Mathematical and Computer Modelling 54 (2011) 152–159

Contents lists available at ScienceDirect

Mathematical and Computer Modelling

journal homepage: www.elsevier.com/locate/mcm

Statistical analysis of the square ratio of two multivariate exponentially correlated α–μ distributions and its application in telecommunications[☆]

Gradimir V. Milovanović[a,*], Mihajlo Č. Stefanović[b], Stefan R. Panić[b], Jelena A. Anastasov[b], Dragana S. Krstić[b]

[a] Faculty of Computer Sciences, Megatrend University, Bulevar umetnosti 29, 11070 Novi Beograd, Serbia

[b] Department of Telecommunications, Faculty of Electronic Engineering, University of Niš, Aleksandra Medvedeva 14, 18000 Niš, Serbia

ARTICLE INFO

Article history:
Received 18 October 2010
Received in revised form 26 January 2011
Accepted 26 January 2011

Keywords:
Multivariate distribution
Square ratios of exponentially correlated variables
Maximal and minimal square ratios
SIR based SC diversity

ABSTRACT

In this paper the exact probability density function of a multivariate α–μ distributed variables with exponentially correlated random variables is derived. Capitalizing on this the joint probability density function (JPDF) is derived for the square ratios of two multivariate exponentially correlated α–μ distributed variables. Closed form expressions are determined for the cumulative distribution function (CDF) and probability density function (PDF) of the maximal and minimal square ratio of two multivariate exponentially correlated α–μ distributions. Using these new formulae, SIR (signal-to-interference) based analysis of selection combining (SC) receiver through standard communication system performance measures can be performed.

1. Introduction

Random variables are of central importance, not only for the statistical but also to the deterministic modeling of mobile radio channels. The multi-path propagation in wireless communications is modeled by several distributions including Weibull, Nakagami-*m*, Hoyt, Rayleigh and Rice. By considering two important phenomena inherent to radio propagation, namely non-linearity and clustering, the α–μ fading model was recently proposed in [1]. The α–μ distribution fading model considers a signal composed of clusters of multipath waves propagating in a non-homogeneous environment, with resulting envelope obtained as a nonlinear function of the modulus of the sum of the multi-path components. This model provides a very good fit to measured data over a wide range of channels conditions. α–μ distribution has the same functional form as the generalized gamma or Stacy distribution [2]. The main advantage of this distribution is that α–μ statistical model includes as special cases, other important distributions such as Weibull and Nakagami-*m* (therefore the one-sided Gaussian and Rayleigh are also special cases of it). Multivariate (correlated) distribution analysis is an important tool in the performance investigation of many receiver structures for multi-path correlative fading channels. The performance analysis of receiving systems concerning to α–μ statistical model of channels is rather scarce in the literature [3–7].

Moreover to the best author's knowledge, no analytical study involving assumed exponentially correlated α–μ distributed variables has been reported in the literature.

2. Joint probability density function of the multivariate exponentially correlated α–μ distributed variables

There is a need to derive the joint statistics for multiple α–μ variables. We are relaying on some results which are already available in the literature for the exponential correlation model of Nakagami-*m* distribution [8].

☆ The authors were supported in part by the Serbian Ministry of Science and Technological Development.

* Corresponding author.
E-mail addresses: gvm@megatrend.edu.rs (G.V. Milovanović), mihajlo.stefanovic@elfak.ni.ac.rs (M.Č. Stefanović), stefanpnc@yahoo.com (S.R. Panić), anastasovjelena@gmail.com (J.A. Anastasov), dragana.krstic@elfak.ni.ac.rs (D.S. Krstić).

0895-7177/$ – see front matter © 2011 Elsevier Ltd. All rights reserved.
doi:10.1016/j.mcm.2011.01.046

Suppose that

(1) $R_{N_1}, \ldots, R_{N_n}$ be n Nakagami-m variables whose marginal statistics are respectively described by the parameters $m_1 = \cdots = m_n = m$;
(2) $R_1, \ldots, R_n$ be $n\alpha$–μ variables whose marginal statistics are respectively described by the parameters $\mu_1 = \cdots = \mu_n = \mu_d, \alpha_1, \ldots, \alpha_n$;
(3) $0 \leq \rho_{\text{Nakagami}-m} \leq 1$ be a Nakagami-m correlation coefficient; and
(4) $\rho_{\alpha-\mu}, 0 \leq \rho_{\alpha-\mu} \leq 1$ be an α–μ correlation coefficient.

The joint probability density function $p_{R_{N_1},\ldots,R_{N_n}}(R_{N_1}, \ldots, R_{N_n})$ of n Nakagami-m variates, $R_{N_1}, \ldots, R_{N_n}$ with marginal statistics just described, is given by [8, Eq. (10)]. We are considering the exponential correlation model of α–μ

distribution. The exponential correlation model can be obtained from [9] by setting $\Sigma_{ij} = 1$ for $i = j$ and $\Sigma_{ij} = \rho_{\text{Nakagami}-m}^{|i-j|}$ for $i \neq j$ in Nakagami-m correlation matrix, where $\rho_{\text{Nakagami}-m}$ is the correlation coefficient. The correlation coefficient $\rho_{\text{Nakagami}-m}$ is defined as

