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A NOTE ON SHARPENING OF THE ERDŐS-LAX INEQUALITY CONCERNING POLYNOMIALS

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A b s t r a c t. In this paper, we establish some upper bound estimates for the modulus of the derivative of a polynomial on the unit disk while taking into account the placement of the zeros and the extremal coefficients of the polynomial. The obtained results sharpen as well as generalize the Erdős-Lax inequality and its extension by Aziz and Dawood and other related inequalities.

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1. *Introduction*

For an arbitrary entire function f, set $\|f\| = \max_{|z|=1} |f(z)|$, the uniform-norm of f on the unit disk $|z| = 1$. Let $P(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ be an algebraic polynomial of degree n in the complex plane and $P'(z)$ its derivative. The Bernstein-type inequalities relating the uniform-norm of the derivative and the underlying polynomial on the unit circle in the plane play a key role in the literature for proving the inverse theorems in approximation theory, and of course have their own intrinsic interest. These inequalities for constrained polynomials have been the subject of many research papers which is witnessed by many recent articles (for example, see [6], [8, 9], [12]–[15]). A classical result due to Bernstein is that: for two polynomials

$P(z)$ and $Q(z)$ with degree of $P(z)$ not exceeding that of $Q(z)$ and $Q(z) \neq 0$ for $|z| > 1$, the inequality $|P(z)| \leq |Q(z)|$ on the unit circle $|z| = 1$ implies the inequality of their derivatives $|P'(z)| \leq |Q'(z)|$ on $|z| = 1$. In particular, this classical result allows one to establish the famous Bernstein inequality [4] for the uniform-norm on the unit circle. Namely, if $P(z)$ of degree n, it is true that

$$\|P'(z)\| \leq n\|P(z)\|. \tag{1.1}$$

Equality holds in (1.1) if and only if $P(z)$ has all its zeros at the origin. It might easily be observed that the restriction on the zeros of $P(z)$ imply an improvement in (1.1). It turns out that to have any hope of an improved upper bound, one must have some control over the location of the zeros of polynomial $P(z)$. It was conjectured by Erdős and later proved by Lax [10] that if $P(z)$ is a polynomial of degree n and $P(z) \neq 0$ in $|z| < 1$, then

$$\|P'(z)\| \leq \frac{n}{2}\|P(z)\|. \tag{1.2}$$

Inequality (1.2) was further sharpened by Aziz and Dawood [2] in the form of

$$\|P'(z)\| \leq \frac{n}{2}\left\{\|P(z)\| - \min_{|z|=1} |P(z)|\right\}. \tag{1.3}$$

Equality in (1.2) and (1.3) holds for any polynomial which has all its zeros on $|z| = 1$.

Various extension of (1.2) and (1.3) are known in the literature on various regions of the complex plane (for example, see Govil [5], Malik [11], Milovanović et al.[13] and Rahman and Schmeisser [16]). The purpose of this paper is to generalize and strengthen the inequalities (1.2), (1.3) and related results, while taking into account the placement of the zeros and the extremal coefficients of the polynomial.

2. *Main results*

We begin by presenting the following extension and refinement of (1.2).

Theorem 2.1. *If* $P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$ *is a polynomial of degree* n *having no zeros in* $|z| < k$*,* $k \geq 1$*, then for* $|z| = 1$*,*

$$|P'(z)| \leq \frac{\|P\|}{2}\left[n - \left\{n\left(\frac{k-1}{k+1}\right) + \frac{2}{k+1}\left(\frac{|a_0| - k^n|a_n|}{|a_0| + k^n|a_n|}\right)\right\}\frac{|P(z)|^2}{\|P\|^2}\right]. \tag{2.1}$$

Equality in (2.1) *holds for* $P(z) = (z+k)^n$*, evaluated at* $z = 1$*.*

Taking $k = 1$ in (2.1), we get the following refinement of (1.2).

Corollary 2.1. *If* $P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$ *is a polynomial of degree* n *and* $P(z) \neq 0$ *in* $|z| < 1$*, then for* $|z| = 1$*,*

$$|P'(z)| \leq \frac{\|P\|}{2}\left\{n - \left(\frac{|a_0| - |a_n|}{|a_0| + |a_n|}\right)\frac{|P(z)|^2}{\|P\|^2}\right\}. \tag{2.2}$$

Equality in (2.2) *holds for* $P(z) = z^n + 1$.

