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SUMMATION OF SLOWLY CONVERGENT SERIES BY THE GAUSSIAN TYPE OF QUADRATURES AND APPLICATION TO THE CALCULATION OF THE VALUES OF THE RIEMANN ZETA FUNCTION

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Dedicated to My Friend Aleksandar Sanja Ivić (1949–2020)

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A b s t r a c t. Beside a short account on the summation/integration methods for slowly convergent series (*Laplace transform method for numerical and trigonometric series and the method of contour integration over a rectangle*)*, a method based on a new transformation of the series to a weighted integral, with respect to the weight function*

$$t \mapsto w(t) = \frac{1}{\sqrt{t}\cosh^2 \frac{\pi\sqrt{t}}{2}}$$

over $\mathbb{R}_+$*, is presented. Such a method is applied to calculation of the values of the Riemann zeta function. Several numerical examples are included.*

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1. *Introduction and preliminaries*

In a joint paper with Walter Gautschi [6] we developed the quadrature formulas of Gaussian type with respect to the Bose-Einstein and the Fermi-Dirac weight functions and their squares ($r = 1$ and $r = 2$),

$$t \mapsto \varepsilon_r(t) = \left(\frac{t}{\mathrm{e}^t - 1}\right)^r \quad \text{and} \quad t \mapsto \varphi_r(t) = \frac{1}{(\mathrm{e}^t + 1)^r} \quad \text{on } (0, \infty), \tag{1.1}$$

respectively, i.e.,

$$\int_0^\infty u(t)\varepsilon_r(t)\,\mathrm{d}t = \sum_{k=1}^n A_k^{\varepsilon_r} u(\tau_k^{\varepsilon_r}) + R_n^{\varepsilon_r}(u) \tag{1.2}$$

and

$$\int_0^\infty u(t)\varphi_r(t)\,\mathrm{d}t = \sum_{k=1}^n A_k^{\varphi_r} u(\tau_k^{\varphi_r}) + R_n^{\varphi_r}(u), \tag{1.3}$$

where $(A_k^{\varepsilon_r}, \tau_k^{\varepsilon_r})$ and $(A_k^{\varphi_r}, \tau_k^{\varphi_r})$, $k = 1, \ldots, n$, are the parameters (weights and nodes) of these quadratures and $R_n^{\varepsilon_r}(u)$ and $R_n^{\varphi_r}(u)$ are the corresponding remainder terms, which are identically equal to zero for all polynomials of degree at most $2n-1$.

This type of weighted integrals with respect to the weight functions $\varepsilon_r(t)$ and $\varphi_r(t)$ frequently occur in connection with the evaluation in the independent particle approximation of thermodynamic variables for solid state physics problems for both boson and fermion systems.

These weighted integrals (1.2) and (1.3) for $r = 1$, with the weight functions $w(t) = \varepsilon_1(t)$ and $w(t) = \varphi_1(t)$, are also found to provide very effective tools for the summation of slowly convergent series, which appear in many problems in applied and computational sciences. For such a purpose a large number of numerical methods have been developed, mainly based on linear and nonlinear transformations. In general, starting from a sequence of partial sums of the series, these transformations give other sequences with faster convergence to the same limit (the sum of the series).

For slowly convergent series

$$T_m = \sum_{k=m}^{+\infty} f(k) \quad \text{and} \quad S_m = \sum_{k=m}^{+\infty} (-1)^k f(k), \tag{1.4}$$

in this paper, we consider another type of summation methods, the so-called summation/integration methods, which are based on the following two steps:

(i) It needs some exact transformations of the series (1.4) to the weighted integrals,

$$T_m = \int_{\mathbb{R}} g_m(t) w(t)\, \mathrm{d}t \quad \text{and} \quad S_m = \int_{\mathbb{R}} \widehat{g}_m(t) \widehat{w}(t)\, \mathrm{d}t, \tag{1.5}$$

where the functions g_m and $\widehat{g}_m$ are connected with the original function f in some way, and the weight functions w and $\widehat{w}$ are supported on $\mathbb{R}$ or $\mathbb{R}_+$;

(ii) Construction of the quadrature formulas for integrals in (1.5), usually of Gaussian type with respect to the weight functions w and $\widehat{w}$ on $\mathbb{R}$ or $\mathbb{R}_+$.