$$\rho_{\text{Nakagami}-m} = \frac{C(R_i^2, R_j^2)}{\sqrt{V(R_i^2)V(R_j^2)}},$$

where $C(.,.)$ denotes the covariance operator. We use the relation between α–μ and the Nakagami-m envelopes ($R_{\alpha-\mu}^{\alpha} = R_{\text{Nakagami}-m}^2$) (cf. [8, Eq. (19)]), so that, $R_1^{\alpha_1} = R_{N_1}^2, \ldots, R_n^{\alpha_n} = R_{N_n}^2$, we find that $\mu = m$. Also, the relation between the correlation coefficient of α–μ distribution and Nakagami-m distribution have been already derived [8, Eq. (31)]. For two Nakagami-m and α–μ variates, it can be seen that $\rho_{\alpha-\mu} = \rho_{\text{Nakagami}-m}\sqrt{\mu_1/\mu_2}$. Now, because in this case we have $\mu_1 = \mu_2 = \mu_d$, from (1), then the correlation coefficient of α–μ distribution is equal to the correlation coefficient of Nakagami-m distribution, i.e., $\rho_{\alpha-\mu} = \rho_{\text{Nakagami}-m}$. Finally, considering all proposed relations and [8, Eq. (10)], the joint probability density $p_{R_{N_1},\ldots,R_{N_n}}(R_{N_1}, \ldots, R_{N_n})$ function of $n\alpha$–μ variates $R_1, \ldots, R_n$ is found as in [8]

$$p_{R_1,\ldots,R_n}(R_1, \ldots, R_n) = |J| p_{R_{N_1},\ldots,R_{N_n}}(R_{N_1}, \ldots, R_{N_n})$$

with J as the Jacobian of the transformation given by

$$|J| = \begin{vmatrix} \frac{\partial R_{N_1}}{\partial R_1} & \frac{\partial R_{N_1}}{\partial R_2} & \cdots & \frac{\partial R_{N_1}}{\partial R_n} \\ \frac{\partial R_{N_2}}{\partial R_1} & \frac{\partial R_{N_2}}{\partial R_2} & \cdots & \frac{\partial R_{N_2}}{\partial R_n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial R_{N_n}}{\partial R_1} & \frac{\partial R_{N_n}}{\partial R_2} & \cdots & \frac{\partial R_{N_n}}{\partial R_n} \end{vmatrix} = \frac{\alpha_1\alpha_2\cdots\alpha_n}{2^n} R_1^{\frac{\alpha_1}{2}-1} R_2^{\frac{\alpha_2}{2}-1} \cdots R_n^{\frac{\alpha_n}{2}-1}.$$

After the standard statistical procedure of transformation of variates and after some mathematical manipulations and simplifications, JPDF can be respectively expressed as

$$\begin{aligned} p_{R_1,\ldots,R_n}(R_1, \ldots, R_n) = &\sum_{k_1,\ldots,k_{n-1}=0}^{\infty} \frac{\alpha_1\cdots\alpha_n \rho^{2(k_1+\cdots+k_{n-1})}}{2^{n\mu_d+2(k_1+\cdots+k_{n-1})}\Gamma(\mu_d)\Gamma(\mu_d+k_1)\cdots\Gamma(\mu_d+k_{n-1})} \\ &\times \frac{R_1^{\alpha_1(\mu_d+k_1)-1} R_n^{\alpha_n(\mu_d+k_{n-1})-1} g_2}{(1-\rho^2)^{(n-1)\mu_d+2(k_1+\cdots+k_{n-1})} k_1!\cdots k_{n-1}!} \exp\left(-\frac{R_1^{\alpha_1}+R_n^{\alpha_n}}{2(1-\rho^2)} - g_1\right), \end{aligned} \tag{2.1}$$

where

$$g_1 = \begin{Bmatrix} 0, & n=2 \\ \dfrac{(\rho^2+1)}{2(1-\rho^2)} \displaystyle\sum_{i=2}^{n-1} R_i^{\alpha_i} & n>2 \end{Bmatrix} \quad \text{and} \quad g_2 = \begin{Bmatrix} 1, & n=2 \\ \displaystyle\prod_{i=2}^{n-1} R_i^{\alpha_i(\mu_d+k_{i-1}+k_i)-1}, & n>2 \end{Bmatrix}.$$

3. Joint probability density function of the square ratio of two multivariate exponentially correlated α–μ distributed variables

Let us define the k-th square ratio of α–μ variates from the exponentially correlated multivariate distributed variables as $\lambda_k = R_k^2/r_k^2$, $k = 1, \ldots, N$. Similarly as (2.1), the JPDF for the denominator, can be presented in the form of

$$\begin{aligned} p_{r_1,\ldots,r_n}(r_1, \ldots, r_n) = &\sum_{\ell_1,\ldots,\ell_{n-1}=0}^{\infty} \frac{\alpha_1\cdots\alpha_n \rho^{2(\ell_1+\cdots+\ell_{n-1})}}{2^{n\mu_c+2(\ell_1+\cdots+\ell_{n-1})}\Gamma(\mu_c)\Gamma(\mu_c+\ell_1)\cdots\Gamma(\mu_c+\ell_{n-1})} \\ &\times \frac{r_1^{\alpha_1(\mu_c+\ell_1)-1} r_n^{\alpha_n(\mu_c+\ell_{n-1})-1} g_4}{(1-\rho^2)^{(n-1)\mu_c+2(\ell_1+\cdots+\ell_{n-1})} \ell_1!\cdots\ell_{n-1}!} \exp\left(-\frac{r_1^{\alpha_1}+r_n^{\alpha_n}}{2(1-\rho^2)} - g_3\right), \end{aligned} \tag{3.1}$$

