RANK FACTORIZATION AND
MOORE-PENROSE INVERSE

Gradimir V. Milovanović and Predrag S. Stanimirović

Abstract. In this paper we develop a few representations of the Moore-Penrose inverse, based on full-rank factorizations of matrices. These representations we divide into the two different classes: methods which arise from the known block decompositions and determinantal representation. In particular cases we obtain several known results.

1. Introduction

The set of $m \times n$ complex matrices of rank $r$ is denoted by $\mathbb{C}^{m \times n}_r = \{ X \in \mathbb{C}^{m \times n} : \text{rank}(X) = r \}$. With $A|_r$ and $A|_c$, we denote the first $r$ columns of $A$ and the first $r$ rows of $A$, respectively. The identity matrix of the order $k$ is denoted by $I_k$, and $\mathbb{O}$ denotes the zero block of an appropriate dimensions.

We use the following useful expression for the Moore-Penrose generalized inverse $A^\dagger$, based on the full-rank factorization $A = PQ$ of $A$ [1-2]:

$$A^\dagger = Q^\dagger P^\dagger = Q^\ast (QQ^\ast)^{-1} (P^\ast P)^{-1} P^\ast = Q^\ast (P^\ast A Q^\ast)^{-1} P^\ast.$$

We restate main known block decompositions [7], [16-18]. For a given matrix $A \in \mathbb{C}^{m \times n}_r$ there exist regular matrices $R$, $G$, permutation matrices $E$, $F$ and unitary matrices $U$, $V$, such that:

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\( (T_1) \quad RAG = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} = N_1, \quad (T_2) \quad RAG = \begin{bmatrix} B & 0 \\ 0 & 0 \end{bmatrix} = N_2, \)

\( (T_3) \quad RAF = \begin{bmatrix} I_r & K \\ 0 & 0 \end{bmatrix} = N_3, \quad (T_4) \quad EAG = \begin{bmatrix} I_r & 0 \\ K & 0 \end{bmatrix} = N_4, \)

\( (T_5) \quad UAG = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} = N_1, \quad (T_6) \quad RAV = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} = N_1, \)

\( (T_7) \quad UAV = \begin{bmatrix} B & 0 \\ 0 & 0 \end{bmatrix} = N_2, \quad (T_8) \quad UAF = \begin{bmatrix} B & 0 \\ K & 0 \end{bmatrix} = N_5, \)

\( (T_9) \quad EAV = \begin{bmatrix} B & 0 \\ 0 & K \end{bmatrix} = N_6, \)

\( (T_{10}) \quad EAF = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = N_7, \) where \( \text{rank}(A_{11}) = \text{rank}(A). \)

\( (T_{11}) \quad \text{Transformation of similarity for square matrices [11]:} \)

\[
RAR^{-1} = RAFF^*R^{-1} = \begin{bmatrix} I_r & K \\ 0 & 0 \end{bmatrix} F^*R^{-1} = \begin{bmatrix} T_1 & T_2 \\ 0 & 0 \end{bmatrix}.
\]

The block form \( (T_{10}) \) can be expressed in two different ways:

\( (T_{10a}) \quad EAF = \begin{bmatrix} A_{11} & A_{12}T \\ SA_{11} & SA_{11}T \end{bmatrix}, \) where the multipliers \( S \) and \( T \) satisfy \( T = A_{11}^{-1}A_{12}, \quad S = A_{21}A_{11}^{-1} \) (see [8]);

\( (T_{10b}) \quad EAF = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22}A_{11}^{-1}A_{12} \end{bmatrix} \) (see [9]).

Block representations of the Moore-Penrose inverse is investigated in [8], [11], [15–19]. In [16], [18] the results are obtained by solving the equations (1)–(4). In [15], [8] the corresponding representations are obtained using the block decompositions \( (T_{10a}) \) and \( (T_{10b}) \) and implied full-rank factorizations.

Also, in [19] is introduced block representation of the Moore-Penrose inverse, based on \( A^\dagger = A^*TA^* \), where \( T \in A^*AA^* \{1\} \).

