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ON CONSTRUCTION OF ONE CUBATURE FORMULA FOR TRIANGLE

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Abstract. Using the transformation (2.1) and applying a generalized product rule, a cubature formula for the triangular domain $T=\{(x,y)\mid x+y\le 1, x,y\ge 0\}$ is derived. The obtained formula contains n^2 knots and its degree of precision is 2n-1. A numerical example is included.

1. Introduction

There are several papers related to numerical integration over triangle, eg. [6],[7],[8],[9](see, also the monographs [2], [10], and [11]).

The cubature formula

$$(1.1)\quad \iint_T f(x,y)w(x,y)dxdy \simeq \sum_{i=1}^{n}\sum_{j=1}^{n} B_{ij}f(x_{ij},y_{ij})$$

of degree 2n-1, with n^2 knots and the weight function

$$(1.2)\quad w(x,y)=x^{p-1}y^{q-1}(x+y)^a(1-x-y)^b,\quad p,q>0,\ p+q+a>0,\ b>-1,$$

over the triangle $T=\{(x,y)\mid x+y\le 1,\ x\ge 0,\ y\ge 0\}$ has been developed in the paper [1]. Namely, using the polynomials of the form

$$z_n(p,q;x,y) = \sum_{j=0}^{n}(-1)^j\binom{n}{j}(p+n-j)_j(q+j)_{n-j}x^{n-j}y^j,$$

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where $(p)_k$ Pochhammer's symbol, $(p)_k = p(p+1)\ldots(p+k-1)$, in the paper [1] the following algorithm is given:

(a) Solve the system of equations

$$
(1.3)\qquad
\begin{aligned}
c_0 I_{\Sigma_0} &+ c_1 I_{\Sigma_1} + \ldots + c_{n-1} I_{\Sigma_{n-1}} + I_{\Sigma_n} = 0,\\
c_0 I_{\Sigma_1} &+ c_1 I_{\Sigma_2} + \ldots + c_{n-1} I_{\Sigma_n} + I_{\Sigma_{n+1}} = 0,\\
&\vdots\\
c_0 I_{\Sigma_{n-1}} &+ c_1 I_{\Sigma_n} + \ldots + c_{n-1} I_{\Sigma_{2n-2}} + I_{\Sigma_{2n-1}} = 0,
\end{aligned}
$$

where $I_{\Sigma_k} = \iint\limits_T (x+y)^k w(x,y)\,dxdy \ (k=0,1,\ldots)$;

(b) Determine all zeros of the polynomial

$$P_n(t) = t^n + c_{n-1}t^{n-1} + \ldots + c_1 t + c_0 \ ;$$

(c) Determine all roots $k_1,\ldots,k_n$ of the algebraic equation

$$(1.4)\qquad Z_n(p,q;k,1) = 0 \qquad (k = x/y);$$

(d) Determine the knots x_{ij} and y_{ij} from $x+y = t_i$, $x/y = k_j$; $(i,j = 1,\ldots,n)$;

(e) Determine the coefficients B_{ij} from the system of equations

$$(1.5)\qquad \sum_{i=1}^{n}\sum_{j=1}^{n} B_{ij} x_{ij}^{k} y_{ij}^{m} = I_{km}, \qquad (k,m) \in S(n),$$

where $I_{km} = \iint\limits_T x^k y^m w(x,y)\,dxdy$ and $S(n) = \{(0,0),(1,0),(0,1),(2,0),(1,1),(0,2),\ldots,(0,2n-1)\}$.

This algorithm is very complicated and numerically unstable. Namely, the systems of equations (1.3) and (1.5) are ill-conditioned. Also, the zeros of the algebraic equation (1.4) are distributed in $(0,\infty)$ and are hardly to determine with the satisfactory accuracy.

In this paper we will give a stable and simple algorithm for construction of these cubature formulas. Also, a numerical example will be included.