with

$$g_3=\left\{\begin{matrix}0, & n=2\\ \dfrac{(\rho^2+1)}{2(1-\rho^2)}\displaystyle\sum_{i=2}^{n-1}r_i^{\alpha_i}, & n>2\end{matrix}\right\}\quad\text{and}\quad g_4=\left\{\begin{matrix}1, & n=2\\ \displaystyle\prod_{i=2}^{n-1}r_i^{\alpha_i(\mu_c+\ell_{i-1}+\ell_i)-1}, & n>2\end{matrix}\right\},$$

where μ_c is the distribution parameter.

The JPDF of the square ratios, $\lambda_k,\ k=1,2,\ldots,n$, can be given by (cf. [8])

$$p_{\lambda_1,\ldots,\lambda_n}(t_1,\ldots,t_n)=\frac{2^{-n}}{\sqrt{t_1\cdots t_n}}\int_0^\infty\int_0^\infty\cdots\int_0^\infty p_{R_1,\ldots,R_n}(r_1\sqrt{t_1},\ldots,r_n\sqrt{t_n})p_{r_1,\ldots,r_n}(r_1,\ldots,r_n)\times r_1r_2\cdots r_n\mathrm{d}r_1\mathrm{d}r_2\cdots\mathrm{d}r_n. \tag{3.2}$$

Substituting (2.1) and (3.1) in (3.2), and using the fact that

$$\int_0^\infty x^{pq-1}\exp(-sx^p)\mathrm{d}x=\frac{\Gamma(q)}{ps^q}\qquad(p,q,s>0),$$

$p_{\lambda_1,\ldots,\lambda_n}(t_1,\ldots,t_n)$ can be expressed in the form

$$\begin{aligned}p_{\lambda_1,\ldots,\lambda_n}(t_1,\ldots,t_n)=&\sum_{k_1,\ldots,k_{n-1}=0}^{\infty}\sum_{\ell_1,\ldots,\ell_{n-1}=0}^{\infty}\frac{\alpha_1\ldots\alpha_n\Gamma(\mu_d+\mu_c+k_1+\ell_1)}{2^n\Gamma(\mu_d)\Gamma(\mu_c)\prod_{i=1}^{n-1}\Gamma(\mu_d+k_i)\Gamma(\mu_c+\ell_i)k_i!\ell_i}\\&\times\Gamma(\mu_d+\mu_c+k_{n-1}+\ell_{n-1})\rho^{2(k_1+\cdots+k_{n-1}+\ell_1+\cdots+\ell_{n-1})}g_5\\&\times(1-\rho^2)^{(n+1)\mu_d+(n+1)\mu_c+2(k_1+\cdots+k_{n-1}+\ell_1+\cdots+\ell_{n-1})}\\&\times\frac{t_1^{\alpha_1(\mu_d+k_1)/2-1}}{\left((1-\rho^2)(t_1^{\alpha_1/2}+1)\right)^{(\mu_c+\mu_d+k_1+\ell_1)}}\frac{t_n^{\alpha_n(\mu_d+k_{n-1})/2-1}}{\left((1-\rho^2)(t_n^{\alpha_n/2}+1)\right)^{(\mu_c+\mu_d+k_{n-1}+\ell_{n-1})}}g_6,\end{aligned}\tag{3.3}$$

where

$$g_5=\left\{\begin{matrix}1, & n=2\\ \displaystyle\prod_{i=2}^{n-1}\Gamma(\mu_d+\mu_c+k_{i-1}+\ell_{i-1}+k_i+\ell_i), & n>2\end{matrix}\right\}$$

and

$$g_6=\left\{\begin{matrix}1, & n=2\\ \displaystyle\prod_{i=2}^{n-1}\frac{t_i^{\alpha_i(\mu_d+k_{i-1}+k_i)/2-1}}{\left((1-\rho^4)(t_i^{\alpha_i/2}+1)\right)^{(\mu_c+\mu_d+k_{i-1}+\ell_{i-1}+k_i+\ell_i)}}, & n>2\end{matrix}\right\}.$$

4. First order statistic of the maximal and minimal square ratio of two multivariate exponentially correlated α–μ distributed variables

In this section, we derive the closed-form expressions for the first order statistics of the maximal and minimal square ratio of two multivariate exponentially correlated distributions. First, we determine the maximal square ratio

$$\lambda=\lambda_{\max}=\max(\lambda_1,\lambda_2,\ldots,\lambda_n).$$

For this case, the cumulative distribution function can be written as (cf. [10])

$$F_\lambda(t)=\int_0^t\int_0^t\cdots\int_0^t p_{\lambda_1,\lambda_2,\ldots,\lambda_n}(x_1,x_2,\ldots,x_n)\mathrm{d}x_1\mathrm{d}x_2\cdots\mathrm{d}x_n. \tag{4.1}$$

