3. Now we shall show how to get the conditions for the inclusion \( \overline{L}_p(r, \mu) \subseteq \overline{L}_p(r, \mu) \).

**Theorem 4.** Let \( 0 < p, q \leq \infty \). The inclusion \( \overline{L}_p(r, \mu) \subseteq \overline{L}_q(r, \mu) \) holds if and only if for each \( m \) there exists a \( k \) such that \( r_{m, p} \) holds if and only if for each \( m \) there exists a \( k \) such that \( r_{m, p} \mu \leq r_{k, q}(\mu) \).

**Proof.** The sufficiency of the condition is obvious. We shall prove the necessity. Let \( \overline{L}_p(r, \mu) \subseteq \overline{L}_q(r, \mu) \).

Let \( 0 < s < \min(p, q) \) and we set \( p^* = p^*(s), q^* = q^*(s) \). According to property 1) of the operation of taking the dual and (2)

\[
\overline{L}_p^*(1/r, \mu) = \left( \overline{L}_p(r, \mu) \right)^{1/q} \subseteq \overline{L}_q(r, \mu) \overline{L}_p(\mu) = \overline{L}_p^*(1/r, \mu).
\]

Now using Theorem 2, we get that for each \( m \) there exists a \( k \) such that

\[
\overline{L}_p^*(1/r, \mu) \subseteq \overline{L}_q^*(r, \mu).
\]

According to property 1) of the operation of taking the dual with respect to \( L_\infty(\mu) \), we arrive at the inclusion \( r_{k, q}(\mu) \mu \subseteq r_{m, p}(\mu) \). But \( p^* = p \) and \( q^* = q \). Thus the theorem is proved.

**Literature Cited**


**Inequalities with Convex Sequences**

G. V. Milovanovich and I. Zh. Milovanovich

In this paper we prove some inequalities with mean powers for convex sequences of order \( k \) and one inequality of Hölder type.

We give some definitions and theorems, which will be used later in the paper.

**Definition.** For a positive sequence \( a = (a_1, \ldots, a_n) \) the mean power of order \( r, r \in \mathbb{R}, r = \pm \infty \), is defined by the formula

\[
M^{\mu r}_p(a, p) = \left( \frac{\sum_{i=1}^{n} p_i a_i^r}{\sum_{i=1}^{n} p_i} \right)^{1/r}, \quad r \neq 0, \quad |r| < +\infty,
\]

where \( p = (p_1, \ldots, p_n) \) is a weight sequence, \( p = \sum_{i=1}^{n} p_i \).

Let us assume that \( S_k \) is the set of all real convex sequences \( a = (a_1, \ldots, a_n) \) of order \( k, 1 < k < n, \)

\[
S_k = \left\{ a \mid \Delta^k a_m = \sum_{i=1}^{k} (-1)^{i} \binom{k}{i} a_{m+k-i} \geq 0, \quad 1 \leq m \leq n-k \right\}.
\]

We define a sequence \( a^{(r)} = (a_1^{(r)}, \ldots, a_n^{(r)}) \) (r is a natural number) as follows:

\[
a_{m+i}^{(r)} = m^{-i} a_{m+i}^{(r-1)}, \quad a_{m}^{(r)} = a_{m}, \quad a_{m}^{(r)} = a_{m}/m^{r-1}.
\]

Let \( S_k^{(r)} = \{ a \mid a \in S_k \land (\Delta^{k-r} a_i^{(r)} \geq 0, 1 \leq i \leq \ldots, p) \} \), where \( p < k \).

In [1] theorems are proved according to which, for each \( k \in \{2, 3, \ldots\} \) one has the implications

\[
a \in S_k^{(r)} \Rightarrow a^{(r)} \in S_{k-1} \quad \text{and} \quad a \in S_k^{(k-1)} \Rightarrow a^{(k)} \in S_k.
\]
Using theorems from [1] we shall prove the following theorem.

**THEOREM 1.** If \( p_i > 0, i = 1, \ldots, n \), and \( x = (x_1, \ldots, x_n) \) is a positive sequence from \( S_k^{(k-1)}, n > k \), then for \( r \geq s \)

\[
M^{(r)}_n(x; p) \geq \alpha_k M^{(s)}_n(x; p) \tag{2}
\]

where \( \alpha_k \) is a constant, calculated according to the formula

\[
\alpha_k = \frac{M^{(r)}_n(a; p)}{M^{(s)}_n(a; p)} \geq 1, \quad a = (1^{k-1}, \ldots, n^{k-1}).
\]

The equality in (2) is achieved for \( x = a \).

