0\leqslant t\leqslant 1.$$

Finally, to prove (27) from (13) and (28), we have

$$\begin{aligned}|x^{(5)}(t)| &\leqslant 120\frac{1}{720}tM+\frac{120}{t}\frac{1}{720}t^2M\\ &\quad+\frac{60}{t^2}\frac{1}{360}t^3M+\frac{20}{t^3}\frac{1}{120}t^4M\\ &\quad+\frac{5}{(1-t)^4}\frac{1}{30}(1-t)^5M+\frac{1}{(1-t)^5}\frac{1}{6}(1-t)^6M\\ &=\frac{2}{3}tM+\frac{2}{3}(1-t)M\\ &=\frac{2}{3}M.\end{aligned}$$

LEMMA 4. *Let $x\in C^{(6)}[a,b]$, and satisfy the conditions* (2). *Then,*

$$\begin{aligned} &|x(t)|\leqslant\frac{1}{32805}(b-a)^6\mu, && |x'(t)|\leqslant\left(\frac{25+34\sqrt{10}}{911250}\right)(b-a)^5\mu,\\ &|x''(t)|\leqslant\frac{1}{360}(b-a)^4\mu, && |x'''(t)|\leqslant\frac{1}{30}(b-a)^3\mu,\\ &|x''''(t)|\leqslant\frac{1}{5}(b-a)^2\mu, && |x^{(5)}(t)|\leqslant\frac{2}{3}(b-a)\mu,\end{aligned} \tag{36}$$

where $\mu=\max_{a\leqslant t\leqslant b}|x^{(6)}(t)|$.

Proof. The proof requires only the transformation $u=a+(b-a)t$, $0\leqslant t\leqslant 1$ in Lemma 3.

Remark. In (36) the inequalities are the best possible as the identity holds for the functions $x_1(t)=(t-a)^4\,(b-t)^2$ and $x_2(t)=(t-a)^2\,(b-t)^4$, and only for these functions up to a constant factor. We also note that these inequalities are of immense value in polynomial interpolation theory, and the problem of computing optimal error bounds for the different problems mainly using Green's function technique has been considered in [1, 2, 4].

Proof of the Theorem. Suppose on the contrary that the boundary value problem (4), (2) has a nontrivial solution $x(t)$. Then, $\mu=\max_{a\leqslant t\leqslant b}|x^{(6)}(t)|\neq 0$. Since, otherwise $x(t)$ would coincide with a polynomial of degree $m<6$ on $[a,b]$ and $x^{(m)}(t)$ would not vanish on $[a,b]$ which contradicts the assumptions that $x(t)$ satisfies (2). Thus, if $\mu=|x^{(6)}(t_1)|$, then from the differential equation (4) we have

$$\mu=\left|\sum_{i=0}^{5} p_i(t_1)\,x^{(i)}(t_1)\right|\leqslant\sum_{i=0}^{5}\|p_i\|\,|x^{(i)}(t_1)|. \tag{37}$$

Now using Lemma 4 in the above inequality, we get

$$\mu\leqslant\theta\mu$$

and hence, it is necessary that

$$\theta\geqslant 1. \tag{38}$$

To exclude the posibility of equality in (38), we note that at least one of the numbers $\|p_i\|$, $0\leqslant i\leqslant 5$ is different from zero, otherwise again $x(t)$ would be a polynomial of degree less than 6 and cannot satisfy the boundary conditions (2). Thus, if in (38) equality holds then equality must hold in (36) for at least one place. However, from the remark, this is possible only if $x(t)$ is a polynomial of degree 6. Thus, equality in (37) holds for any point t_1 in $[a,b]$. But $|x^{(i)}(t_1)|$ is not constant on $[a,b]$ for any $0\leqslant i\leqslant 5$ ensures the strict inequality in (38). This completes the proof of our theorem.

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