Construction of the Gaussian quadrature rules can be done by the Golub-Welsch algorithm [7], knowing the corresponding symmetric tridiagonal Jacobi matrix

$$J_n(w) = \begin{bmatrix} \alpha_0 & \sqrt{\beta_1} & & & \mathbf{O} \\ \sqrt{\beta_1} & \alpha_1 & \sqrt{\beta_2} & & \\ & \sqrt{\beta_2} & \alpha_2 & \ddots & \\ & & \ddots & \ddots & \sqrt{\beta_{n-1}} \\ \mathbf{O} & & & \sqrt{\beta_{n-1}} & \alpha_{n-1} \end{bmatrix},$$

i.e., the coefficients α_k and β_k in the recurrence relation for the corresponding monic orthogonal polynomials $\pi_k(t) \equiv \pi_k(w; t)$,

$$\begin{aligned} &\pi_{k+1}(t) = (t - \alpha_k)\pi_k(t) - \beta_k \pi_{k-1}(t), \quad k = 0, 1, 2, \dots, \\ &\pi_{-1}(t) = 0, \ \ \pi_0(t) = 1. \end{aligned} \tag{1.6}$$

Nodes of such a n-point Gaussian quadrature rule are eigenvalues of the Jacobi matrix $J_n(w)$ (or zeros of $\pi_n(t)$), and the weight coefficients (Christoffel numbers) can be calculated from the the first components of the corresponding normalized eigenvectors of this Jacobi matrix $J_n(w)$ (cf. Mastroianni and Milovanović [11, p. 326]).

Remark 1.1. The function $z \mapsto f(z)$ in the series T_m and S_m, given in (1.4), can depend on several other parameters, e.g., $f(z; x, y, \dots)$, so that this kind of summation processes can be applied also to some classes of functional series, not only to numerical series.

Beside a short account on these summation/integration methods in Section 2, we present in Section 3 a method based on a new transformation of the series to the integral, with respect to the weight function

$$t \mapsto w(t) = \frac{1}{\sqrt{t} \cosh^2 \frac{\pi\sqrt{t}}{2}}$$

over $\mathbb{R}_+$. This method is applied to calculation of the values of the Riemann zeta function in Section 4.

2. *Summation/integration methods for slowly convergent series*

In this section we give an account on the summation/integration methods, for which the first step (i) is very important. Two typical approaches will be analyzed: *Laplace transform method* for numerical and trigonometric series ([6], [17], [19]) and the *method of contour integration over a rectangle* [12].

The step (ii) can be complicated if it is previously necessary to numerically determine the coefficients α_k and β_k in the recurrence relation (1.6).

Remark 2.1. An interesting observation on sums and integrals was given by famous Swiss mathematician of Ukrainian origin, Alexander Ostrowski (1893–1986) and it can be found in the book [3, p. 14] on Spirals (from Theodorus to Chaos), which was written by also very famous American applied mathematician Philip J. Davis (1923–2018): "... *In the seventeenth and eighteenth centuries, mathematicians tried to express integrals as sums. In the nineteenth century, they began to express sums as integrals. So math goes in spirals* ..." (see also [23]).

2.1. Laplace transform method for numerical series

We consider the series $T = T_1$ and $S = S_1$ from (1.4). Let for $\operatorname{Re} s \geq 1$

$$f(s) = \mathcal{L}[g(t)] = \int_0^{+\infty} \mathrm{e}^{-st} g(t)\,\mathrm{d}t. \tag{2.1}$$