Remark 2.1. In fact, excepting the case when all the zeros of $P(z)$ lie on unit disk $|z| = 1$, the bound obtained in (2.2) is always sharp than the bound obtained in (1.2).

The following result which was recently proved by Ahanger and Shah [1] immediately follows from Theorem 2.1 as a special case.

Corollary 2.2. *If* $P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$ *is a polynomial of degree* n *having no zeros in* $|z| < k,\ k \geq 1$*, then for* $|z| = 1$*,*

$$|P'(z)| \leq \frac{n\|P\|}{2}\left\{1 - \frac{k-1}{k+1}\cdot\frac{|P(z)|^2}{\|P\|^2}\right\}. \tag{2.3}$$

Equality in (2.3) *holds for* $P(z) = (z + k)^n$*, evaluated at* $z = 1$.

Next, we prove the following extension and sharpening of (1.3), which also provides a generalization of Theorem 2.1.

Theorem 2.2. *If* $P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$ *is a polynomial of degree* n *having no zeros in* $|z| < k,\ k \geq 1$*, then for* $0 \leq t \leq 1$ *and* $|z| = 1$*,*

$$|P'(z)| \leq \frac{\|P\| - tm}{2}\left\{n - \left[\frac{(k-1)n}{k+1} + \frac{2C_n(k,t)}{k+1}\right]\left(\frac{|P(z)| - tm}{\|P\| - tm}\right)^2\right\}, \tag{2.4}$$

where

$$m = \min_{|z|=k}|P(z)| \quad \textit{and} \quad C_n(k,t) = \frac{|a_0| - tm - k^n|a_n|}{|a_0| - tm + k^n|a_n|}.$$

Equality in (2.4) *holds for* $P(z) = (z + k)^n$*, evaluated at* $z = 1$.

For $k = 1$, Theorem 2.2 reduces to the following result which is a refinement of (1.3).

Corollary 2.3. *If* $P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$ *is a polynomial of degree* n *and* $P(z) \neq 0$ *in* $|z| < 1$*, then for* $0 \leq t \leq 1$ *and* $|z| = 1$*,*

$$|P'(z)| \leq \frac{\|P\| - tm'}{2}\left\{n - \left(\frac{|a_0| - tm' - |a_n|}{|a_0| - tm' + |a_n|}\right)\left(\frac{|P(z)| - tm'}{\|P\| - tm'}\right)^2\right\}, \tag{2.5}$$

where

$$m' = \min_{|z|=1} |P(z)|.$$

Equality in (2.5) *holds for* $P(z) = z^n + 1$.

Remark 2.2. For $t = 0$, (2.4) reduces to (2.1) and (2.5) reduces to (2.2).

The following result which is a refinement of (2.3) immediately follows from Theorem 2.2 as a special case.

Corollary 2.4. *If* $P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$ *is a polynomial of degree* n *having no zeros in* $|z| < k$*,* $k \geq 1$*, then for* $0 \leq t \leq 1$ *and* $|z| = 1$*,*

$$|P'(z)| \leq \frac{n(\|P\| - tm)}{2} \left\{ 1 - \frac{k-1}{k+1} \left(\frac{|P(z)| - tm}{\|P\| - tm} \right)^2 \right\}, \tag{2.6}$$

where

$$m = \min_{|z|=k} |P(z)|.$$

Equality in (2.6) *holds for* $P(z) = (z+k)^n$*, evaluated at* $z = 1$*.*

3. *Auxiliary results*

For the proofs of the theorems, we shall make use of the following lemmas.

Lemma 3.1. *If* $x_\nu,\ \nu = 1, 2, \ldots, n$*, is a sequence of real numbers such that for each* $n \in \mathbb{N}$*,* $x_\nu \geq 1$*, then*

$$(\forall\, n \in \mathbb{N}) \qquad \sum_{\nu=1}^{n} \frac{x_\nu - 1}{x_\nu + 1} \geq \frac{\prod_{\nu=1}^{n} x_\nu - 1}{\prod_{\nu=1}^{n} x_\nu + 1}.$$

PROOF. The above lemma follows by using induction on n. We omit the details.

