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A RAMANUJAN INTEGRAL AND ITS DERIVATIVES: COMPUTATION AND ANALYSIS

WALTER GAUTSCHI AND GRADIMIR V. MILOVANOVIĆ

Abstract. The principal tool of computation used in this paper is classical Gaussian quadrature on the interval [0,1], which happens to be particularly effective here. Explicit expressions are found for the derivatives of the Ramanujan integral in question, and it is proved that the latter is completely monotone on $(0,\infty)$. As a byproduct, known series expansions for incomplete gamma functions are examined with regard to their convergence properties.

The paper also pays attention to another famous integral, the Euler integral — better known as the gamma function — revitalizing a largely neglected part of the function, the part corresponding to negative values of the argument, which plays a prominent role in our work.

1. The Ramanujan integral

Among the many special definite integrals that Ramanujan has studied, see, e.g., his collected papers [2], the one considered here is

$$I_R(t)=\int_0^\infty \mathrm{e}^{-tx}x^{-1}\left(\pi^2+\log^2 x\right)^{-1}\,\mathrm{d}x,\quad t>0.$$

According to [6, Eq. (4)] we can write, equivalently,

$$I_R(t)=\mathrm{e}^{\,t}\int_0^1\frac{\Gamma(a,t)}{\Gamma(a)}\,\mathrm{d}a,\quad t>0, \tag{1.1}$$

where $\Gamma(a,t)$ is the (upper) incomplete gamma function [4, Ch. 8]. Note that the integrand in (1.1) is singular at $a=0$, where $\Gamma(0)=\infty$, but this singularity can easily be removed by using $\Gamma(a)=\Gamma(a+1)/a$ in (1.1) and thereby avoiding any infinities. Similar remarks apply to all integrals that will come up later in the paper, such as those in (2.1) or (3.1).

As a function of a, therefore, the integrand in (1.1) is analytic on $[0,1]$ and has the values 0 and e^{-t} at the left and right endpoints. Graphs for selected values of t are shown in Fig. 1. They are produced by the Matlab functions `plot_Raman1.m`, `plot_Raman2.m`, …, `plot_Raman4.m`.[1]

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[1]All Matlab codes referred to in this paper are accessible at the website `https://www.cs.purdue.edu/archives/2002/wxg/codes/RAMANUJAN.html`.

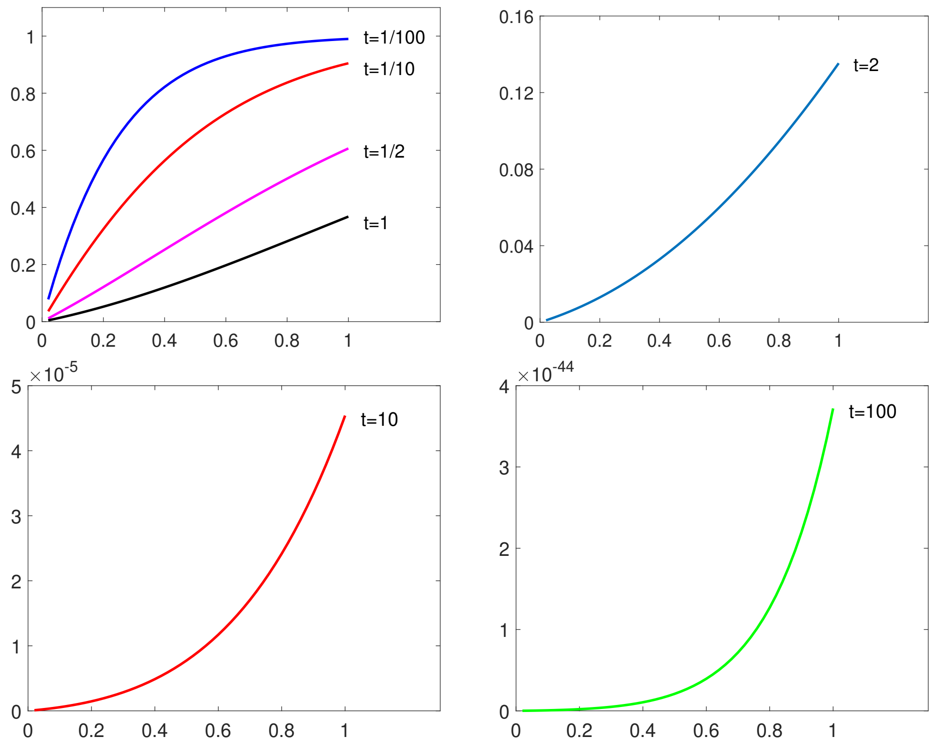

FIGURE 1. The integrand in (1.1) as a function of a for selected values of t

Our approach for computing the integral in (1.1) is to apply Gauss-Legendre quadrature on the interval $[0, 1]$, which is expected to work particularly well. Indeed, Table 1, for the same values of t as in Fig. 1, shows the number n of quadrature nodes required to achieve an accuracy of 15 decimal digits in Matlab double precision, along with the results obtained for I_R. As can be seen, numerical quadrature here is indeed very efficient, n never exceeding 14. It is also very stable since the quadrature summation involves only positive terms.