Substituting expression (3.3) in (4.1) and after n successive integrations, the CDF becomes

$$\begin{aligned}F_\lambda(t)=&\sum_{k_1,\ldots,k_{n-1}=0}^{\infty}\sum_{\ell_1,\ldots,\ell_{n-1}=0}^{\infty}\frac{\Gamma(\mu_d+\mu_c+k_1+\ell_1)\Gamma(\mu_d+\mu_c+k_{n-1}+\ell_{n-1})}{\Gamma(\mu_d)\Gamma(\mu_c)\prod_{i=1}^{n-1}\Gamma(\mu_d+k_i)\Gamma(\mu_c+\ell_i)k_i!\ell_i!}\\&\times\rho^{2(k_1+\cdots+k_{n-1}+\ell_1+\cdots+\ell_{n-1})}(1-\rho^2)^{\mu_d+\mu_c}\\&\times B\left(\frac{t^{\alpha_1/2}}{t^{\alpha_1/2}+1},\mu_d+k_1,\mu_c+\ell_1\right)B\left(\frac{t^{\alpha_n/2}}{t^{\alpha_n/2}+1},\mu_d+k_{n-1},\mu_c+\ell_{n-1}\right)\frac{g_5g_9}{g_8},\end{aligned}\tag{4.2}$$

Table 4.1
Values of the difference $E_N(t) - E_{N+1}(t)$ for the case of the square ratio of two three-variate exponentially correlated α–μ distributed variables, with parameters $\alpha_1 = 1.8, \alpha_2 = 2, \alpha_3 = 2.2, \mu_d = 1.2, \mu_c = 3$, and $\rho = 0.2$.

N	$t = 0.01$	$t = 0.1$	$t = 1$	$t = 10$	$t = 100$
3	4.48(−4)	3.20(−4)	9.49(−5)	7.50(−5)	7.51(−5)
4	3.45(−5)	2.27(−5)	5.38(−6)	4.10(−6)	4.10(−6)
5	2.44(−6)	1.47(−6)	2.89(−7)	2.15(−7)	2.15(−7)
6	1.62(−7)	9.00(−8)	1.50(−8)	1.09(−8)	1.09(−8)
7	1.02(−8)	5.22(−9)	7.52(−10)	5.41(−10)	5.43(−10)
8	6.12(−10)	2.91(−10)	3.69(−11)	2.63(−11)	2.64(−12)
9	3.56(−11)	1.57(−11)	1.78(−12)	1.26(−12)	1.26(−12)

Table 4.2
Number of terms in the summation (4.2) so that $|F_\lambda^N(t) - F_\lambda(t)| < 10^{-6}$, for the case of square ratio of two three-variate exponentially correlated α–μ distributed variables with parameters $\alpha_1 = 1.8, \alpha_2 = 2, \alpha_3 = 2.2, \mu_d = 1.2, \mu_c = 3$.

ρ	$t = 0.1$	$t = 1$	$t = 10$
0.2	4	5	5
0.3	6	7	7
0.4	8	9	10
0.5	11	13	13
0.6	16	18	18

where

$$g_8 = \left\{\begin{array}{ll} 1, & n = 2 \\ (1+\rho^2)^{\mu_d+\mu_c+k_1+k_2+\ell_1+\ell_2}, & n = 3 \\ (1+\rho^2)^{(n-2)\mu_d+(n-2)\mu_c+k_1+\ell_1+2\sum_{i=2}^{n-2} k_i+k_{n-1}+\ell_i+\ell_{n-1}}, & n > 3 \end{array}\right\},$$

$$g_9 = \left\{\begin{array}{ll} 1, & n = 2 \\ \prod_{i=2}^{n-1} B\left(\frac{t^{\alpha_i/2}}{t^{\alpha_i/2}+1}, \mu_d + k_{i-1} + k_i, \mu_c + \ell_{i-1} + \ell_i\right), & n > 2 \end{array}\right\},$$

and $B(z, a, b)$ is the incomplete Beta function (see [11, Eq. (8.39)]).

By $F_\lambda^N(t)$ we denote the corresponding finite series of (4.2), so that all indices run from zero to N, i.e.,

$$F_\lambda^N(t) = \sum_{k_1,\ldots,k_{n-1}=0}^{N} \sum_{\ell_1,\ldots,\ell_{n-1}=0}^{N} \cdots .$$

Let $E_N(t)$ denote the corresponding relative error $E_N(t) = (F_\lambda(t) - F_\lambda^N(t))/F_\lambda(t)$. It is clear that all terms of this series are nonnegative, so that

$$E_N(t) \ge E_{N+1}(t) \ge 0.$$

Numerical examples show a quite satisfactory convergence $F_\lambda^N(t) \to F_\lambda(t)$ as $N \to \infty$ for $t > 0$. To illustrate the speed of convergence we consider the behavior of the difference $E_N(t) - E_{N+1}(t)$, for sufficiently large N, supposing that $F_\lambda(t) \approx F_\lambda^{N+1}(t)$. Thus,

$$E_N(t) - E_{N+1}(t) = \frac{F_\lambda^{N+1}(t) - F_\lambda^N(t)}{F_\lambda(t)} \approx \frac{F_\lambda^{N+1}(t) - F_\lambda^N(t)}{F_\lambda^{N+1}(t)}.$$

These values, for $3 \le N \le 9$ and some selected values of t, are displayed in Table 4.1. Numbers in parentheses indicate decimal exponents.