Lemma 3.2. *If* $P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$ *is a polynomial of degree* n *having no zeros in* $|z| < k$*,* $k \geq 1$*, then for each point* z *on* $|z| = 1$ *for which* $P(z) \neq 0$*, we have*

$$\mathit{Re}\left(\frac{zP'(z)}{P(z)} \right) \leq \frac{1}{1+k} \left\{ n - \frac{|a_0| - k^n |a_n|}{|a_0| + k^n |a_n|} \right\}. \tag{3.1}$$

PROOF. Recall that $P(z)$ has all its zeros in $|z| \geq k$, $k \geq 1$. If $z_1, z_2, \dots, z_n$, are the zeros of $P(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ of degree n, then $|z_\nu| \geq k$, $k \geq 1$, and we can write $P(z) = a_n \prod_{\nu=1}^{n} (z - z_\nu)$. This gives

$$\frac{zP'(z)}{P(z)} = \sum_{\nu=1}^{n} \frac{z}{z - z_\nu}.$$

Now for the points $\mathrm{e}^{\mathrm{i}\theta}$, $0 \leq \theta \leq 2\pi$, with $P(\mathrm{e}^{\mathrm{i}\theta}) \neq 0$, we have

$$\begin{aligned}
\mathrm{Re}\left(\frac{\mathrm{e}^{\mathrm{i}\theta} P'(\mathrm{e}^{\mathrm{i}\theta})}{P(\mathrm{e}^{\mathrm{i}\theta})}\right) &= \sum_{\nu=1}^{n} \mathrm{Re}\left(\frac{\mathrm{e}^{\mathrm{i}\theta}}{\mathrm{e}^{\mathrm{i}\theta} - z_\nu}\right) \\
&\leq \sum_{\nu=1}^{n} \frac{1}{1 + |z_\nu|} \\
&= \frac{n}{1+k} - \frac{1}{1+k} \sum_{\nu=1}^{n} \frac{|z_\nu| - k}{|z_\nu| + 1} \\
&\leq \frac{n}{1+k} - \frac{1}{1+k} \sum_{\nu=1}^{n} \frac{|z_\nu| - k}{|z_\nu| + k} \qquad (\text{ as } k \geq 1) \\
&= \frac{n}{1+k} - \frac{1}{1+k} \sum_{\nu=1}^{n} \frac{\frac{|z_\nu|}{k} - 1}{\frac{|z_\nu|}{k} + 1}.
\end{aligned}$$

Since $|z_\nu|/k \geq 1$, $\nu = 1, 2, \dots, n$, we get on using Lemma 3.1 for the points $\mathrm{e}^{\mathrm{i}\theta}$, $0 \leq \theta \leq 2\pi$, with $P(\mathrm{e}^{\mathrm{i}\theta}) \neq 0$,

$$\begin{aligned}
\mathrm{Re}\left(\frac{\mathrm{e}^{\mathrm{i}\theta} P'(\mathrm{e}^{\mathrm{i}\theta})}{P(\mathrm{e}^{\mathrm{i}\theta})}\right) &\leq \frac{n}{1+k} - \frac{1}{1+k}\left(\frac{\prod_{\nu=1}^{n} \frac{|z_\nu|}{k} - 1}{\prod_{\nu=1}^{n} \frac{|z_\nu|}{k} + 1}\right) \\
&= \frac{n}{1+k} - \frac{1}{1+k}\left(\frac{\frac{|a_0|}{k^n |a_n|} - 1}{\frac{|a_0|}{k^n |a_n|} + 1}\right),
\end{aligned}$$

which is equivalent to (3.1). This completes the proof of Lemma 2.

Lemma 3.3. *If $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leq k$, $k \leq 1$, then for $|z| = 1$ and $R > 1$,*

$$|P(Rz) - P(z)| \geq \left(\frac{R^n - 1}{k^n}\right) \min_{|z|=k} |P(z)|. \tag{3.2}$$

The above lemma is due to Aziz and Rather [3].

If we divide both sides of (3.2) by $R-1$ and let $R \to 1$, we immediately get the following result.