Block decomposition \( (T_{11}) \) is investigated in [11], but only for square matrices and the group inverse.

The notion \textit{determinantal representation} of the Moore-Penrose inverse of \( A \) means representation of elements of \( A^\dagger \) in terms of minors of \( A \). Determinantal representation of the Moore-Penrose inverse is examined in [1–2], [4–6], [12–14]. For the sake of completeness, we restate here several notations and the main result. For an \( m \times n \) matrix \( A \) let \( \alpha = \{\alpha_1, \ldots, \alpha_r\} \) and \( \beta = \{\beta_1, \ldots, \beta_s\} \) be subsets of \( \{1, \ldots, m\} \) and \( \{1, \ldots, n\} \), respectively.
Then $A\left(\alpha_1 \ldots \alpha_r \beta_1 \ldots \beta_r\right) = |A_{\alpha \beta}|$ denotes the minor of $A$ determined by the rows indexed by $\alpha$ and the columns indexed by $\beta$, and $A_{\alpha \beta}$ represents the corresponding submatrix. Also, the algebraic complement of $A_{\alpha \beta}$ is defined by

$$\frac{\partial}{\partial a_{ij}} |A_{\alpha \beta}| = A_{ij} \left(\alpha_1 \ldots \alpha_{p-1} i \alpha_{p+1} \ldots \alpha_r \beta_1 \ldots \beta_{q-1} j \beta_{q+1} \ldots \beta_r\right) = (-1)^{p+q} A_{ij} \left(\alpha_1 \ldots \alpha_{p-1} \alpha_{p+1} \ldots \alpha_r \beta_1 \ldots \beta_{q-1} \beta_{q+1} \ldots \beta_r\right).$$

Adjoint matrix of a square matrix $B$ is denoted by $\text{adj}(B)$, and its determinant by $|B|$. Determinantal representation of full rank matrices is introduced in [1], and for full-rank matrices in [4–6]. In [12–14] is introduced an elegant derivation for determinantal representation of the Moore-Penrose inverse, using a full-rank factorization and known results for full-rank matrices. Main result of these papers is:

**Proposition 1.1.** The $(i, j)$th element of the Moore-Penrose inverse $G = (g_{ij})$ of $A \in \mathbb{C}^{m \times n}$ is given by

$$g_{ij} = \frac{\sum_{\alpha : j \in \alpha; \beta : i \in \beta} |A_{\alpha \beta}| \frac{\partial}{\partial a_{ji}} |A_{\alpha \beta}|}{\sum_{\gamma, \delta} |A_{\gamma \delta}|}. $$

In this paper we investigate two different representations of the Moore-Penrose inverse. The first class of representations is a continuation of the papers [8] and [15]. In other words, from the presented block factorizations of matrices find corresponding full-rank decompositions $A = PQ$, and then apply $A^\dagger = Q^* (P^* AQ^*)^{-1} P^*$. In the second representation, $A^\dagger$ is represented in terms of minors of the matrix $A$. In this paper we describe an elegant proof of the well-known determinantal representation of the Moore-Penrose inverse. Main advantages of described block representations are their simply derivation and computation and possibility of natural generalization. Determinantal representation of the Moore-Penrose inverse can be implemented only for small dimensions of matrices ($n \leq 10$).

**2. Block representation**

**Theorem 2.1.** The Moore-Penrose inverse of a given matrix $A \in \mathbb{C}^{m \times n}$ can be represented as follows, where block representations $(G_i)$ correspond to the block decompositions $(T_i)$, $i \in \{1, \ldots, 9, 10a, 10b, 11\}$:

$$(G_1) \quad A^\dagger = (G^{-1\mid r})^*(\left(R^{-1\mid r}\right)^* A (G^{-1\mid r})^*)^{-1} \left(R^{-1\mid r}\right)^*,$$
\( G_2 \)  
\[ A^\dagger = (G^{-1}\mid_r)^* \left( (R^{-1\mid_r}B)^* \right) \left( A (G^{-1}\mid_r)^* \right)^{-1} \left( R^{-1\mid_r}B \right)^* \],