Then

$$\sum_{k=1}^{+\infty} (\pm 1)^k f(k) = \sum_{k=1}^{+\infty} (\pm 1)^k \int_0^{+\infty} \mathrm{e}^{-kt} g(t)\,\mathrm{d}t = \int_0^{+\infty} \left(\sum_{k=1}^{+\infty} (\pm \mathrm{e}^{-t})^k \right) g(t)\,\mathrm{d}t,$$

i.e.,

$$\sum_{k=1}^{+\infty} (\pm 1)^k f(k) = \int_0^{+\infty} \frac{\pm \mathrm{e}^{-t}}{1 \mp \mathrm{e}^{-t}}\, g(t)\,\mathrm{d}t = \pm \int_0^{+\infty} \frac{1}{\mathrm{e}^t \mp 1}\, g(t)\,\mathrm{d}t,$$

so that we transform series to the weighted integrals with respect to the weight functions $t \mapsto \varepsilon_1(t)$ and $t \mapsto \varphi_1(t)$, defined in (1.1),

$$T = \sum_{k=1}^{+\infty} f(k) = \int_0^{+\infty} \frac{g(t)}{t} \varepsilon_1(t)\,\mathrm{d}t \tag{2.2}$$

and

$$S = \sum_{k=1}^{+\infty} (-1)^k f(k) = \int_0^{+\infty} (-g(t)) \varphi_1(t)\,\mathrm{d}t \tag{2.3}$$

for "alternating" series. Thus, the obtained integrals (2.2) and (2.3) are just ones on the left hand side in (1.2) and (1.3), respectively, for $r = 1$, so that the summation of the series T and S is transformed to the integration problems with respect to the weight functions $w_1(t) = \varepsilon(t)$ and $w_2(t) = \varphi(t)$, respectively. If these series are slowly convergent and the respective functions $t \mapsto u(t) = g(t)/t$ and $t \mapsto u(t) = g(t)$ in the integrals (2.2) and (2.3), respectively, are smooth, then low-order Gaussian quadrature rules (1.2) and (1.3) can provide an efficient summation procedure (cf. [6], [17]).

Remark 2.2. If g in (2.2) and (2.3) is no a smooth function, for example, if its behaviour as $t \to 0$ is such that $g(t) = t^\gamma h(t)$, where $0 < \gamma < 1$ and $h(0)$ is a constant, then the previous formulas for series T and S should be reduced to the following forms

$$T = \int_0^{+\infty} h(t) \frac{t^\gamma}{\mathrm{e}^t - 1}\, \mathrm{d}t \quad \text{and} \quad S = \int_0^{+\infty} (-h(t)) \frac{t^\gamma}{\mathrm{e}^t + 1}\, \mathrm{d}t,$$

respectively. The construction of the corresponding Gaussian formulas with respect to such modified weights was given in [17].

2.2. Laplace transform method for trigonometric series

Recently, we have considered the corresponding method for summation of slowly convergent trigonometric series [19]

$$C(x) = \sum_{k=1}^{+\infty} f(k) \cos k\pi x = \frac{\pi}{2} \int_0^{+\infty} \frac{\cos \pi x - \mathrm{e}^{-\pi t}}{\cosh \pi t - \cos \pi x} g(\pi t)\, \mathrm{d}t,$$
$$S(x) = \sum_{k=1}^{+\infty} f(k) \sin k\pi x = \frac{\pi}{2} \int_0^{+\infty} \frac{\sin \pi x}{\cosh \pi t - \cos \pi x} g(\pi t)\, \mathrm{d}t,$$

where $g(t) = \mathcal{L}^{-1}[f(s)]$, and developed the weighted Gaussian quadrature formulas for the above integrals, as well as the corresponding orthogonal polynomials. Since the weight $t \mapsto w_I(t; x) = \sin x/(\cosh \pi t - \cos \pi x)$ is a positive function on $\mathbb{R}_+$ for each $0 < x < 1$, and all its moments are

$$\mu_k^I(x) = \int_0^{+\infty} t^k w_I(t; x)\, \mathrm{d}t = \begin{cases} 1 - x, & k = 0, \\ \dfrac{2k!}{\pi^{k+1}} \operatorname{Im}\bigl\{\operatorname{Li}_{k+1}(\mathrm{e}^{\mathrm{i}\pi x})\bigr\}, & k \in \mathbb{N}, \end{cases}$$