Lemma 3.4. *If $P(z)$ is a polynomial of degree n having all its zeros in $|z| \le k$, $k \le 1$, then for $|z| = 1$, we have*

$$|zP'(z)| \ge \frac{n}{k^n} \min_{|z|=k} |P(z)|.$$

Lemma 3.5. *If $P(z)$ is a polynomial of degree n, then for $|z| = 1$, we have*

$$|P'(z)| + |Q'(z)| \le n\,||P(z)||,$$

where here and throughout $Q(z) = z^n \overline{P(1/\overline{z})}$.

The above lemma is due to Govil and Rahman [7].

4. *Proofs of main results*

PROOF OF THEOREM 2.1. Recall that $P(z)$ has no zeros in $|z| < k,\ k \ge 1$. First suppose that $k > 1$, therefore by Lemma 3.2, we have for $|z| = 1$,

$$2\,\mathrm{Re}\left(\frac{zP'(z)}{P(z)}\right) \le \frac{2}{1+k}\left\{n - \frac{|a_0| - k^n|a_n|}{|a_0| + k^n|a_n|}\right\}. \tag{4.1}$$

If $Q(z) = z^n \overline{P(1/\overline{z})}$, then it easily follows that

$$|Q'(z)| = |nP(z) - zP'(z)| \quad \text{for} \ \ |z| = 1.$$

This implies for $|z| = 1$,

$$\begin{aligned}\left|\frac{Q'(z)}{P(z)}\right|^2 &= \left|n - \frac{zP'(z)}{P(z)}\right|^2 \\ &= n^2 + \left|\frac{zP'(z)}{P(z)}\right|^2 - 2n\,\mathrm{Re}\left(\frac{zP'(z)}{P(z)}\right),\end{aligned}$$

which by (4.1) yields for $|z| = 1$,

$$\left|\frac{zQ'(z)}{P(z)}\right|^2 \ge \left|\frac{zP'(z)}{P(z)}\right|^2 + n^2\left(\frac{k-1}{k+1}\right) + \frac{2n}{1+k}\left(\frac{|a_0| - k^n|a_n|}{|a_0| + k^n|a_n|}\right).$$

This gives for $|z| = 1$,

$$\left[|P'(z)|^2 + n\left\{n\,\frac{k-1}{k+1} + \frac{2}{k+1}\cdot\frac{|a_0| - k^n|a_n|}{|a_0| + k^n|a_n|}\right\}|P(z)|^2\right]^{1/2} \le |Q'(z)|. \quad (4.2)$$

Inequality (4.2) gives, by using Lemma 3.5, that for $|z| = 1$,

$$|P'(z)|^2 + n\left\{n\left(\frac{k-1}{k+1}\right) + \frac{2}{k+1}\left(\frac{|a_0| - k^n|a_n|}{|a_0| + k^n|a_n|}\right)\right\}|P(z)|^2 \le (n\|P\| - |P'(z)|)^2.$$

Equivalently for $|z| = 1$,

$$2n\|P\||P'(z)| \le n^2\|P\|^2 - n\left\{n\,\frac{k-1}{k+1} + \frac{2}{k+1}\,\frac{|a_0| - k^n|a_n|}{|a_0| + k^n|a_n|}\right\}|P(z)|^2,$$

which after simplification gives (2.1). For $k = 1$, the above inequality is trivially true for points on $|z| = 1$ for which $P(z) = 0$ by (1.2). This completes the proof of Theorem 2.1.

PROOF OF THEOREM 2.2. By hypothesis $P(z)$ has all its zeros in $|z| \ge k$, where $k \ge 1$. If $P(z)$ has a zero on $|z| = k$, then

$$m = \min_{|z|=k} |P(z)| = 0,$$

and Theorem 2.2 follows from Theorem 2.1 in this case. Henceforth, we suppose that all the zeros of $P(z)$ lie in $|z| > k$, $k \ge 1$. It follows by Rouche's theorem that for each β with $|\beta| < 1$, the polynomial $G(z) = P(z) - \beta m$, will have all its zeros in $|z| > k$. On applying inequality (4.2) to the polynomial $G(z) = P(z) - \beta m$, we get for every $|\beta| < 1$ and $|z| = 1$,