\( G_3 \)  
\[ A^\dagger = F \left[ I_{r R} \right] \left( (R^{-1\mid_r})^* AF \left[ I_{K^*} \right] \right)^{-1} \left( R^{-1\mid_r} \right)^* , \]

\( G_4 \)  
\[ A^\dagger = (G^{-1}\mid_r)^* \left( [I_{r R}, K^*] \right) \left( E A (G^{-1}\mid_r)^* \right)^{-1} [I_{r R}, K^*] E , \]

\( G_5 \)  
\[ A^\dagger = (G^{-1}\mid_r)^* \left( U_{r R} A (G^{-1}\mid_r)^* \right)^{-1} U_{r R} , \]

\( G_6 \)  
\[ A^\dagger = V^{\mid_r} \left( (R^{-1\mid_r})^* AV^{\mid_r} \right)^{-1} (R^{-1\mid_r})^* , \]

\( G_7 \)  
\[ A^\dagger = V^{\mid_r} (B^{\mid_r} U_{r R} A V^{\mid_r})^{-1} B^{\mid_r} U_{r R} , \]

\( G_8 \)  
\[ A^\dagger = F \left[ B^{\mid_r} \right] \left( U_{r R} AF \left[ B^{\mid_r} \right] \right)^{-1} U_{r R} , \]

\( G_9 \)  
\[ A^\dagger = V^{\mid_r} \left( [B^{\mid_r}, K^*] E AV^{\mid_r} \right)^{-1} [B^{\mid_r}, K^*] E , \]

\( G_{10a} \)  
\[ A^\dagger = F \left[ I_{T^*} \right] \left( A_{11}^* \left[ I_{r R}, S^* \right] E A F \left[ I_{T^*} \right] \right)^{-1} A_{11}^* \left[ I_{r R}, S^* \right] E \]
\[ = F \left[ I_{r T^*} \right] \left( I_{r R} + T T^* \right)^{-1} A_{11}^* \left[ I_{r R} + S^* S \right]^{-1} \left[ I_{r R}, S^* \right] E , \]

\( G_{10b} \)  
\[ A^\dagger = F \left[ A_{11}^* \right] \left( A_{11}^* \right)^{-1} \left[ A_{11}^*, A_{21}^* \right] E A F \left[ A_{11}^* \right]^{-1} \left( A_{11}^* \right)^{-1} \left[ A_{11}^*, A_{21}^* \right] E \]
\[ = F \left[ A_{11}^* \right] \left( A_{11} + A_{12} A_{21}^* \right)^{-1} A_{11}^* \left[ A_{11} + A_{21} A_{21}^* \right] \left[ A_{11}^*, A_{21}^* \right] E , \]

\( G_{11} \)  
\[ A^\dagger = R^* \left\{ \left[ \frac{I_{r}}{T_1^{-1} T_2^*} \right] \left( (R^{-1\mid_r})^* A R^* \left[ \frac{I_{r}}{T_1^{-1} T_2^*} \right] \right) \right\}^{-1} (R^{-1\mid_r} T_1)^* . \]

**Proof.** (\( G_1 \)) Starting from \( (T_1) \), we obtain
\[ A = R^{-1} \left[ \frac{I_{r}}{\circ} \right] G^{-1} = R^{-1} \left[ \frac{I_{r}}{\circ} \right] [I_{r R}, \circ] G^{-1} , \]
which implies
\[ P = R^{-1} \left[ \frac{I_{r}}{\circ} \right] = R^{-1\mid_r} , \quad Q = [I_{r R}, \circ] G^{-1} = G^{-1} \mid_r . \]

Now, we get
\[ A^\dagger = Q^* \left( P^* A Q^* \right)^{-1} P^* = (G^{-1}\mid_r)^* \left( (R^{-1\mid_r})^* A (G^{-1}\mid_r)^* \right)^{-1} (R^{-1\mid_r})^* . \]
The other block decompositions can be obtained in a similar way.  