where Li_n is the polylogarithm function defined by

$$\mathrm{Li}_n(z) = \sum_{\nu=1}^{+\infty} \frac{z^\nu}{\nu^n},$$

we conclude that the orthogonal polynomials with respect to the inner product

$$(p, q) = \int_0^{+\infty} p(t)q(t)w_I(t; x)\,\mathrm{d}t,$$

as well as the corresponding Gaussian quadrature formulas for $S(x)$ exist for each $n \in \mathbb{N}$. However, the weight

$$t \mapsto w_R(t; x) = \frac{\cos \pi x - \mathrm{e}^{-\pi t}}{\cosh \pi t - \cos \pi x}$$

changes its sign at the point

$$t = \pi^{-1} \log(1/\cos \pi x) \in (0, +\infty), \quad \text{when } 0 < x < 1/2,$$

while for $1/2 \le x \le 1$ this function is negative for each $t \in \mathbb{R}_+$. Because of that, the orthogonal polynomials with respect to $t \mapsto w_R(t; x)$ on $\mathbb{R}_+$ exists for each $1/2 \le x \le 1$, and for $0 \le x < 1/2$ the existence is not guaranteed. In [19] we have obtained the Gaussian quadratures for an equivalent transformed problem

$$S(x) = \frac{\pi}{4} \int_0^{+\infty} \frac{\sin \pi x}{\sqrt{t}\left(\cosh \pi\sqrt{t} - \cos \pi x\right)} g\big(\pi\sqrt{t}\,\big)\,\mathrm{d}t, \tag{2.4}$$

as well as the coefficients $\alpha_k^M(x)$ and $\beta_k^M(x)$ in the three-term recurrence relation for the monic polynomials $M_k(t; x)$,

$$M_{k+1}(t; x) = (t - \alpha_k^M(x))M_k(t; x) - \beta_k^M(x)M_{k-1}(t; x), \quad k = 0, 1, \ldots,$$

with $M_0(t; x) = 1$ and $M_{-1}(t; x) = 0$, orthogonal with respect to the weight function in the integral on the right hand side in (2.4). It is interesting that these coefficients are obtained in the explicit form [19, Theorem 4.1]

$$\begin{cases} \alpha_0^M(x) = \dfrac{1}{3}x(2 - x), \\[2mm] \alpha_k^M(x) = \dfrac{32(k+1)k^3 - 8k^2(x-2)x - 4k(x-1)^2 + (x-2)x}{(4k-1)(4k+3)}, \quad k \in \mathbb{N}; \\[2mm] \beta_0^M(x) = 2(1 - x), \\[2mm] \beta_k^M(x) = \dfrac{4k^2(2k-1)^2\left(4k^2 - (1-x)^2\right)\left((2k-1)^2 - (1-x)^2\right)}{(4k-3)(4k-1)^2(4k+1)}, \quad k \in \mathbb{N}, \end{cases}$$

and the polynomials $t \mapsto M_k(t;x)$ can be expressed in terms of the so-called Pasternak polynomials, represented by the hypergeometric function ${}_3F_2$ (see [1, pp. 191–193], [19])

$$M_k(t;x) = \frac{(-1)^k (2k)!(2-x)_{2k}}{4^k \left(\frac{1}{2}\right)_{2k}}\, {}_3F_2 \left[\begin{matrix} -2k,\ 2k+1,\ \frac{1}{2}\big(2-x+\mathrm{i}\sqrt{t}\big) \\ 1,\ 2-x \end{matrix} \,\middle|\, 1 \right].$$

2.3. Method of contour integration over a rectangle

As we have seen in the Laplace transform method, the function g in (2.2) and (2.3) is connected with the original function f over its Laplace transform (2.1), while the weight functions are $\varepsilon(t) = t/(\mathrm{e}^t - 1)$ and $\varphi(t) = 1/(\mathrm{e}^t + 1)$ (or their generalized forms).

In 1994 we developed a method based on an integration over a rectangular contour in the complex plane [12], in which the weight w in (1.5) is one of the hyperbolic functions

$$w_1(t) = \frac{1}{\cosh^2 t} \quad \text{and} \quad w_2(t) = \frac{\sinh t}{\cosh^2 t}, \tag{2.5}$$

and the functions g_m and $\widehat{g}_m$ can be expressed in terms of the indefinite integral F of f chosen so as to satisfy certain decay properties.