Table 1 is produced by the Matlab function `nquad_Ram.m`.

TABLE 1. Quadrature perfomance

t	n	$I_R(t)$
1/100	13	0.778309402566200
1/10	14	0.648177757201035
1/2	13	0.514259531970136
1	13	0.451747320759197
2	13	0.391476469754981
10	12	0.276334325659172
100	12	0.179175804574065

High-precision 32-digit results, produced by `snquad_Ram.m`, are shown in Table 2 and plots of the Ramanujan integral in Fig. 2, the one on the left produced by

TABLE 2. 32-digit values of the Ramanujan integrals for the same values of t as in Table 1

t	$I_R(t)$
1/100	0.77830940256619957857954222245003
1/10	0.64817775720103525381784770829289
1/2	0.51425953197013601784165144344007
1	0.45174732075919640028832361367444
2	0.39147646975498101142310011505978
10	0.27633432565917153284405429812094
100	0.17917580457406502430704593379152

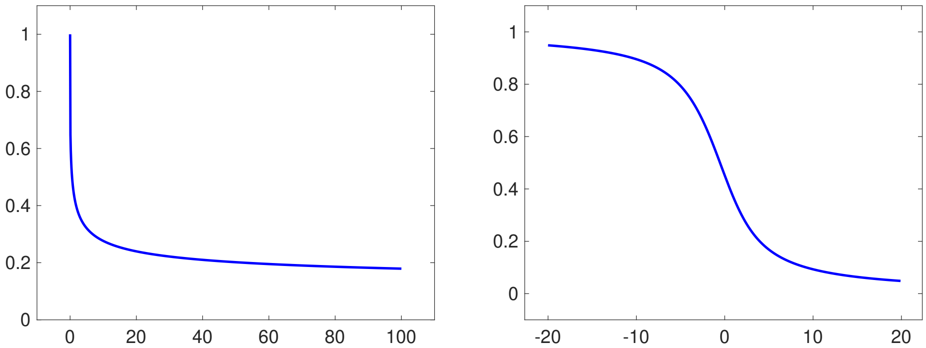

FIGURE 2. The Ramanujan integral $I_R(t)$ for $0 \le t \le 100$ (left) and $I_R(\mathrm{e}^t)$ for $-20 \le t \le 20$ (right)

plot_Ram.m and the one on the right by plot_sRam1.m. The value of $I_R(t)$ at $t = 0$ is clearly 1, as confirmed by the figure on the left and suggested by the figure on the right. The limit as $t \to \infty$ is less obvious from the figures, although the one on the right suggests it to be 0. This in fact can be confirmed if we write

$$I_R(t) = \frac{\displaystyle\int_0^1 \frac{\Gamma(a,t)}{\Gamma(a)}\,\mathrm{d}a}{\mathrm{e}^{-t}} \tag{1.2}$$

and use Bernoulli–L'Hôpital's rule,

$$\lim_{t\to\infty} \int_0^1 \frac{t^{a-1}}{\Gamma(a)}\,\mathrm{d}a = 0.$$

Convergence, however, is extremely slow, as the graph on the left of Fig. 2 indicates and the following analysis, in particular (1.5) and Table 3, will show.

Large-t asymptotics for the Ramanujan integral indeed can be obained [3] by combining the 1-term asymptotic expansion of the incomplete gamma function [4, Eq. 8.11.2]

$$\Gamma(a,t) = t^{a-1}\mathrm{e}^{-t}\left(1 + O(t^{-1})\right), \quad t \to \infty$$

(which is uniform for bounded a) with the $(m+1)$-term Taylor expansion of $f(a) = 1/\Gamma(a)$ at $a = 1$ (where the main contributions are coming from),

$$f(a) = \sum_{k=0}^{m} \frac{(a-1)^k}{k!}\, f^{(k)}(1) + \cdots . \tag{1.3}$$

Inserting this in (1.1) yields

$$\begin{aligned} I_R(t) &= \mathrm{e}^t \int_0^1 \frac{t^{a-1}\mathrm{e}^{-t}}{\Gamma(a)} \left(1 + O(t^{-1})\right) \mathrm{d}a = \int_0^1 \frac{t^{a-1}}{\Gamma(a)} \left(1 + O(t^{-1})\right) \mathrm{d}a \\ &= \int_0^1 t^{a-1} \left(\sum_{k=0}^{m} \frac{(a-1)^k}{k!}\, f^{(k)}(1) + \cdots \right) \mathrm{d}a \left(1 + O(t^{-1})\right), \end{aligned}$$

that is,

$$I_R(t) = \left(\sum_{k=0}^{m} \frac{f^{(k)}(1)}{k!} \int_0^1 (a-1)^k\, t^{a-1} \mathrm{d}a + \cdots \right) \left(1 + O(t^{-1})\right). \tag{1.4}$$