(G5) Block decomposition (T5) implies  
\[ A = U^* \begin{bmatrix} I_r & \mathbb{O} \\ \mathbb{O} & \mathbb{O} \end{bmatrix} G^{-1} = U^* \begin{bmatrix} I_r \\ \mathbb{O} \end{bmatrix} [I_r, \mathbb{O}] G^{-1}, \]
which means  
\[ P = U^* \begin{bmatrix} I_r \\ \mathbb{O} \end{bmatrix} = U^{s\uparrow r}, \quad Q = [I_r, \mathbb{O}] G^{-1} = G^{-1}_{\uparrow r}. \]

(G7) It is easy to see that (T7) implies  
\[ A = U^* \begin{bmatrix} B & \mathbb{O} \\ \mathbb{O} & \mathbb{O} \end{bmatrix} V^* = U^* \begin{bmatrix} B \\ \mathbb{O} \end{bmatrix} [I_r, \mathbb{O}] V^*. \]
Thus,  
\[ P = U^* \begin{bmatrix} B \\ \mathbb{O} \end{bmatrix} = U^{s\uparrow r} B, \quad Q = [I_r, \mathbb{O}] V^* = V^*_{\uparrow r}, \]
which means  
\[ P^* = B^* U_{\uparrow r}, \quad Q^* = V^r. \]

(G10a) From (T10a) we obtain  
\[ A = E^* \begin{bmatrix} A_{11} & A_{11}T \\ S A_{11} & S A_{11}T \end{bmatrix} F^* = E^* \begin{bmatrix} I_r \\ S \end{bmatrix} A_{11} [I_r, T] F^*, \]
which implies, for example, the following full rank factorization of A:  
\[ P = E^* \begin{bmatrix} I_r \\ S \end{bmatrix} A_{11}, \quad Q = [I_r, T] F^*. \]
Now,  
\[ A^\dagger = F \begin{bmatrix} I_r; \\ T^* \end{bmatrix} \left(A_{11}^* [I_r, S^*] E A F \begin{bmatrix} I_r \\ T^* \end{bmatrix}\right)^{-1} A_{11}^* [I_r, S^*] E. \]
The proof can be completed using  
\[ E A F = \begin{bmatrix} I_r \\ S \end{bmatrix} A_{11} [I_r, T]. \]  

Remarks 2.1.  
(i) A convenient method for finding the matrices S, T and A_{11}^{-1}, required in (T10a) was introduced in [8], and it was based on the following extended Gauss-Jordan transformation:
\[
\begin{bmatrix}
A_{11} & A_{12} & I \\
A_{21} & A_{22} & 0
\end{bmatrix} \Rightarrow \begin{bmatrix}
I & T & A_{11}^{-1} \\
0 & 0 & -S
\end{bmatrix}.
\]

(ii) In [3] it was used the following full-rank factorization of \(A\), derived from \(A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{21}A_{11}^{-1}A_{12} \end{bmatrix}\):

\[
P = \begin{bmatrix} A_{11} \\ A_{21} \end{bmatrix}, \quad Q = [I_r, A_{11}^{-1}A_{12}].
\]

3. Determinantal representation

In the following definition are generalized concepts of determinant, algebraic complement, adjoint matrix and determinantal representation of generalized inverses. (see also [14].)

**Definition 3.1.** Let \(A\) be \(m \times n\) matrix of rank \(r\).

(i) The generalized determinant of \(A\), denoted by \(N_r(A)\), is equal to

\[
N_r(A) = \sum_{\alpha, \beta} |A^\alpha_{\beta}| |A^\beta_{\alpha}|,
\]

(ii) **Generalized algebraic complement** of \(A\) corresponding to \(a_{ij}\) is

\[
A^\dagger_{ij} = \sum_{\alpha : j \in \alpha ; \beta : i \in \beta} |A^\alpha_{\beta}| \frac{\partial}{\partial a_{ji}} |A^\beta_{\alpha}|.
\]

(iii) Generalized adjoint matrix of \(A\), denoted by \(\text{adj}^\dagger(A)\) is the matrix whose elements are \(A^\dagger_{ij}\).