Define the finite sums $T_{m,n}$ and $S_{m,n}$ $(m, n \in \mathbb{Z};\ m \le n)$ by

$$T_{m,n} = \sum_{k=m}^{n} f(k) \quad \text{and} \quad S_{m,n} = \sum_{k=m}^{n} (-1)^k f(k)$$

and assume that $z \mapsto f(z)$ is a holomorphic function in the region

$$D = \big\{ z \in \mathbb{C} \ \big|\ \mathrm{Re}\, z \ge \alpha,\ m-1 < \alpha < m \big\}. \tag{2.6}$$

Also, we take the rectangular contour $\Gamma = \Gamma_{\alpha,\beta,\delta} = \partial G$, where

$$G = \left\{ z \in \mathbb{C} \ \middle|\ \alpha \le \mathrm{Re}\, z \le \beta,\ |\mathrm{Im}\, z| \le \frac{\delta}{\pi} \right\} \subset D,$$

with $m-1 < \alpha < m$, $n < \beta < n+1$, and $\delta > 0$, so that by an integration of the functions $z \mapsto f(z)(\pi/\tan \pi z)$ and $z \mapsto f(z)(\pi/\sin \pi z)$ over Γ, and after integration by parts, we obtain

$$T_{m,n} = \sum_{k=m}^{n} f(k) = \frac{1}{2\pi i} \oint_{\Gamma} \left(\frac{\pi}{\sin \pi z} \right)^2 F(z)\, \mathrm{d}z \tag{2.7}$$

and

$$S_{m,n} = \sum_{k=m}^{n} (-1)^k f(k) = \frac{1}{2\pi i} \oint_\Gamma \Big(\frac{\pi}{\sin \pi z}\Big)^2 \cos \pi z \, F(z) \, \mathrm{d}z, \tag{2.8}$$

where F is an integral of f.

Suppose now the following decay properties (see [10], [12], [11]):

(C1) *F is a holomorphic function in the region D;*

(C2) $\lim_{|t|\to+\infty} \mathrm{e}^{-c|t|} F\,(x + \mathrm{i}t/\pi) = 0$*, uniformly for $x \ge \alpha$;*

(C3) $\lim_{x\to+\infty} \int_{\mathbb{R}} \mathrm{e}^{-c|t|} \,\big|\, F\,(x + \mathrm{i}t/\pi)\big| \,\mathrm{d}t = 0,$

where $c = 2$ (or $c = 1$ for "alternating" series).

Setting $\alpha = m - 1/2$, $\beta = n + 1/2$, and letting $\delta \to +\infty$ and $n \to +\infty$, under conditions (C1), (C2), and (C3), in [12] (see also [14]) we transformed the integrals (2.7) and (2.8) to

$$T_m = T_{m,\infty} = \sum_{k=m}^{+\infty} f(k) = \int_0^{+\infty} w_1(t) \Phi \left(m - \frac{1}{2}, \frac{t}{\pi}\right) \mathrm{d}t \tag{2.9}$$

and

$$S_m = S_{m,\infty} = \sum_{k=m}^{+\infty} (-1)^k f(k) = \int_0^{+\infty} w_2(t) \Psi \left(m - \frac{1}{2}, \frac{t}{\pi}\right) \mathrm{d}t, \tag{2.10}$$

respectively, where the weight functions w_1 and w_2 are given by (2.5), and Φ and Ψ by

$$\Phi(x, y) = -\frac{1}{2}\left[F(x + \mathrm{i}y) + F(x - \mathrm{i}y)\right]$$

and

$$\Psi(x, y) = \frac{(-1)^m}{2\mathrm{i}}\left[F(x + \mathrm{i}y) - F(x - \mathrm{i}y)\right].$$

The integrals (2.9) and (2.10) can be calculated by using Gaussian quadrature rules, with respect to the hyperbolic weights w_1 and w_2. In [12, §3] we constructed such quadratures, with the previous generation of the recursion coefficients α_k and β_k in (1.6) for the corresponding monic orthogonal polynomials, using the discretized Stieltjes-Gautschi procedure, with the discretization based on the Gauss-Laguerre quadrature rule.