**Proof.** To prove (2) we first set \( p_i = p_i^{(k-1)} \), \( x_i = x_i^{1/k-1} \), \( i = 1, \ldots, n \) in the inequality [2]

\[
M^{(r)}_n(x; p) \geq M^{(s)}_n(x; p), \quad r \geq s.
\]

Then, defining for any sequences \( a = (a_1, \ldots, a_n), b = (b_1, \ldots, b_n) \) the sequence \( ab = (a_1b_1, \ldots, a_nb_n) \), we get

\[
M^{(r)}_n(xa; pa) \geq M^{(s)}_n(xa; pa), \quad a = (1^{k-1}, \ldots, n^{k-1}).
\]

To complete the proof we shall show that one has

\[
\sum_{i=1}^n p_i x_i^{1/k-1} \geq \sum_{i=1}^n p_i x_i^{1/(k-1)} \sum_{i=1}^n p_i x_i^{(k-1)/r}.
\]

It is obtained from Chebyshev's inequality [3]

\[
\sum_{i=1}^n p_i x_i \geq \sum_{i=1}^n q_i u_i \geq \sum_{i=1}^n q_i v_i
\]

for \( q_i = p_i^{(k-1)}, u_i = i^{(r-1)/(k-1)}, v_i = x_i^{1/r} \), \( i = 1, \ldots, n \).

Since the sequence \( x = (x_1, \ldots, x_n) \) belongs to \( S_k^{(k-1)}, k \geq 2 \), according to a theorem from [1] the sequence \( (x_1^{1/k-1}, \ldots, x_n^{1/k-1}) \) is nondecreasing.

If in (3) one sets \( x_i = i^{k-1} \), then \( \alpha_k \geq 1 \).

**COROLLARY 1.** Since \( \alpha_k \geq 1 \), (2) is more precise than (3).

**COROLLARY 2.** For \( k = 2 \), from Theorem 1 we get the theorem connected with Theorem 3 of [4].

We note that this theorem is proved in [4], and later also proved in [5].

**COROLLARY 3.** We introduce \( x_i = i^k, i = 1, \ldots, n \) in (2). Then \( \alpha_{k+1} \geq \alpha_k \), so (2) becomes more precise with increasing \( k \).

**COROLLARY 4.** If \( p_i = 1, i = 1, \ldots, n \), then (2) assumes the form

\[
\left( \sum_{i=1}^n x_i \right)^{1/r} \geq M(k) \left( \sum_{i=1}^n x_i \right)^{1/s},
\]

where

\[
M(k) = \left( \sum_{i=1}^n i^{(k-1)/r} \right)^{1/r} \left( \sum_{i=1}^n i^{(k-1)/s} \right)^{1/s}.
\]

Since \( \lim_{n \to +\infty} (x^{r-1/s} M(k)) = (s(k-1) + 1)^{1/r} (r(k-1) + 1)^{1/s} \), as \( n \to +\infty \) from (4) one can get the inequalities for convex functions of order \( k \) proved in [6].

**Remark.** On an integral analog of Theorem 1 cf. [7].

Analogously to Theorem 1, one can prove the following theorem.

**THEOREM 2.** Let the sequence \( p = (p_1, \ldots, p_n), x = (x_1, \ldots, x_n), b = (b_1, \ldots, b_n) \) be such that \( p_i > 0, \ldots, p_n > 0; x_1 > 0, \ldots, x_n > 0; b_1 > 0, \ldots, b_n > 0; b_1 \leq b_2 \leq \ldots \leq b_n \) and \( (x_1/b_1, \ldots, x_n/b_n), n > k \) be a sequence from \( S_k^{(k-1)}, k \geq 2 \). Then for \( r \geq s \) one has

*Numbered as in Russian original - Publisher.*
where \( H_k = M^{(k)}(a; p)/M^{(k)}(b; p) = H_k M^{(k)}(x; p)/M^{(k)}(b; p) \).

**Theorem 3.** Let the sequence \( p = (p_1, \ldots, p_n) \) be positive. Let the \( r \) sequences \( a = (a_1, \ldots, a_n), b = (b_1, \ldots, b_n), \ldots, l = (l_1, \ldots, l_n) \) be positive and belong to the set \( S^{(k-1)}_k, n > k \). Then for \( 0 \leq m_i \leq 1, i = 1, \ldots, r \) one has

\[
\sum_{i=1}^{n} p_i a_i p_i \ldots l_i \leq \left( \sum_{i=1}^{n} p_i a_i^{m_i} \right)^{1/m_i} \ldots \left( \sum_{i=1}^{n} p_i l_i^{m_i} \right)^{1/m_i} \geq Q_r,
\]

where

\[
Q_r = \left( \sum_{i=1}^{n} p_i l_i^{m_i} \right)^{1/m_i} \ldots \left( \sum_{i=1}^{n} p_i l_i^{m_i} \right)^{1/m_i}.
\]

The equality in (6) is achieved for \( a_i = b_i = \ldots = l_i = 1^{k-1}, i = 1, \ldots, n \).