2. Algorithm

Let Q be a square, $Q = \{(u,v) \mid -1 \leq u,v \leq 1\}$. Using the transformation $F:(u,v) \to (x,y)$, given by

$$x = \frac{1}{4}(1+u)(1+v), \quad y = \frac{1}{4}(1+u)(1-v), \tag{2.1}$$

the square Q maps to the triangle T, so that we can consider an integration over the square Q instead of the triangle T. Then we can apply the standard theory of Gauss-Christoffel quadratures (see Gautschi [3],[4],[5]) and use a generalized product rules (GPR).

At first, we note

(a) The Jacobian of the transformation (2.1) is

$$J = \frac{D(x,y)}{D(u,v)} = -\frac{1}{8}(1+u);$$

(b) $w(x,y)dxdy = 2^{-s}w_1(u)du\, w_2(v)dv$, where $s = a+b+2p+2q-1$ and

$$w_1(u) = (1-u)^b(1+u)^{p+q+a-1}, \quad w_2(v) = (1-v)^{q-1}(1+v)^{p-1}.$$

These weights correspond to classical Jacobi orthogonal polynomials. Let $\hat{P}_n^{(\alpha,\beta)}(t)$ be monic Jacobi polynomials orthogonal on (-1,1) with respect to the weight function $t \to (1-t)^{\alpha}(1+t)^{\beta}$, where $\alpha,\beta > -1$. Then the above mentioned polynomials are

$$\hat{P}_n^{(b,p+q+a-1)}(u) \quad \text{and} \quad \hat{P}_n^{(q-1,p-1)}(v).$$

The monic Jacobi polynomials satisfy the three-term recurrence relation

$$\hat{P}_{k+1}^{(\alpha,\beta)}(t) = (t-\alpha_k)\hat{P}_k^{(\alpha,\beta)}(t) - \beta_k\hat{P}_{k-1}^{(\alpha,\beta)}(t), \quad k=0,1,\ldots,$$

$$\hat{P}_{-1}^{(\alpha,\beta)}(t) = 0, \quad \hat{P}_0^{(\alpha,\beta)}(t) = 1,$$

where the coefficients α_k and β_k are given by

$$\alpha_k = \frac{\beta^2 - \alpha^2}{(2k+\alpha+\beta)(2k+\alpha+\beta+2)} \quad \text{and} \quad \beta_k = \frac{4k(k+\alpha)(k+\beta)(k+\alpha+\beta)}{(2k+\alpha+\beta)^2((2k+\alpha+\beta)^2-1)}.$$

In the n-point Gaussian quadrature

$$(2.2) \qquad \int_{-1}^{1} g(t)(1-t)^{\alpha}(1+t)^{\beta}dt = \sum_{i=1}^{n} A_i g(t_i) + R_n(g), \; R_n(\mathbb{P}_{2n-1}) = 0,$$

the nodes t_i and the weights A_i can be easily obtained from the corresponding Jacobi matrix

$$(2.3) \qquad J_n = \begin{bmatrix} \alpha_0 & \sqrt{\beta_1} & & & O \\ \sqrt{\beta_1} & \alpha_1 & \sqrt{\beta_2} & & \\ & \sqrt{\beta_2} & \ddots & \ddots & \\ & & \ddots & \ddots & \sqrt{\beta_{n-1}} \\ O & & & \sqrt{\beta_{n-1}} & \alpha_{n-1} \end{bmatrix}.$$

The nodes t_i are the eigenvalues of J_n and the weights are given by $A_i = \mu_0 v_{i1}^2$, where μ_0 is the moment of degree zero, and v_{i1} is the first component of the normalized eigenvector v_i corresponding to the eigenvalue t_i. The eigensystem of (2.3) is efficiently calculated by the QL algorithm with shifts.

For our product cubature formula we need two Gauss-Jacobi quadrature formulas (2.2) for

(1) $\alpha = b$, $\beta = p+q+a-1$;

(2) $\alpha = q-1$, $\beta = p-1$.