The large-t behavior of $I_R(t)$ is thus determined by the behavior, as functions of t, of the integrals

$$i_k(t) = \int_0^1 (a-1)^k\, t^{a-1} \mathrm{d}a, \quad k = 0, 1, 2, \ldots ,$$

which can also be expressed in terms of the incomplete gamma function,

$$i_k(t) = (-1)^k\, k!\, \log^{-(k+1)} t \left(1 - \frac{\Gamma(k+1, \log t)}{k!} \right).$$

Since (cf. [4, Eqs. 8.4.8 and 8.4.11])

$$\frac{\Gamma(k+1, \log t)}{k!} = \frac{1}{t} \sum_{\mu=0}^{k} \frac{\log^{\mu} t}{\mu!} = O\left(t^{-1} \log^k t\right) \quad \text{as t} \to \infty,$$

we get

$$i_k(t) = (-1)^k\, k!\, \log^{-(k+1)} t \left(1 + O\left(t^{-1} \log^k t\right)\right), \quad k = 0, 1, 2, \ldots .$$

By (1.4), therefore, we obtain the following asymptotic approximations of $I_R(t)$ for large t in terms of the reciprocal of $\log t$,

$$I_R(t) = \left(\sum_{k=0}^{m} c_k \log^{-(k+1)} t \right) \left(1 + O(t^{-1} \log^m t)\right), \quad m = 0, 1, 2, \ldots , \tag{1.5}$$

where $c_k = (-1)^k\, f^{(k)}(1)$, $k = 0, 1, \ldots, m$, and $f^{(k)}(1)$ are the derivatives in Taylor's expansion (1.3).

Using Mathematica, we find

$$
\begin{aligned}
c_0 &= f(1) = 1,\\
c_1 &= -f^{'}(1) = -\gamma,\\
c_2 &= f^{''}(1) = \gamma^2 - \tfrac{1}{6}\,\pi^2,\\
c_3 &= -f'''(1) = -2\,\zeta(3) - \gamma^3 + \tfrac{1}{2}\gamma\pi^2,\\
c_4 &= f^{(4)}(1) = 8\,\gamma\,\zeta(3) + \gamma^4 - \gamma^2\pi^2 + \tfrac{1}{60}\pi^4,\\
c_5 &= -f^{(5)}(1) = -24\,\zeta(5) - \left(20\gamma^2 - \tfrac{10}{3}\pi^2\right)\zeta(3) - \gamma\left(\gamma^4 - \tfrac{5}{3}\gamma^2\pi^2 + \tfrac{1}{12}\,\pi^4\right),
\end{aligned}
$$

where γ is Euler's constant and ζ the Riemann zeta function.

TABLE 3. Asymptotic approxmations for large t of the Ramanujan integral $I_R(t)$ obtained from (1.5)

$t = 10^{40}$	$t = 10^{80}$	$t = 10^{160}$
.010857362047581	0.005428681023791	0.002714340511895
.010789318531270	0.005411670144713	0.002710087792126
.010787639626486	0.005411460281615	0.002710061559239
.010787643128558	0.005411460500494	0.002710061572919
.010787643731600	0.005411460519339	0.002710061573507
.010787643739895	0.005411460519469	0.002710061573509

Taking successively the first six terms in (1.5) (i.e., $m = 5$), when $t = 10^{40}$, $t = 10^{80}$, and $t = 10^{160}$, the approximations obtained (with `RamIntAs.m`) are shown in Table 3. They demonstrate, in particular, the extreme slowness with which the Ramanujan integral coverges to zero as $t \to \infty$. Roughly speaking (i.e., $m = 0$), to bring down $I_R(t)$ to 10^{-p} would require $t = \exp(10^p)$, that is, $t \approx 10^{4343}$ when $p = 4$, and $t \approx 10^{43429448}$ when $p = 8$.

2. Derivatives of the Ramanujan integral

Derivatives of the Ramanujan integral are of interest in a number of different fields, including heat conduction and fluid mechanics; cf. [6, Introduction, Eqs. (2a)–(2g)].