For the generation of the recursion coefficients α_k and β_k and the construction of those Gaussian rules on $\mathbb{R}_+$ today we use a recent progress in symbolic computation and variable-precision arithmetic, as well as the MATHEMATICA package

`OrthogonalPolynomials` (see [2], [21]). The package is downloadable from Web Site: `http://www.mi.sanu.ac.rs/~gvm/`. The approach enables us to overcome the numerical instability in generating recursion coefficients (cf. [4], [**?**], [11], [20]). In this construction we need only a procedure for the symbolic calculation of moments or their calculation in variable-precision arithmetic (see [21], [15]).

Remark 2.3. Regarding the properties of the function f is often appropriate to extract a finite number of first terms in the series (1.4). For example, if we need to calculate T_1, we first take

$$T_1 = \sum_{k=1}^{m-1} f(k) + \sum_{k=m}^{+\infty} f(k),$$

and then apply the procedure only to the series starting with the index $k = m$, and the first finite sum calculate directly (see [17]). The rapidly increasing speed of convergence of a summation process is achieved as m increases (see [12]).

Remark 2.4. Instead of integration over $\mathbb{R}_+$ in (2.9) and (2.10), one might keep integration over $\mathbb{R}$ as in Abel-Plana summation formula (cf. Milovanović [18, §4], [16]) and note that (see [12])

$$T_m = \int_{\mathbb{R}} \Phi\Big(m - \frac{1}{2}, \frac{t}{2\pi}\Big) \frac{\mathrm{e}^{-t}}{(1+\mathrm{e}^{-t})^2}\, \mathrm{d}t$$

and

$$S_m = \int_{\mathbb{R}} \Psi\Big(m - \frac{1}{2}, \frac{t}{2\pi}\Big) \sinh\frac{t}{2} \cdot \frac{\mathrm{e}^{-t}}{(1+\mathrm{e}^{-t})^2}\, \mathrm{d}t.$$

For this so-called *logistic weight function* $t \mapsto \mathrm{e}^{-t}/(1+\mathrm{e}^{-t})^2$, the recursion coefficients for the respective orthogonal polynomials are explicitly known (cf. [11, p. 159]):

$$\alpha_k = 0, \quad \beta_0 = 1, \quad \beta_k = \frac{k^2\pi^2}{4k^2-1}, \quad k = 1, 2, \dots . \tag{2.11}$$

It means no procedure is required to generate the recursion coefficients. However, as we noted in [12, p. 469] the Gaussian quadrature over the full real line with respect to the logistic weight function converges considerably more slowly than one for one-sided integration (over $\mathbb{R}_+$).

In the next section, we construct a new summation process based on the Gaussian quadrature rule over $\mathbb{R}_+$, where no procedure is required to generate the recursion coefficients.

3. *A simple construction of a new summation method*

In our construction of this new summation/integration method we start with two requirements:

(1) Transform the series to an integral over semiaxis $\mathbb{R}_+$ in order to have faster convergence of the summation process;
(2) Avoid generating recursive coefficients, i.e., use such a weight function on $\mathbb{R}_+$ for which the coefficients in the three-term recurrence relation for the corresponding orthogonal polynomials are known in an explicit form.

The second requirement is provided by the following result:

Theorem 3.1. *The polynomials* $\{p_k\}_{k\in\mathbb{N}_0}$ *orthogonal with respect to the weight function*

$$x \mapsto w(x) = \frac{1}{\sqrt{x}\cosh^2 \frac{\pi\sqrt{x}}{2}} \tag{3.1}$$

on $(0,+\infty)$ *satisfy the three-term recurrence relation*

$$p_{k+1}(x) = (x-a_k)p_k(x) - b_k p_{k-1}(x), \quad k = 0,1,\ldots$$

where $p_0(x) = 1$ *and* $p_{-1}(x) = 0$*, with the recurrence coefficients*

$$\left.\begin{aligned} a_0 &= \frac{1}{3}, \qquad a_k = \frac{32k^4+32k^3+8k^2-1}{(4k-1)(4k+3)} \qquad (k\in\mathbb{N}),\\ b_0 &= \frac{4}{\pi}, \qquad b_k = \frac{16(2k-1)^4k^4}{(4k-3)(4k-1)^2(4k+1)} \qquad (k\in\mathbb{N}).\end{aligned}\right\} \tag{3.2}$$