**Proof.** In [8] for sequences from the set \( S^{(k-1)}_k, k \geq 2 \) there is proved Chebyshev's inequality

\[
\sum_{i=1}^{n} p_i a_i p_i \ldots l_i \leq \frac{\sum_{i=1}^{n} p_i l_i^{m_i}}{\sum_{i=1}^{n} p_i^{m_i}} \left( \sum_{i=1}^{n} p_i a_i \right) \ldots \left( \sum_{i=1}^{n} p_i l_i \right).
\]

From (7) we get

\[
\frac{\sum_{i=1}^{n} p_i a_i p_i \ldots l_i}{\left( \sum_{i=1}^{n} p_i a_i^{m_i} \right)^{1/m_i} \ldots \left( \sum_{i=1}^{n} p_i l_i^{m_i} \right)^{1/m_i}} \leq \frac{\sum_{i=1}^{n} p_i l_i^{m_i}}{\left( \sum_{i=1}^{n} p_i l_i^{m_i} \right)^{1/m_i}} \left( \sum_{i=1}^{n} p_i a_i \right) \ldots \left( \sum_{i=1}^{n} p_i l_i \right).
\]

Using Theorem 1 we get

\[
\sum_{i=1}^{n} p_i a_i \left( \sum_{i=1}^{n} p_i x_i \right)^{m_i} \leq \sum_{i=1}^{n} p_i l_i^{m_i} \left( \sum_{i=1}^{n} p_i l_i \right)^{1/m_i}.
\]

From (8) and (9) follows (6).

**Corollary 7.** For \( k = 2, r = 2, p_1 = 1, i = 1, \ldots, n \), from Theorem 3 we get a theorem similar to Theorem 3 of [5].

**Literature Cited**

The effect of a random disturbance on mechanical systems can be properly studied by the method of Fokker-Planck-Kolmogorov (FPK) equations, especially when the latter is combined with the asymptotic method of nonlinear mechanics [1]. In the nonautonomous case, however, it was noted in [1] that the corresponding FPK equation will be complicated. In this paper we shall solve the FPK equation for an important class of nonautonomous systems. On the basis of [2] we shall seek the solution in the form of a series for the amplitude. We obtain a system of separable differential equations that makes it possible to successively find the series coefficients of any order.

1. Let us consider a nonautonomous mechanical system with one degree of freedom whose equation of motion has the form

\[ \ddot{x} + \omega^2 x = \epsilon f(x, \dot{x}) + \epsilon P \cos \omega t + \sqrt{\epsilon} \xi(t) \]  
\[ \omega^2 = \omega^2 + \epsilon \Delta, \] (2)

where \( \xi(t) \) is white noise of unit intensity, and

\[ f(x, \dot{x}) = \sum_{i=1}^{m} a_i \left( \sum_{j=0}^{m} \gamma_{ij} x^i \dot{x}^j \right), \quad a_i, \gamma_{ij} = \text{const} \] (3)

is a polynomial in \( x \) and \( \dot{x} \).

With the use of (2) let us rewrite (1) in the form

\[ \ddot{x} + \omega^2 x = \epsilon f_1(x, \dot{x}, \omega t) + \sqrt{\epsilon} \xi(t), \] (4)

where

\[ f_1(x, \dot{x}, \omega t) = f(x, \dot{x}) - \Delta x + P \cos \omega t. \] (5)

By a change of variables [1]

\[ x = a \cos \psi, \quad \dot{x} = -av \sin \psi, \quad \psi = \omega t + \theta \] (6)

we can transform Eq. (4) with the aid of Ito's formula to standard form

\[ da = \left[ -\frac{\epsilon}{v} f_1(x, \dot{x}, \omega t) \sin \psi + \frac{\epsilon \sigma}{2v^2 a} \cos^2 \psi \right] dt - \frac{\epsilon \sigma}{v} \sin \psi d\xi(t), \] (7a)

\[ d\theta = \left[ -\frac{\epsilon}{av} f_1(x, \dot{x}, \omega t) \cos \psi - \frac{\epsilon \sigma^2}{a^2 v^2} \sin \psi \cos \psi \right] dt - \frac{\epsilon \sigma^2}{av} \cos \psi d\xi(t). \] (7b)