Differentiating both sides of (1.1) with respect to the variable t, using the product rule of differentiation, yields

$$
\begin{aligned}
I_R'(t) &= \mathrm{e}^t \int_0^1 \frac{\Gamma(a,t)}{\Gamma(a)}\,\mathrm{d}a + \mathrm{e}^t \int_0^1 \frac{\partial\Gamma(a,t)}{\partial t}\,\frac{\mathrm{d}a}{\Gamma(a)}\\
&= I_R(t) - \mathrm{e}^t \int_0^1 t^{a-1}\mathrm{e}^{-t}\,\frac{\mathrm{d}a}{\Gamma(a)}\,,
\end{aligned}
$$

that is,

$$
I_R'(t) = I_R(t) - \int_0^1 \frac{t^{a-1}}{\Gamma(a)}\,\mathrm{d}a.
$$

Higher-order derivatives can then be obtained as follows,

$$I_R''(t) = I_R(t) - \left(\int_0^1 \frac{t^{a-1}}{\Gamma(a)}\, \mathrm{d}a + \int_0^1 \frac{t^{a-2}}{\Gamma(a-1)}\, \mathrm{d}a \right),$$

$$I_R'''(t) = I_R(t) - \left(\int_0^1 \frac{t^{a-1}}{\Gamma(a)}\, \mathrm{d}a + \int_0^1 \frac{t^{a-2}}{\Gamma(a-1)}\, \mathrm{d}a + \int_0^1 \frac{t^{a-3}}{\Gamma(a-2)}\, \mathrm{d}a \right), \text{ etc.}$$

Thus, rather remarkably,

$$I_R^{(k)}(t) = I_R(t) - \sum_{j=1}^{k} \int_0^1 \frac{t^{a-j}}{\Gamma(a-j+1)}\, \mathrm{d}a, \quad k = 0, 1, 2, \dots . \tag{2.1}$$

For an integral representation of $I_R^{(k)}(t)$ resembling (1.1), see (3.1).

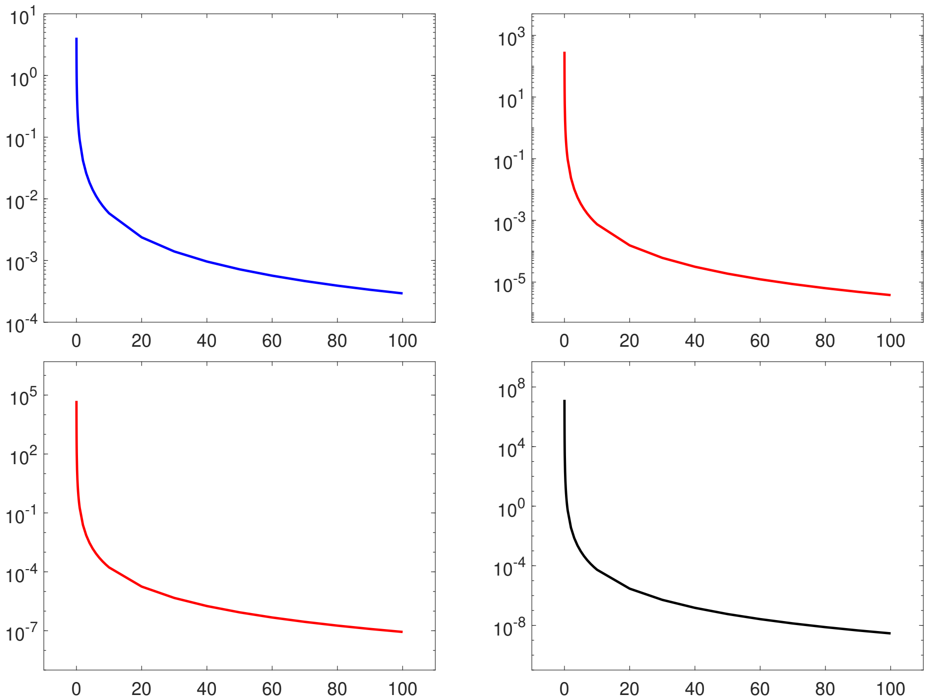

FIGURE 3. The functions $(-1)^k I_R^{(k)}(t)$, $k = 1, 2, 3, 4$, $0 \le t \le 100$, on a logarithmic scale from top left to bottom right

Computing the Ramanujan integral and its derivatives is a matter of computing the integrals

$$h_0(t) = \int_0^1 \frac{\Gamma(a,t)}{\Gamma(a)}\, \mathrm{d}a, \quad h_j(t) = \int_0^1 \frac{t^{a-j}}{\Gamma(a-j+1)}\, \mathrm{d}a, \quad j = 1, 2, \dots , \tag{2.2}$$

to obtain

$$I_R^{(k)}(t) = \mathrm{e}^t\, h_0(t) - \sum_{j=1}^{k} h_j(t), \quad k = 1, 2, 3, \dots .$$

Integration in (2.2) is over the variable a on $[0,1]$ and can be effectively accomplished, as in §1, by Gauss–Legendre quadrature. If the order n of quadrature is chosen so as to obtain consistently an accuracy of 15 decimal places, then as before, the n required is only moderately large, typically about 20.