In Table 4.2 we present the number of terms N in the finite sum $F_\lambda^N(t)$ in order to achieve the accuracy $|F_\lambda^N(t) - F_\lambda(t)| < 10^{-6}$. It is obvious that the number of terms increases as the correlation coefficient increases.

The PDF can be obtained easily from the previous expression. Namely, we get

$$p_\lambda(t) = \frac{\mathrm{d}}{\mathrm{d}t} F_\lambda(t) = \sum_{k_1,\ldots,k_n=0}^{\infty} \sum_{\ell_1,\ldots,\ell_n=0}^{\infty} \frac{1}{2t} \frac{\Gamma(\mu_d+\mu_c+k_1+\ell_1)}{\Gamma(\mu_d)\Gamma(\mu_c)\prod_{i=1}^{n-1}\Gamma(\mu_d+k_i)\Gamma(\mu_c+\ell_i)k_i!\ell_i!}$$
$$\times \Gamma(\mu_d+\mu_c+k_{n-1}+\ell_{n-1})\rho^{2(k_1+\cdots+k_{n-1}+\ell_1+\cdots+\ell_{n-1})}(1-\rho^2)^{\mu_d+\mu_c}\frac{g_5}{g_8}\left(\sum_{i=1}^{n} A_i(t)\right),$$

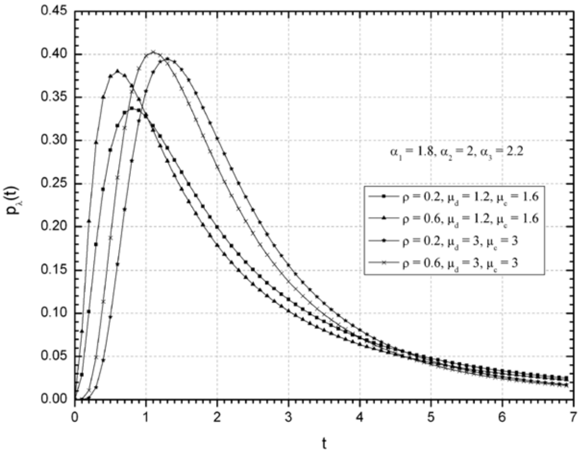

Fig. 4.1. PDF of the maximal square ratio of two three-variate exponentially correlated α–μ distribution for various values of correlation coefficients and distribution parameters.

where

$$
\begin{aligned}
A_1(t) &= \alpha_1 \frac{t^{\alpha_1/2(\mu_d+k_1)}}{(t^{\alpha_1/2}+1)^{(\mu_c+\ell_1+\mu_d+k_1)}} B\left(\frac{t^{\alpha_2/2}}{t^{\alpha_2/2}+1}, \mu_d+k_1+k_2, \mu_c+\ell_1+\ell_2\right) \\
&\quad \times \cdots \times B\left(\frac{t^{\alpha_n/2}}{t^{\alpha_n/2}+1}, \mu_d+k_{n-1}, \mu_c+\ell_{n-1}\right), \\
A_i(t) &= \alpha_i \frac{t^{\alpha_i/2(\mu_d+k_{i-1}+k_i)}}{(t^{\alpha_1/2}+1)^{(\mu_c+\ell_{i-1}+\ell_i+\mu_d+k_{i-1}+k_i)}} B\left(\frac{t^{\alpha_1/2}}{t^{\alpha_1/2}+1}, \mu_d+k_1, \mu_c+\ell_1\right) B\left(\frac{t^{\alpha_n/2}}{t^{\alpha_n/2}+1}, \mu_d+k_{n-1}, \mu_c+\ell_{n-1}\right) \\
&\quad \times \prod_{j=2,\ldots,n-1; j\neq i} B\left(\frac{t^{\alpha_i/2}}{t^{\alpha_i/2}+1}, \mu_d+k_{i-2}+k_{i-1}, \mu_c+\ell_{i-2}+\ell_{i-1}\right)
\end{aligned}
$$

for $j = 2, \ldots, n-1$, and

$$
\begin{aligned}
A_n(t) &= \alpha_n \frac{t^{\alpha_n/2(\mu_d+k_{n-1})}}{(t^{\alpha_n/2}+1)^{(\mu_c+\ell_{n-1}+\mu_d+k_{n-1})}} B\left(\frac{t^{\alpha_1/2}}{t^{\alpha_1/2}+1}, \mu_d+k_1, \mu_c+\ell_1\right) \\
&\quad \times \cdots \times B\left(\frac{t^{\alpha_{n-1}/2}}{t^{\alpha_{n-1}/2}+1}, \mu_d+k_{n-2}+k_{n-1}, \mu_c+\ell_{n-2}+\ell_{n-1}\right).
\end{aligned}
$$

Fig. 4.1 shows the pdf of the maximal square ratio of two three-variate exponentially correlated α–μ distributions for some values of correlation coefficients and distribution parameters.