Proof. According to Remark 2.4 we have that the orthogonal polynomials with respect to the logistic weight function

$$W(t) = \frac{\mathrm{e}^{-t}}{(1+\mathrm{e}^{-t})^2} = \frac{1}{4}\cdot\frac{1}{\cosh^2(t/2)} \quad \text{on } \mathbb{R} \tag{3.3}$$

satisfy the recurrence relation (1.6) with the coefficients given in (2.11). Further, we omit the multiplicative constant $1/4$ in (3.3). Putting $t = \pi x$ we have that the recurrence relation for the monic polynomials $\Pi_k(x) = \pi_k(\pi x)/\pi^k$, orthogonal with respect to the weight function $x \mapsto W(\pi x)$ is given by

$$\Pi_{k+1}(x) = x\Pi_k(x) - \widehat{\beta}_k\Pi_{k-1}(x), \quad k = 1,2,\ldots,$$

where $\Pi_0(x) = 1$ and $\Pi_{-1}(x) = 0$, and $\widehat{\beta}_k = k^2/(4k^2-1)$, $k\in\mathbb{N}$.

Using Theorems 2.2.11 and 2.2.12 from [11, p. 102] we get the sequence of monic polynomials $\{p_k(x)\}_{k\in\mathbb{N}_0}$, where $p_k(x) = \Pi_{2k}(\sqrt{x})$, which are orthogonal with respect to the weight function $x \mapsto W\left(\pi\sqrt{x}\right)/\sqrt{x}$, i.e., (3.1), and the corresponding recursive coefficients are

$$a_0 = \widehat{\beta}_1 = \frac{1}{3}, \quad a_k = \widehat{\beta}_{2k} + \widehat{\beta}_{2k+1}, \quad b_k = \widehat{\beta}_{2k-1}\widehat{\beta}_{2k} \quad (k \in \mathbb{N}),$$

i.e., (3.2). Finally, for and b_0 we take

$$b_0 = \int_0^{+\infty} \frac{1}{\sqrt{x}\cosh^2 \frac{\pi\sqrt{x}}{2}}\,\mathrm{d}x = \frac{4}{\pi}.$$

To obtain a transformation of series to the one-side integral (over $\mathbb{R}_+$) we start with the integral representation (2.9) and by change variables $t = \pi\sqrt{x}/2$,

$$\begin{aligned} T_m = \sum_{k=m}^{+\infty} f(k) &= \int_0^{+\infty} \Phi\left(m - \frac{1}{2}, \frac{t}{\pi}\right) \frac{1}{\cosh^2 t}\,\mathrm{d}t \\ &= \int_0^{+\infty} \Phi\left(m - \frac{1}{2}, \frac{\sqrt{x}}{2}\right) \frac{1}{\cosh^2 \frac{\pi\sqrt{x}}{2}} \frac{\pi\,\mathrm{d}t}{4\sqrt{x}} \\ &= \frac{\pi}{4}\int_0^{+\infty} \Phi\left(m - \frac{1}{2}, \frac{\sqrt{x}}{2}\right) w(x)\,\mathrm{d}x, \end{aligned} \tag{3.4}$$

where the weight function w is given by (3.1), and Φ is defined by

$$\Phi(x, y) = -\frac{1}{2}\left[F(x + \mathrm{i}y) + F(x - \mathrm{i}y)\right]. \tag{3.5}$$

Formula (3.4) suggests to apply Gaussian quadrature formula to the last integral, using the weight function w, i.e.,

$$\int_0^{+\infty} g(x)w(x)\,\mathrm{d}x = \sum_{\nu=1}^{n} A_\nu^{(n)} g\big(\xi_\nu^{(n)}\big) + R_n(g), \tag{3.6}$$

where $\big(A_\nu^{(n)}, \xi_\nu^{(n)}\big)$, $\nu = 1, \ldots, n$, are the parameters (weights and nodes) of this quadrature rule and $R_n(g)$ is the corresponding remainder term, which is identically equal to zero for all algebraic polynomials of degree at most $2n - 1$. The required recursion coefficients for the corresponding orthogonal polynomials $\{p_k\}_{k\in\mathbb{N}_0}$ are known in the explicit form and given in Theorem 3.1 by (3.2).