It will be shown in §3 that $I_R(t)$ is a completely monotone function on $(0,\infty)$, i.e., $(-1)^k I_R^{(k)}(t) > 0$ on $(0,\infty)$ for all $k = 0,1,2,3,\ldots$. The four functions for $k = 1,2,3,4$, are plotted in Fig. 3 on a logarithmic scale using `plot_Ram.der.m`.

We note that the reason for the alternating signs of the derivatives is the gamma function in the denominators of the integrands in (2.1). When $j = 1$, the argument of the gamma function in the first integral is a, which is in (0,1), thus positive. The first integral therefore is positive. In the next integral, the argument of the gamma function is $a-1$, thus in the interval $(-1,0)$ where the gamma function is negative and hence the second integral is negative. From there on, the signs of the integrals alternate because of the alternating signs of the gamma function on the intervals $(-\lambda-1,-\lambda)$, $\lambda = 0,1,2,\ldots$. For more details, see the next section.

3. Complete monotonicity

The purpose of this section is to prove complete monotonicity of the Ramanujan integral and to state another useful integral representation of its kth derivative.

Theorem 3.1. *The Ramanujan integral $I_R(t)$ is completely monotone on $(0,\infty)$, that is,*

$$(-1)^k I_R^{(k)}(t) > 0 \quad \text{on} \quad (0,\infty), \quad k = 0,1,2,\ldots .$$

Proof. According to [4, Eq. 8.8.19],

$$\frac{\mathrm{d}^k}{\mathrm{d}t^k}\left[\mathrm{e}^t\Gamma(a,t)\right] = (-1)^k(1-a)_k\,\mathrm{e}^t\,\Gamma(a-k,t),$$

where $(1-a)_k = \Gamma(1-a+k)/\Gamma(1-a)$ is the Pochhammer symbol.

We have

$$\begin{aligned} I_R^{(k)}(t) &= \frac{\mathrm{d}^k}{\mathrm{d}t^k}\left(\mathrm{e}^t\int_0^1 \frac{\Gamma(a,t)}{\Gamma(a)}\,\mathrm{d}a\right) \\ &= \int_0^1 \frac{(-1)^k}{\Gamma(a)}(1-a)_k\,\mathrm{e}^t\,\Gamma(a-k,t)\,\mathrm{d}a \\ &= (-1)^k\mathrm{e}^t\int_0^1 \frac{\Gamma(1-a+k)}{\Gamma(a)\Gamma(1-a)}\,\Gamma(a-k,t)\,\mathrm{d}a. \end{aligned}$$

Applying the identity $\Gamma(a)\Gamma(1-a) = \pi/\sin\pi a$, once as is and once with the argument a replaced by $a-k$, we obtain

$$\frac{\Gamma(1-a+k)}{\Gamma(a)\Gamma(1-a)} = \frac{(-1)^k}{\Gamma(a-k)}\,,$$

so that

$$I_R^{(k)}(t) = \mathrm{e}^t\int_0^1 \frac{\Gamma(a-k,t)}{\Gamma(a-k)}\,\mathrm{d}a, \quad t>0,\ k=0,1,2,\ldots, \tag{3.1}$$

an intriguing result of independent interest. Now noting that the numerator in (3.1) is positive and the denominator has constant sign $(-1)^k$ establishes complete monotonicity of $I_R(t)$. $\square$

Note that when $a \leq k$ (which is the case for all $k = 1, 2, \dots$) the incomplete gamma function $\Gamma(a-k, t)$ in (3.1) tends to ∞ as $t \downarrow 0$, and therefore also $I_R^{(k)}(t)$. The limit of $I_R^{(k)}(t)$ as $t \uparrow \infty$ is 0, as shown in the same way as in (1.2).

Remark 3.2. An alternative (to (2.1)) and simpler way of computing derivatives is to evaluate the integral (3.1) by Gauss quadrature, as was done repeatedly in this paper for similar integrals. Nevertheless, (2.1) can be used to obtain another integral representation of $I_R^{(k)}(t)$, namely

$$I_R^{(k)}(t) = \int_0^1 \left(e^t \Gamma(a,t) - \sum_{j=1}^k a_j t^{a-j} \right) \frac{\mathrm{d}a}{\Gamma(a)}\,, \tag{3.2}$$

where

$$a_1 = 1,\ a_j = (a-1)(a-2)\cdots(a-j+1), \quad j = 2, 3, \dots . \tag{3.3}$$

Interestingly, the sum in (3.2) is identical to the k-term asymptotic approximation [4, Eq. 8.11.2] of $\mathrm{e}^t\Gamma(a,t)$ for large t, its coefficients a_j in this reference being given both in terms of the Pochhammer symbol as $(-1)^{j-1}(1-a)_{j-1}$ and as in (3.3).[2]

4. Applications

Some of the preceding work has implications to the theory of the incomplete gamma function, as will be discussed in this section.