Now let us determine the minimal square ratio

$$\lambda = \lambda_{\min} = \min(\lambda_1, \lambda_2 \ldots, \lambda_n).$$

For this case, the CDF can be written in the form (see [10])

$$F_\lambda(t) = 1 - \int_t^\infty \int_t^\infty \cdots \int_t^\infty p_{\lambda_1,\lambda_2,\ldots,\lambda_n}(x_1, x_2, \ldots, x_n)\mathrm{d}x_1\mathrm{d}x_2\cdots\mathrm{d}x_n. \tag{4.3}$$

Substituting expression (3.3) in (4.3) and after n successive integrations, the CDF becomes:

$$
\begin{aligned}
F_\lambda(t) &= 1 - \sum_{k_1,\ldots,k_{n-1}=0}^{\infty} \sum_{\ell_1,\ldots,\ell_{n-1}=0}^{\infty} \frac{\Gamma(\mu_d+\mu_c+k_1+\ell_1)\Gamma(\mu_d+\mu_c+k_{n-1}+\ell_{n-1})}{\Gamma(\mu_d)\Gamma(\mu_c)\prod_{i=1}^{n-1}\Gamma(\mu_d+k_i)\Gamma(\mu_c+\ell_i)k_i!\ell_i!} \\
&\quad \times \rho^{2(k_1+\cdots+k_{n-1}+\ell_1+\cdots+\ell_{n-1})}(1-\rho^2)^{\mu_d+\mu_c} \\
&\quad \times \left[B(\mu_d+k_1, \mu_c+\ell_1) - B\left(\frac{t^{\alpha_1/2}}{t^{\alpha_1/2}+1}, \mu_d+k_1, \mu_c+\ell_1\right)\right] \\
&\quad \times \left[B\left(\mu_d+k_{n-1}, \mu_c+\ell_{n-1}\right) - B\left(\frac{t^{\alpha_n/2}}{t^{\alpha_n/2}+1}, \mu_d+k_{n-1}, \mu_c+\ell_{n-1}\right)\right] \frac{g_5 g_{10}}{g_8},
\end{aligned}
$$

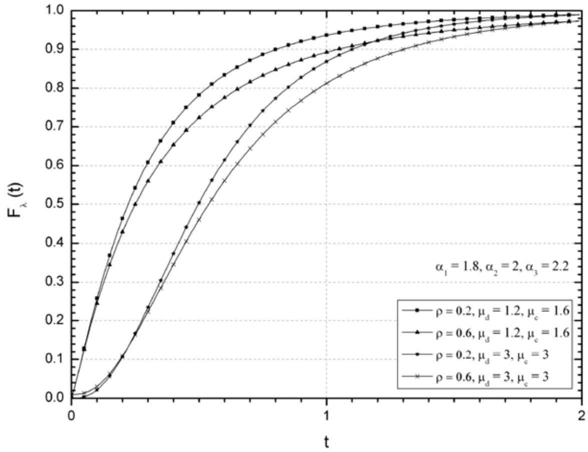

Fig. 4.2. CDF of the minimal square ratio of two three-variate exponentially correlated α–μ distribution for various values of correlation coefficients and distribution parameters.

where

$$g_{10} = \left\{ \begin{matrix} 1, & n = 2 \\ \prod_{i=2}^{n-1} (g_{11} - g_{12}), & n > 2 \end{matrix} \right\}, \qquad g_{11} = B(\mu_d + k_{i-1} + k_i, \mu_c + \ell_{i-1} + \ell_i),$$

and

$$g_{12} = B\left(\frac{t^{\alpha_i/2}}{t^{\alpha_i/2}+1}, \mu_d + k_{i-1} + k_i, \mu_c + \ell_{i-1} + \ell_i\right).$$

Similar conclusions about the rapid convergence of the nested infinite sum from this expression, as for the convergence of expression (4.2) are valid.

The PDF can be obtained easily from the previous expression. Namely,

$$p_\lambda(t) = \frac{\mathrm{d}}{\mathrm{d}t} F_\lambda(t) = \sum_{k_1,\ldots,k_n=0}^{\infty} \sum_{\ell_1,\ldots,\ell_n=0}^{\infty} \frac{1}{2t} \frac{\Gamma(\mu_d + \mu_c + k_1 + \ell_1)}{\Gamma(\mu_d)\Gamma(\mu_c) \prod_{i=2}^{n-1} \Gamma(\mu_d + k_i)\Gamma(\mu_c + \ell_i) k_i! \ell_i!}$$
$$\times \Gamma(\mu_d + \mu_c + k_{n-1} + \ell_{n-1}) \rho^{2(k_1+\cdots+k_{n-1}+\ell_1+\cdots+\ell_{n-1})} (1-\rho^2)^{\mu_d+\mu_c} \left(\sum_{i=1}^{n} A_i(t)\right) \frac{g_5}{g_8},$$