For full-rank matrix \(A\) the following results can be proved:

**Lemma 3.1.** [14] If \(A\) is an \(m \times n\) matrix of full-rank, then:

(i) \(N_r(A) = \begin{cases} |AA^*|, & r = m \\ |R^*A|, & r = n. \end{cases}\)

(ii) \(A^\dagger_{ij} = \begin{cases} (A^* \text{adj}(AA^*))_{ij}, & r = m \\ (\text{adj}(A^*A)A^*)_{ij}, & r = n. \end{cases}\)

(iii) \(A^\dagger = \begin{cases} A^*(AA^*)^{-1}, & r = m \\ (A^*A)^{-1}A^*, & r = n. \end{cases}\)

(iv) \(\text{adj}^\dagger(A) = \begin{cases} A^* \text{adj}(AA^*), & r = m \\ \text{adj}(A^*A)A^*, & r = n. \end{cases}\)

Main properties of the generalized adjoint matrix, generalized algebraic complement and generalized determinant are investigated in [14].
Lemma 3.2. [14] If $A = PQ$ is a full-rank factorization of an $m \times n$ matrix $A$ of rank $r$, then

(i) $\text{adj}^\dagger(Q) \cdot \text{adj}^\dagger(P) = \text{adj}^\dagger(A)$;
(ii) $N_r(Q) \cdot N_r(P) = N_r(P) \cdot N_r(Q) = N_r(A)$;

From Lemma 3.1 and Lemma 3.2 we obtain an elegant proof for the determinantal representation of the Moore-Penrose inverse.

Theorem 3.1. Let $A$ be an $m \times n$ matrix of rank $r$, and $A = PQ$ be its full-rank factorization. The Moore-Penrose inverse of $A$ possesses the following determinantal representation:

$$a_{ij}^\dagger = \frac{\sum_{\alpha:j \in \alpha; \beta:i \in \beta} |A_{\alpha\beta}^\dagger| \frac{\partial}{\partial a_{ji}} |A_{\alpha\beta}^\dagger|}{\sum_{\gamma,\delta} |A_{\gamma\delta}^\dagger| |A_{\gamma\delta}^\dagger|}.$$ 

Proof. Using $A^\dagger = Q^\dagger P^\dagger$ [3] and the results of Lemma 3.1 and Lemma 3.2, we obtain:

$$A^\dagger = Q^\dagger P^\dagger = Q^* (QQ^*)^{-1} (P^* P)^{-1} P^* = \frac{Q^* \text{adj}(QQ^*)}{|QQ^*|} \frac{\text{adj}(P^* P) P^*}{|P^* P|} = \frac{\text{adj}^\dagger(Q) \text{adj}^\dagger(P)}{N_r(Q) N_r(P)} = \frac{\text{adj}^\dagger(A)}{N_r(A)}.$$

4. Examples

Example 4.1. Block decomposition $(T_1)$ can be obtained by applying transformation $(T_3)$ two times:

$$R_1 AF_1 = \begin{bmatrix} I_r & K \\ \O & \O \end{bmatrix} = N_3,$$

$$R_2 N_3^T F_2 = \begin{bmatrix} I_r & \O \\ \O & \O \end{bmatrix} = N_1.$$

Then, the regular matrices $R, G$ can be computed as follows:

$$N_1 = N_1^T = F_2^T N_3 R_2^T = F_2^T R_1 A F_1 R_2^T \Rightarrow R = F_2^T R_1, \quad G = F_1 R_2^T.$$ 

For the matrix $A = \begin{pmatrix} -1 & 1 & 3 & 5 & 7 \\ 1 & 0 & -2 & 0 & 4 \\ 1 & 1 & -1 & 5 & 15 \\ -1 & 2 & 4 & 10 & 18 \end{pmatrix}$ we obtain
\[ R_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -1 & -2 & 1 & 0 \\ -2 & -1 & 0 & 1 \end{pmatrix}, \quad F_1 = I_5, \]

\[ R_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & -1 & 1 & 0 \\ 0 & -5 & 0 & 1 \\ -4 & -11 & 0 & 0 & 1 \end{pmatrix}, \quad F_2 = I_4. \]