4.1. **Inequalities.** Just like in (3.1), where the integrand was shown to have the constant sign $(-1)^k$, we conjecture that the same is true for the integrand in (3.2). This gives rise to the following infinite set of inequalities,

$$\mathrm{e}^t\,\Gamma(a,t) \begin{cases} < \displaystyle\sum_{j=1}^k a_j t^{a-j}, & t > 0,\ 0 \leq a < 1,\ k \text{ odd}, \\ > \displaystyle\sum_{j=1}^k a_j t^{a-j}, & t > 0,\ 0 \leq a < 1,\ k \text{ even}. \end{cases} \tag{4.1}$$

(Note that we left out $a = 1$ in (4.1) because the sums on the right then collapse to their very first term $a_1 = 1$, all a_j, $j \geq 2$, being zero by (3.3). The same value appears on the left: $\mathrm{e}^t\,\Gamma(1,t) = \mathrm{e}^t\,\mathrm{e}^{-t} = 1$.) The first inequality (for $k = 1$) is known, see [4, Eq. 8.10.1], and so is the second (for $k = 2$), see the upper inequality in [4, Eq. 8.10.3] when $0 \leq a < 1$. The others have been verified by computation for $a = 0 : .1 : .9$, $t = .2 : .2 : 5,\ 5 : .5 : 15,\ 15 : 1 : 30,\ 30 : 5 : 100$, and $k = 3 : 50$; see `sconj_Ram.m`. The calculations were done in high precision (36-digit arithmetic) since for large t and large k the two sides of (4.1) may agree to as many as 30 digits. (Cf. the observation at the end of §3.)

[2] We may call $(x)_n$ the *ascending* Pochhammer symbol, in contrast to the *descending* Pochhammer symbol $[x]_n$ defined by $[x]_0 = 1$, $[x]_n = x(x-1)\cdots(x-n+1)$, $n = 1, 2, \dots$. Then, (3.3) may be written as $a_1 = [a-1]_0$, $a_j = [a-1]_{j-1}$, $j = 2, 3, \dots$.

4.2. **Power series expansion for the upper incomplete gamma function.** The last remark in §4.1 suggests the validity for large t of the series expansion (cf. the paragraph immediately after (3.3))

$$t^{1-a}\,\mathrm{e}^t\,\Gamma(a,t) = \sum_{n=0}^{\infty} a_{n+1} t^{-n}, \quad 0 \le a < 1, \tag{4.2}$$

where the coefficients a_j are given in (3.3). Actually, this can be proved, not only for $0 \le a < 1$ but for all $a \ge 0$, by combining Eq. 8.5.3 in [4] with Eq. 13.7.10 in [5]. Convergence of the series, if (4.1) holds true, is alternately from above and below. Being a power series in the reciprocal of the variable t, it will converge only for t sufficiently large, say $t \ge t_0$. Values of t_0 as well as the number n of terms required to obtain an accuracy of 15 decimal places are shown in Table 4, produced by `sconv_Ram.m`, for equally spaced values of a in the interval $[0,1)$. Also shown are 15-digit values of the respective limit of the series. While convergence takes place only for t relatively large (about ≥ 37.3), but when it does, it is reasonably fast,

Table 4. Convergence data for the series (4.2)

a	t_0	n	limit	a	t_0	n	limit
0	37.3	37	.974523208086786	.5	34.9	35	.986248931351758
.1	36.9	36	.976776156468323	.6	34.4	33	.988815207100577
.2	36.5	35	.979087803103952	.7	33.7	33	.991419930512530
.3	36.0	35	.981411177293594	.8	32.9	33	.994129155922825
.4	35.5	34	.983809577147394	.9	31.9	31	.996966798745332

as shown by the values of n. (When $a = 0$, the limit in Table 4 should be equal to $t_0 \mathrm{e}^{t_0} E_1(t_0)$, where E_1 is the exponential integral, which in fact is true, to all 15 digits.) When a is a positive integer, the series on the right is finite, having exactly a terms. Otherwise, when a increases, t_0 generally decreases but remains ≥ 16.5, whereas the value of n is between 24 and 55 in the range $0 \le a \le 40$.

Recall that Table 4 has been computed under the assumption that an accuracy of 15 decimal digits is desired. Lowering this accuracy requirement, say to 8-digit accuracy, yields more favorable results for t_0, ranging from 16.0 (for $a = .9$) to 20.9 (for $a = 0$) and correspondingly smaller values of n.