$$\left[|P'(z)|^2 + n\left\{n\frac{k-1}{k+1} + \frac{2}{k+1}\left(\frac{|a_0 - \beta m| - k^n|a_n|}{|a_0 - \beta m| + k^n|a_n|}\right)\right\}|P(z) - \beta m|^2\right]^{1/2}$$

$$\le |Q'(z) - \overline{\beta} mnz^{n-1}|. \quad (4.3)$$

By triangle inequality, we have

$$|P(z) - \beta m| \ge \big||P(z)| - |\beta| m\big|, \quad (4.4)$$

and note that

$$\big||P(z)| - |\beta| m\big|^2 = \big(|P(z)| - |\beta| m\big)^2, \quad (4.5)$$

so that, from (4.4) and (4.5), we obtain that

$$|P(z) - \beta m|^2 \geq \big(|P(z)| - |\beta| m\big)^2. \tag{4.6}$$

Also, all the zeros of $Q(z) = z^n \overline{P(1/\overline{z})}$ lie in $|z| \leq 1/k$, where $1/k \leq 1$, it follows by applying Lemma 3.4 to $Q(z)$, that for $|z| = 1$,

$$|Q'(z)| \geq nk^n \min_{|z|=1/k} |Q(z)|.$$

Since

$$\min_{|z|=1/k} |Q(z)| = \frac{1}{k^n} \min_{|z|=k} |P(z)| = \frac{m}{k^n},$$

therefore, we have

$$|Q'(z)| \geq mn. \tag{4.7}$$

From (4.7), it is possible to choose the argument of β on the right hand side of (4.3) such that for $|z| = 1$,

$$|Q'(z) - \overline{\beta} mnz^n| = |Q'(z)| - |\beta| mn. \tag{4.8}$$

Also, for every β, we have

$$|a_0 - \beta m| \geq |a_0| - |\beta| m,$$

and since the function

$$x \mapsto \frac{x - k^n |a_n|}{x + k^n |a_n|} \quad (x \geq 0)$$

is non-decreasing, it follows that

$$\frac{|a_0 - \beta m| - k^n |a_n|}{|a_0 - \beta m| + k^n |a_n|} \geq \frac{|a_0| - |\beta| m - k^n |a_n|}{|a_0| - |\beta| m + k^n |a_n|} = C_n(k, |\beta|), \tag{4.9}$$

where

$$C_n(k, t) = \frac{|a_0| - tm - k^n |a_n|}{|a_0| - tm + k^n |a_n|}.$$

Using (4.6), (4.8) and (4.9) in (4.3), we get for every $|\beta| < 1$ and $|z| = 1$,

$$\left\{ |P'(z)|^2 + n\left[n \frac{k-1}{k+1} + \frac{2}{k+1} C_n(k, |\beta|) \right] (|P(z)| - |\beta| m)^2 \right\}^{1/2} \\ \leq |Q'(z)| - |\beta| mn,$$

which gives by using Lemma 3.5 for $|z| = 1$, that

$$|P'(z)|^2 + n\left[n\,\frac{k-1}{k+1} + \frac{2}{k+1}C_n(k,|\beta|)\right](|P(z)| - |\beta|m)^2 \\ \le \left[n\big(\|P\| - |\beta|m\big) - |P'(z)|\right]^2.$$

Equivalently for $|z| = 1$,

$$2n\big(\|P\| - |\beta|m\big)|P'(z)| \le n^2\big(\|P\| - |\beta|m\big)^2 \\ - n\left[n\,\frac{k-1}{k+1} + \frac{2}{k+1}C_n(k,|\beta|)\right]\big(|P(z)| - |\beta|m\big)^2,$$

which gives for $|z| = 1$ and $|\beta| < 1$,

$$|P'(z)| \le \frac{|P(z)| - |\beta|m}{2}\left\{n - \left[\frac{(k-1)n}{k+1} + \frac{2C_n(k,|\beta|)}{k+1}\right]\left(\frac{|P(z)| - |\beta|m}{\|P\| - |\beta|m}\right)^2\right\}.$$

Finally, for β with $|\beta| = 1$, the above inequality follows by continuity. Taking $|\beta| = t$, so that $0 \le t \le 1$, we get (2.4), and Theorem 2.2 is proved.

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