From \( R = R_1, \ G = R_2^T, \) we get

\[ R^{-1|2} = \begin{pmatrix} -1 & 1 \\ 1 & 0 \\ 1 & 1 \\ -1 & 2 \end{pmatrix}, \quad G^{-1|2} = \begin{pmatrix} 1 & 0 & -2 & 0 & 4 \\ 0 & 1 & 1 & 5 & 11 \end{pmatrix}. \]

Using formula \((G_1)\), we obtain

\[ A^\dagger = \begin{pmatrix} -169 & 6720 & 233 & -137 & -6720 \\ -128 & -128 & -128 & 128 & 128 \\ 781 & 2240 & -1037 & 653 & 13440 \\ 5 & -5 & -5 & 5 & 128 \\ -197 & 151 & 709 & 59 & 13440 \\ -13440 & -13440 & -13440 & -13440 & -13440 \end{pmatrix}. \]

**Example 4.2.** For the matrix \( A = \begin{pmatrix} -1 & 0 & 1 & 2 \\ -1 & 1 & 0 & -1 \\ 0 & -1 & 1 & 3 \\ 0 & 1 & -1 & -3 \\ 1 & -1 & 0 & 1 \\ 1 & 0 & -1 & -2 \end{pmatrix} \) we obtain

\[ A_{11}^{-1} = \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix}, \quad S = \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}, \quad T = \begin{pmatrix} -1 & -2 \\ -1 & -3 \end{pmatrix} [5]. \]

Using \((G_{10a})\) we obtain

\[ A^\dagger = \begin{pmatrix} -5 & 3 & -17 & 3 & -17 \\ 5 & 13 & -102 & 5 & -102 \\ 7 & 102 & -102 & -102 & -102 \\ 1 & -3 & 3 & 3 & 3 \\ 1 & -17 & -3 & 3 & -17 \end{pmatrix}. \]
Example 4.3. For the matrix \( A = \begin{pmatrix} 4 & -1 & 1 & 2 \\ -2 & 2 & 0 & -1 \\ 6 & -3 & 1 & 3 \\ -10 & 4 & -2 & -5 \end{pmatrix} \) we obtain

\[
R = \begin{pmatrix} -1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ 2 & -1 & 0 & 1 \end{pmatrix}, \quad T_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad T_2 = \begin{pmatrix} -1 & -2 \\ -1 & -3 \end{pmatrix}, \quad F = I_4.
\]

Then, the following results can be obtained:

\[
(R^{-1v}T_1)^* = \begin{pmatrix} -1 & -1 & 0 & 1 \\ 0 & 1 & -1 & 1 \end{pmatrix}, \quad R^* \left[ \frac{I_r}{T_1^{-1}T_2} \right]^* = \begin{pmatrix} -4 & -6 \\ 1 & 3 \\ -1 & -1 \\ -2 & -3 \end{pmatrix},
\]

Finally, using \((G_{11})\), we get

\[
A^\dagger = \begin{pmatrix} \frac{8}{81} & \frac{10}{102} & -\frac{2}{31} & -\frac{2}{27} \\ \frac{47}{162} & \frac{79}{162} & \frac{16}{31} & -\frac{5}{54} \\ \frac{7}{54} & \frac{11}{54} & -\frac{2}{27} & -\frac{1}{18} \\ \frac{4}{81} & \frac{5}{81} & -\frac{1}{31} & -\frac{1}{27} \end{pmatrix}.
\]

References


Department of Mathematics, Faculty of Electronic Engineering, P.O.Box 73, Beogradska 14, 18000 Niš

University of Niš, Faculty of Philosophy, Departement of Mathematics, Čirila i Metodija 2, 18000 Niš