Once the function

$$g(a,t) = t^{1-a}\,\mathrm{e}^t\,\Gamma(a,t)$$

has been obtained on the interval $0 \le a < 1$ by means of (4.2), one can, in principle, use the recurrence relation [4, Eq. 8.8.2]

$$\Gamma(a+1,t) = a\Gamma(a,t) + t^a\,\mathrm{e}^{-t},$$

either as written, or as solved for $\Gamma(a,t)$ on the right, to obtain $g(a,t)$ for any real value of a and appropriately large t. Specifically, we find

$$g(a+1,t) = 1 + \frac{a}{t}\,g(a,t), \quad g(a-1,t) = \frac{t}{1-a}\,(1 - g(a,t))\,. \tag{4.3}$$

Thus, given $a \ge 0$, let

$$a0 = \mathrm{floor}(a), \quad r = a - a0. \tag{4.4}$$

Then the sequence of numbers $\{r+m\}$, $m = 0, 1, \ldots, a0$, is such that the first (for $m = 0$) is in the interval $[0, 1)$ and the last is a. Therefore, computing $y_0 = g(r, t)$ by (4.2), and then, if $a0 > 0$, using the first relation in (4.3), continuing with

$$y_{m+1} = 1 + \frac{r+m}{t}\, y_m, \quad m = 0, 1, \ldots, a0-1, \tag{4.5}$$

yields $y_{a0} = g(a, t)$. (If $a0 = 0$, no recursion is needed since $r = a$ and $g(a, t) = y_0$.) Similarly, if $a < 0$, using the second relation in (4.3), and (4.4),

$$y_0 = g(r, t),\; y_{m+1} = \frac{t}{1-(r-m)}\,(1-y_m), \quad m = 0, 1, \ldots, |a0|-1, \tag{4.6}$$

yields $y_{|a0|} = g(a, t)$.

TABLE 5. $g(a, 40)$ obtained by (4.5) (left) and (4.6) (right)

a	$g(a,40)$	relative error	a	$g(a,40)$	relative error
1.5	1.0123493	6.07e−18	−1.5	.94242650	1.11e−12
3.5	1.0648727	1.35e−20	−3.5	.90084515	2.13e−10
5.5	1.1229823	1.26e−22	−5.5	.86271701	1.44e−08
10.5	1.2992070	2.31e−26	−10.5	.78001330	3.94e−05
20.5	1.8765490	7.27e−31	−20.5	1.0000000	5.29e−01

Table 5 shows the results to 8 digits for selected positive and negative values of a midway between integers (similar results hold otherwise) and $t = 40$. As can be seen, the procedure for positive a is not only very stable but in fact improving in accuracy as one proceeds to larger values of a. Unfortunately, just the opposite is the case when a is negative. Accuracy is rapidly lost, presumably because of small, but continuing, cancellation errors committed in computing $1 - y_m$ in the second equation of (4.6). Using increased-precision arithmetic is not an entirely viable option because it would require a more accurate y_0, which in turn would call for a larger t_0 for the series in (4.2) to converge. See, however, [1, §6.2].

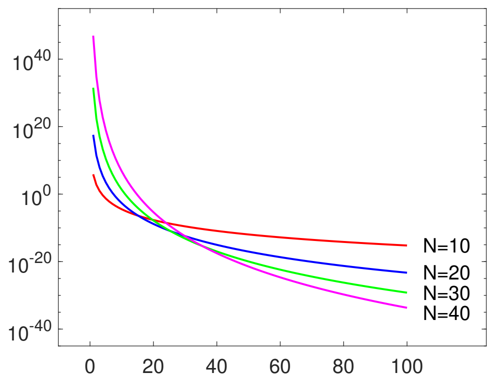

FIGURE 4. Relative error of the Nth partial sum of the series in (4.2) for $a = 1/2$, $0 \le t \le 100$, and $N = 10 : 10 : 40$

The relative error of the partial sum from $n = 0$ to $n = N$ of the series in (4.2) is shown in Fig. 4 for $a = 1/2$, $1 \le t \le 100$, and $N = 10 : 10 : 40$. The figure was produced by `plot_Ram_series.m` in 32-digit arithmetic. For comparison, in the next two subsections we also look briefly at other known series expansions.

4.3. **Power series expansion for the lower incomplete gamma function.** The series of interest here is the power series [4, Eq. 8.7.3], written in the form

$$\mathrm{e}^t\,\gamma^*(a,t) = \sum_{n=0}^{\infty} \frac{t^n}{\Gamma(a+n+1)}, \quad t>0,\ a\neq -1,\ a\neq -2,\ldots . \tag{4.7}$$

(For $\gamma^*(a,t)$ see [4, Eq. 8.2.7].) The convergence properties of this series are illustrated in Fig. 5, on the left for $a=1$ and on the right for $a=10$. In either case it looks like $N=100$ terms suffice to get 15-digit accuracy, or better, when $t\leq 40$. For larger values of t we have the series (4.2), which yields the same accuracy even more efficiently; see §4.2.