where

$$\begin{aligned} A_1(t) &= \alpha_1 \frac{t^{\alpha_1/2(\mu_d+k_1)}}{(t^{\alpha_1/2}+1)^{(\mu_c+\ell_1+\mu_d+k_1)}} \\ &\quad \times \left[B(\mu_d + k_1 + k_2, \mu_c + \ell_1 + \ell_2) - B\left(\frac{t^{\alpha_2/2}}{t^{\alpha_2/2}+1}, \mu_d + k_1 + k_2, \mu_c + \ell_1 + \ell_2\right)\right] \\ &\quad \times \cdots \times \left[B(\mu_d + k_{n-1}, \mu_c + \ell_{n-1}) - B\left(\frac{t^{\alpha_n/2}}{t^{\alpha_n/2}+1}, \mu_d + k_{n-1}, \mu_c + \ell_{n-1}\right)\right], \\ A_i(t) &= \alpha_i \frac{t^{\alpha_i/2(\mu_d+k_{i-1}+k_i)}}{(t^{\alpha_1/2}+1)^{(\mu_c+\ell_{i-1}+\ell_i+\mu_d+k_{i-1}+k_i)}} \left[B(\mu_d + k_1, \mu_c + \ell_1) - B\left(\frac{t^{\alpha_1/2}}{t^{\alpha_1/2}+1}, \mu_d + k_1, \mu_c + \ell_1\right)\right] \\ &\quad \times \left[B(\mu_d + k_{n-1}, \mu_c + \ell_{n-1}) - B\left(\frac{t^{\alpha_n/2}}{t^{\alpha_n/2}+1}, \mu_d + k_{n-1}, \mu_c + \ell_{n-1}\right)\right] \prod_{j=2,\ldots,n-1; j\neq i} [g_{10} - g_{11}] \end{aligned}$$

for $i = 2, \ldots, n-1$, and

$$\begin{aligned} A_n(t) &= \alpha_n \frac{t^{\alpha_n/2(\mu_d+k_{n-1})}}{(t^{\alpha_n/2}+1)^{(\mu_c+\ell_{n-1}+\mu_d+k_{n-1})}} \left[B(\mu_d + k_1, \mu_c + \ell_1) - B\left(\frac{t^{\alpha_1/2}}{t^{\alpha_1/2}+1}, \mu_d + k_1, \mu_c + \ell_1\right)\right] \\ &\quad \times \left[B(\mu_d + k_{n-2} + k_{n-1}, \mu_c + \ell_{n-2} + \ell_{n-1}) - B\left(\frac{t^{\alpha_{n-1}/2}}{t^{\alpha_{n-1}/2}+1}, \mu_d + k_{n-2} + k_{n-1}, \mu_c + \ell_{n-2} + \ell_{n-1}\right)\right]. \end{aligned}$$

Fig. 4.2 shows the CDF of the minimal square ratio of two three-variate exponentially correlated α–μ distribution for some values of correlation coefficients and distribution parameters.

5. Application in telecommunications: the performance analysis of the multibranch SIR based SC diversity

Performances of mobile radio systems are remarkably affected by fading phenomena [12]. In practice, fading is not independent due to the insufficient antenna spacing. Multivariate (correlated) distribution analysis is an important tool in the performance investigation of many receiver structures for multiple-path correlative fading channels. Hence, characterizing the diversity system performance over correlated fading channels is important from both a theoretical and practical viewpoint. There have been proposed several correlation models and used for the performance analysis of various wireless systems, corresponding to specific modulation, detection, and diversity combining schemes. Though useful in mathematics and some situations in engineering, the assumption of constant or exponential correlation generally matches the practical environment in mobile communications. The assumption of exponential correlation is somewhat close to the situation in a linear array, but it requires equispaced diversity antennas [13]. By considering two important phenomena inherent to radio propagation, namely non-linearity and clustering, it was shown, that α–μ fading model provides a very good fit to measured data over a wide range of fading conditions. The α–μ distribution is written in terms of physically-based fading parameters, namely α and μ. Roughly speaking, α is related to the non-linearity of the environment whereas μ is associated with the number of multi-path clusters.

Various techniques are used to combine the signals from multiple diversity branches, representing different levels of performance [14]. As one of the classic combining methods, selection combining (SC) has been widely used in practice due to its simplicity. In the interference-limited fading environments (where the level of the cochannel interference is sufficiently high as compared to the thermal noise), it picks the branch with the maximum instantaneous signal-to-interference ratio (SIR) for a symbol decision. Instantaneous values of SIR at the k-th diversity branch input can be defined as $\lambda_k = R_k^2/r_k^2$. The selection combiner chooses and outputs the branch with the largest SIR

$$\lambda = \lambda_{\text{out}} = \max(\lambda_1, \lambda_2 \dots, \lambda_n).$$

This type of combining technique, in which the branch with the highest SIR is selected, can be measured in real time both in base stations (up-link) and in mobile stations (down-link) using specific SIR estimators, as well as those for both analogue and digital wireless systems (e.g. GSM) [15,16].

Based on the previous section, the proposed analysis is carried out assuming correlative α–μ fading for both the desired signals and co-channel interferers, and useful closed-form of formulae for the pdf and cdf are derived. The correlation between desired signals and correlation between interferences are modeled with the same correlation coefficient ρ because the arrival angles of the contribution with the broadside directions of antennas are assumed to be the same [14]. The main contribution of this analysis is that we observed multi-branch selection combining diversity system and it has been done for general case of α–μ distributed fading. Capitalizing on this standard system performance measures can be determined like the outage probability (OP).

5.1. Outage probability

The outage probability P_{out} is a standard measure of the communication system's performance and is commonly used to control the noise or cochannel interference level in wireless communication systems.

