A CHARACTERIZATION OF THE CLASSICAL ORTHOGONAL POLYNOMIALS

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The classical orthogonal polynomials \( Q_n \) can be specified as the Jacobi polynomials \( P_n^{(\alpha,\beta)}(t) \) \((\alpha,\beta > -1)\), the generalized Laguerre polynomials \( L_n^s(t) \) \((s > -1)\), and finally as the Hermite polynomials \( H_n(t) \). Their weight functions \( t \mapsto w(t) \) on an interval of orthogonality \((a,b)\) satisfy the differential equation

\[
\frac{d}{dt}(A(t)w(t)) = B(t)w(t),
\]

where the functions \( t \mapsto A(t) \) and \( t \mapsto B(t) \) are defined as in Table 1.

Table 1. The Classification of the Classical Orthogonal Polynomials

<table>
<thead>
<tr>
<th>((a,b))</th>
<th>(w(t))</th>
<th>(A(t))</th>
<th>(B(t))</th>
<th>(\lambda_n)</th>
<th>(Q_n(t))</th>
</tr>
</thead>
<tbody>
<tr>
<td>((-1,1))</td>
<td>((1-t)\alpha(1+t)\beta)</td>
<td>(1-t^2\beta - \alpha - (\alpha + \beta + 2)t)</td>
<td>(n(n + \alpha + \beta + 1))</td>
<td>(P_n^{(\alpha,\beta)}(t))</td>
<td></td>
</tr>
<tr>
<td>((0,\infty))</td>
<td>(t^s e^{-t})</td>
<td>(t)</td>
<td>(s + 1 - t)</td>
<td>(n)</td>
<td>(L_n^s(t))</td>
</tr>
<tr>
<td>((-\infty,\infty))</td>
<td>(e^{-t^2})</td>
<td>(1)</td>
<td>(-2t)</td>
<td>(2n)</td>
<td>(H_n(t))</td>
</tr>
</tbody>
</table>

The classical orthogonal polynomial \( t \mapsto Q_n(t) \) is a particular solution of the following differential equation of the second order

\[
L[y] = A(t)y'' + B(t)y' + \lambda_n y = 0,
\]

where \(\lambda_n\) is given in the above table.

Let \( (f,g) = \int_a^b f(t)g(t)w(t)dt \) and \( \|f\|^2 = (f,f) \), and let \( P_n \) be the set of all algebraic polynomials of degree at most \(n\). Similarly to the well-known Landau inequality [5] for continuously-differentiable functions and other generalizations (see, for example, [1–4] and [6–8]), in this short note we state the following characterization of the classical orthogonal polynomials.

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Theorem. For all $P_n \in \mathcal{P}_n$ the inequality
\begin{equation}
(2) \quad (2\lambda_n + B'(0))\|\sqrt{A}P'_n\|\leq \lambda_n^2 \|P_n\|^2 + \|AP''_n\|^2
\end{equation}
holds, with equality if only if $P_n(t) = cQ_n(t)$, where $Q_n$ is the classical orthogonal polynomial of degree $n$ orthogonal to all polynomials of degree $\leq n - 1$ with respect to the weight function $t \mapsto w(t)$ on $(a, b)$, and $c$ is an arbitrary real constant. The $\lambda_n$, $A(t)$ and $B(t)$ are given in Table 1.

Proof. Using (1) we have
\begin{align*}
\|L[P_n]\|^2 &= \|AP''_n\|^2 + \|BP'_n\|^2 + \lambda_n^2 \|P_n\|^2 \\
&\quad + 2(AP''_n, BP'_n) + 2\lambda_n(AP''_n, P_n) + 2\lambda_n(BP'_n, P_n).
\end{align*}

A simple application of integration by parts gives
\begin{align*}
2(AP''_n, BP'_n) &= -B'(0)\|\sqrt{A}P'_n\|^2 - \|BP'_n\|^2
\end{align*}
and
\begin{align*}
\|\sqrt{A}P'_n\|^2 &= -(AP''_n, P_n) - (BP'_n, P_n).
\end{align*}

Then, we find
\begin{align*}
\|L[P_n]\|^2 &= \|AP''_n\|^2 - B'(0)\|\sqrt{A}P'_n\|^2 + \lambda_n^2 \|P_n\|^2 - 2\lambda_n \|\sqrt{A}P'_n\|^2.
\end{align*}

Since $\|L[P_n]\| \geq 0$, we obtain (2).

It is easy to see that the equality case is given by $P_n(t) = cQ_n(t)$. Namely, the polynomial solution of the equation (1) is only $cQ_n(t)$, where $c$ is a constant. □

Now, we give the special cases.

First, for $w(t) = e^{-t^2}$ on $(-\infty, +\infty)$, the inequality (2) reduces to Varma’s result [9]:
\begin{equation}
\|P'_n\|^2 \leq \frac{1}{2(2n-1)} \|P''_n\|^2 + \frac{2n^2}{2n-1} \|P_n\|^2.
\end{equation}

In the generalized Laguerre case, the inequality (2) becomes
\begin{equation}
\|\sqrt{t}P'_n\|^2 \leq \frac{n^2}{2n-1} \|P_n\|^2 + \frac{1}{2n-1} \|tP''_n\|^2,
\end{equation}
where $w(t) = t^se^{-t}$ on $(0, +\infty)$.

In the Jacobi case $(A(t) = 1-t^2, w(t) = (1-t)\alpha(1+t)\beta$ on $(-1, 1))$ the inequality (2) reduces to the following inequality
\begin{align*}
&((2n-1)(\alpha + \beta) + 2(n^2 + n - 1))\|\sqrt{1-t^2}P'_n\|^2 \\
&\leq n^2(n + \alpha + \beta + 1)^2 \|P_n\|^2 + \|(1-t^2)P''_n\|^2.
\end{align*}
In the simplest case, when \( \alpha = \beta = 0 \) (Legendre case), we obtain
\[
\| \sqrt{1 - t^2} P'_n \| \leq \frac{n^2(n + 1)^2}{2(n^2 + n - 1)} \| P_n \|^2 + \frac{1}{2(n^2 + n - 1)} \| (1 - t^2) P''_n \|^2.
\]

In Chebyshev case \( (\alpha = \beta = -1/2) \), we get
\[
\| \sqrt{1 - t^2} P'_n \| \leq \frac{n^4}{2n^2 - 1} \| P_n \|^2 + \frac{1}{2n^2 - 1} \| (1 - t^2) P''_n \|^2,
\]
where \( \| f \|^2 = \int_{-1}^{1} (1 - t^2)^{-1/2} f(t)^2 \, dt \).

References

8. S. B. Steckhin, Inequalities between the upper bounds of the derivatives of an arbitrary function on the half-line, Mat. Zametki 1 (1967), 665-574. (Russian)