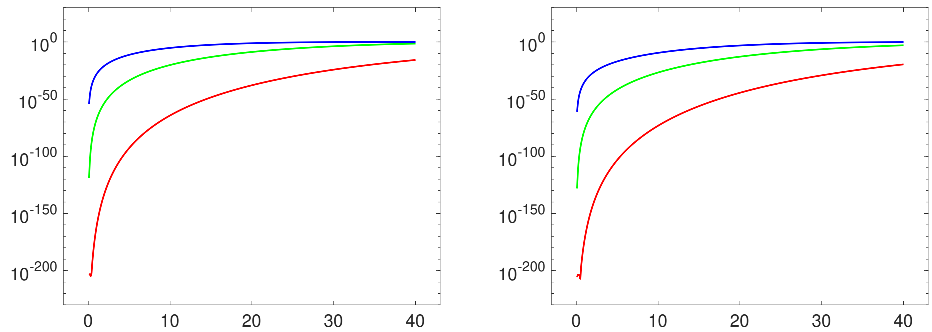

FIGURE 5. Relative error of the Nth partial sum of the series in (4.7) for $a=1$ (left) and $a=10$ (right) where $0\leq t\leq 40$ and $N=25$ (blue), $N=50$ (green), and $N=100$ (red)

Fig. 5 was produced in very high-precision arithmetic (`dig` $=200$) by the Matlab function `plot_serr_ser_pow.m` (which may take about 15 minutes to run). Very similar results are obtained for negative a, say $a=-1.5$ or $a=-10.5$.

4.4. **Series expansion in Laguerre polynomials.** The series in question is [4, Eq. 8.7.6]

$$t^{-a}\,\mathrm{e}^t\,\Gamma(a,t) = \sum_{n=0}^{\infty} \frac{L_n^{(a)}(t)}{n+1}, \quad a<1/2,\ t>0. \tag{4.8}$$

Here, there are serious problems with convergence when $|a|$ is small; see the case $a=0$ (which is typical) on the left in Fig. 6, where a thousand terms are seen to barely produce two correct digits when t is small, and none at all when $t=10$. Convergence improves somewhat, though remains slow, when a is larger negative; see the figure on the right for $a=-10$. Obviously, the series is unsuitable for computational purposes.

The figure on the left of Fig. 6 was generated by the Matlab function `plot_err_Lagser.m` and the one on the right by `plot_serr_Lagser.m` in 32-digit arithmetic (which may take about 19 minutes to run).

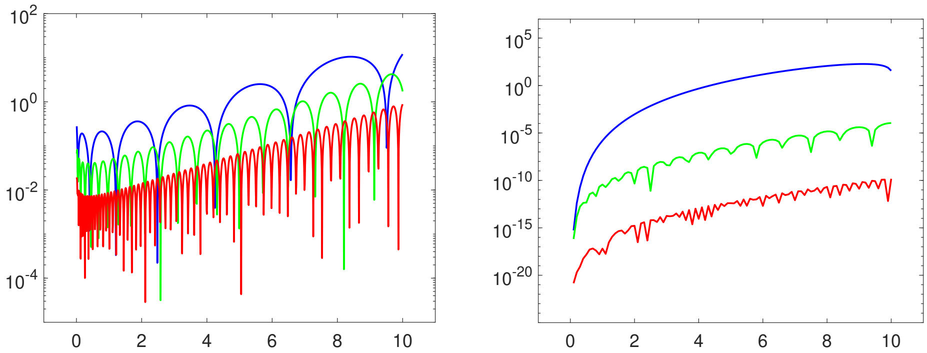

FIGURE 6. Relative error of the partial sum from $n = 0$ to $n = N$ in (4.8), where $N = 10$ (blue), $N = 100$ (green), and $N = 1000$ (red), and $0 \le t \le 10$. The figure on the left is for $a = 0$ and the one on the right for $a = -10$

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PURDUE UNIVERSITY, WEST LAFAYETTE, IN 47907-2066, USA
E-mail address: wgautschi@purdue.edu

SERBIAN ACADEMY OF SCIENCES AND ARTS, 11000 BELGRADE, KNEZA MIHAILA 35, SERBIA AND UNIVERITY OF NIŠ, FACULTY OF SCIENCES AND MATHEMATICS, 18000 NIŠ, SERBIA
E-mail address: gvm@mi.sanu.ac.rs