Let their parameters be (t_i', A_i') and (t_i'', A_i''), $i=1,\dots,n$. Then we have

$$(2.4) \qquad \iint_T f(x,y)w(x,y)dxdy \simeq C_n(f) = \sum_{i=1}^{n}\sum_{j=1}^{n} B_{ij} f(x_{ij}, y_{ij}),$$

where

$$B_{ij} = \frac{1}{2^s} A_i' A_j'', \quad x_{ij} = \frac{1}{4}(1+t_i')(1+t_j''), \quad y_{ij} = \frac{1}{4}(1+t_i')(1-t_j'').$$

The formula (2.4) is exact for all polynomials of degree at most 2n-1, i.e., for all monomials 1; x, y; x^2, xy, y^2; ...; x^{2n-1}, $x^{2n-2}y$, ..., y^{2n-1}. Of course, this formula is not with minimal number of knots.

For $w(x,y)=1$, the formula (2.4) can be found in the book of Stroud [11, pp. 28-31].

In 1976 F. Lether [7] gave a family of Gauss-Legendre GPR for the triangle T. These cubature rules require n^2 evaluations of f and are exact whenever f is a polynomial in x and y of degree $\leq 2n-2$. It is one less than Stroud's formula. However, these formulas are more convinient to program for a computer, because they require the storage of 2n fewer weights and abscissas.

This advantage have our formulas (2.4) for a restricted class of the weight functions. Of course, the degree of precision is still 2n-1.

If we put $a=-q$, $b=q-1$, then (1.2) becomes

$$\text{(2.5)} \qquad w(x,y) = x^{p-1}y^{q-1}\,\frac{(1-x-y)^{q-1}}{(x+y)^q} \qquad (p,q>0).$$

Then, in our formula (2.4),

$$t_i' = t_i'' = t_i \quad \text{and} \quad A_i' = A_i'' = A_i \qquad (i=1,\dots,n)$$

so that

$$B_{ij} = 2^{-s}A_iA_j,\ x_{ij} = \frac{1}{4}(1+t_i)(1+t_j),\ y_{ij} = \frac{1}{4}(1+t_i)(1-t_j),$$

where $s = 2(p+q-1)$. Of course, Stroud's case $w(x,y)=1$ ($p=q=1$, $a=b=0$) can not be got from (2.5).

At the end we give a numerical example. Let

$$I(f) = \iint_T \sqrt{\frac{x}{y}}\cdot\frac{(x+y)^{3/2}}{\sqrt[4]{1-x-y}}\sin\pi x\sin\pi y\,dxdy = 0.16929936085881\dots$$

Using the formula (2.4) for n=2(1)7 we obtain the results with the corresponding errors $R_n(f) = I(f) - C_n(f)$ given in the following table. (Numbers in parentheses denote decimal exponents).

n	Error $R_n(f)$
2	1.6(-3)
3	-2.8(-5)
4	2.9(-7)
5	-1.9(-9)
6	8.5(-12)
7	-2.8(-14)

REMARK. This algorithm can be applied to more general weight functions, for example $w(x,y)=x^{p-1}y^{q-1}U(x+y)$, where p, q, and U have to be such that the all moments of weight function exist.

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O KONSTRUKCIJI KUBATURNE FORMULE ZA TROUGAONU OBLAST

Korišćenjem transformacije $x=(1+u)(1+v)/4$, $y=(1+u)(1-v)/4$ i primenom generalisanog produktnog pravila Gauss-Jacobievog tipa, izvedena je kubaturna formula za trougaonu oblast $T = \{(x,y) \mid x+y \leq 1,\ x,y \geq 0\}$. Dobijena kubaturna formula sadrži n^2 čvorova, a njen stepen tačnosti je $2n-1$. Primena formule se ilustruje numeričkim primerom.

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