In the interference-limited environment, outage probability P_{out} is defined as the probability which combined-SIR falls below a given outage threshold γ, also known as a protection ratio [17]. Protection ratio depends on modulation technique and expected quality-of-service (QoS).

$$P_{\text{out}} = P_R(\lambda < \gamma) = \int_0^{\gamma} p_\lambda(t)\mathrm{d}t = F_\lambda(\gamma).$$

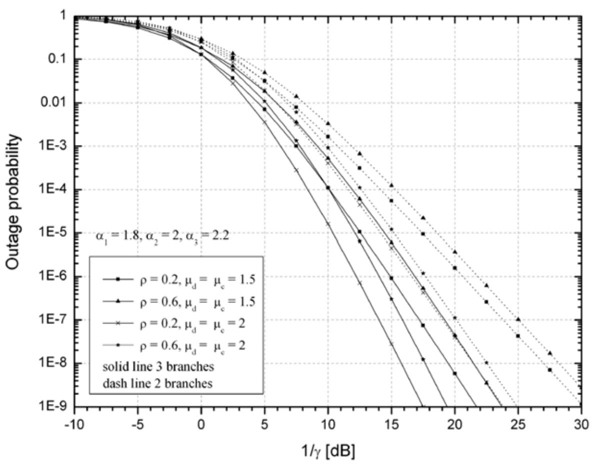

Fig. 5.1. Outage probability versus $1/\gamma$.

In Fig. 5.1, the outage probability is plotted versus $1/\gamma$ for several values of μ_d and μ_c and the correlation coefficient ρ. Diversity systems with two and three branches are observed. It is evident that a system with three branches has better performance (lower values of outage probability for the same parameters). Also it is interesting to note here that for low values of $1/\gamma$ (<2 dB) due to interference dominance, the outage probability increases (performance deteriorates) when the fading severity of the interferers decreases (μ_c increases). But, for higher values of $1/\gamma$ (dominance of the desired signal), the outage probability decreases when μ_c increase.

6. Conclusion

Derivation of probability density function of a multivariate α–μ distributed variables with exponentially correlated random variables is derived. Also JPDF is derived for the square ratio of two multivariate exponentially correlated α–μ distributed variables. Statistical properties of the maximal and minimal square ratio of two multivariate exponentially correlated α–μ distributed variables are determined. Capitalizing on this, an approach to the performance analysis of SIR based SC over α–μ fading channels in the presence of co-channel interference is presented.

Acknowledgements

We are very grateful to the reviewers for their suggestions, which have improved the quality of the paper.

References

[1] M.D. Yacoub, The α–μ distribution: a physical fading model for the stacy distribution, IEEE Transactions on Vehicular Technology 56 (2007) 27–34.
[2] E.W. Stacy, A generalization of the gamma distribution, Annals of Mathematical Statistics 33 (1962) 1187–1192.
[3] N.C. Sagias, P.T. Mathiopulous, Switched diversity receivers over generalized gamma fading channels, IEEE Communications Letters 9 (2005) 871–873.
[4] N.C. Sagias, G.K. Karagiannidis, P.T. Mathiopulous, T.A. Tsiftsis, On the performance analysis of equal-gain diversity receivers over generalized gamma fading channels, IEEE Transactions on Wireless Communications 5 (2006) 2967–2975.
[5] T. Piboongungon, V.A. Aalo, C.D. Iskander, Average error rate of linear diversity reception schemes over generalized gamma fading channels, in: Proc. IEEE Southeastcon, Southeast Conf., 2005, Lauderdale, FL, USA, 2005, pp. 265–270.
[6] S.R. Panic, M.C. Stefanovic, A.V. Mosic, Performance analyses of selection combining diversity receiver over α–μ fading channels in the presence of co-channel interference, IET Communications 3 (2009) 1769–1777.
[7] J.A. Anastasov, S.R. Panic, M.C. Stefanovic, A.V. Mosic, Generalized approach for performance analysis of SIR-based multiple selection diversity over constant correlated α–μ fading channels, European Transactions on Telecommunications 21 (2010) 655–662.
[8] G.K. Karagiannidis, D.A. Zogas, S.A. Kotsopoulos, On the multivariate Nakagami-*m* distribution with exponential correlation, IEEE Transactions on Communications 51 (2003) 1240–1244.
[9] V. Aalo, Performance of maximal-ratio diversity systems in a correlated Nakagami-fading environment, IEEE Transactions on Communications 43 (1995) 2360–2369.
[10] G.L. Stuber, Principles of Mobile Communication, Kluwer, 2003.
[11] I. Gradshteyn, I. Ryzhik, Tables of Integrals, Series, and Products, Academic Press, 1980.
[12] M.K. Simon, M.S. Alouini, Digital Communication Over Fading Channels, Wiley, 2005.
[13] M. Schwartz, W.R. Bennett, S. Stein, Communication Systems and Techniques, McGraw-Hill, New York, 1966.
[14] W.C.Y. Lee, Mobile Communications Engineering, McGraw-Hill, New York, 2001.
[15] M.D. Austin, G.L. Stuber, In-service signal quality estimation for TDMA cellular systems, in: IEEE International Symposium on Personal, Indoor and Mobile Radio Communications, PIMRC'95, Toronto, ON, Canada, September 1995, pp. 836–840.
[16] A.L. Brandao, L.B. Lopez, D.C. McLernon, Co-channel interference estimation for *M*-ary PSK modulated signals, IEEE Wireless Personal Communications 1 (1994) 23–32.
[17] M. Stefanović, D. Milović, A. Mitić, M. Jakovljević, Performance analysis of system with selection combining over correlated Weibull fading channels in the presence of cochannel interference, International Journal of Electronics and Communications 62 (2007) 695–700.