The following procedure in the MATHEMATICA package `OrthogonalPolynomials` (see [2, 21]) gives parameters of the Gaussian quadratures for n nodes, with `Precision->PR`) (here, `n=10` and `PR=20`):

```
<< orthogonalPolynomials`
    n=10; PR=20;
    ak=Table[If[k==0,1/3,(32k^4+32k^3+8k^2-1)/
                ((4k-1)(4k+3))],{k,0,n-1}];
    bk=Table[If[k==0,4/Pi,16(2k-1)^4 k^4/
                ((4k-3)(4k-1)^2(4k+1))],{k,0,n-1}];
    {xn,An}=aGaussianNodesWeights[n,ak,bk,
             WorkingPrecision->PR+5,Precision->PR];
```

According to the previous facts we obtain the following statement:

Theorem 3.2. *Let F be an integral of f such that conditions* (C1)–(C3) *are satisfied with $c = 2$. If $\big(A_\nu^{(n)}, \xi_\nu^{(n)}\big)$, $\nu = 1,\ldots,n$, are the parameters* (*weights and nodes*) *of the Gaussian quadrature rule* (3.6)*, then*

$$T_m = \sum_{k=m}^{+\infty} f(k) = \frac{\pi}{4}\sum_{\nu=1}^{n} A_\nu^{(n)}\Phi\left(m-\frac{1}{2}, \frac{\sqrt{\xi_\nu^{(n)}}}{2}\right) + R_n(\Phi),$$

where the function Φ is given by (3.5).

Remark 3.1. This method can be applied also to summation of the alternating series S_m, including the hyperbolic sine as a factor in the integrand, so that

$$\begin{aligned} S_m = \sum_{k=m}^{+\infty}(-1)^k f(k) &= \int_0^{+\infty} \Psi\left(m-\frac{1}{2}, \frac{t}{\pi}\right)\frac{\sinh t}{\cosh^2 t}\,\mathrm{d}t \\ &= \frac{\pi}{4}\int_0^{+\infty} \Psi\left(m-\frac{1}{2}, \frac{\sqrt{x}}{2}\right)\sinh\frac{\pi\sqrt{x}}{2} w(x)\,\mathrm{d}x \\ &= \frac{\pi}{4}\sum_{\nu=1}^{n} A_\nu^{(n)}\Psi\left(m-\frac{1}{2}, \frac{\sqrt{\xi_\nu^{(n)}}}{2}\right)\sinh\frac{\pi\sqrt{\xi_\nu^{(n)}}}{2} + R_n(\widetilde{\Psi}), \end{aligned}$$

where Ψ is given by

$$\Psi(x,y) = \frac{(-1)^m}{2\mathrm{i}}\left[F(x+\mathrm{i}y) - F(x-\mathrm{i}y)\right],$$

and $\widetilde{\Psi}$ is the $\sinh$-modification of Ψ.

Example 3.1. We consider the simple series

$$T_1(p) = \sum_{k=1}^{+\infty} \frac{1}{k^{1/p}(k+1)}, \quad p > 0, \tag{3.7}$$

for which we have

$$f(z) = \frac{1}{z^{1/p}(z+1)}, \tag{3.8}$$

$$F(z) = \frac{pz^{1-1/p}}{p-1}\, {}_2F_1\Big(1, 1-\frac{1}{p}; 2-\frac{1}{p}; -z\Big) - \frac{\pi}{\sin(\pi/p)}, \tag{3.9}$$

where ${}_2F_1$ is the well-known Gauss hypergeometric function [22, p. 384], and the integration constant is taken so that $F(\infty) = 0$. The series (3.7) is slowly convergent. When $p \to +\infty$ this series is divergent.

For $p = 2$ the series $T_1(2) = 1.86002507922119030718069 59\ldots$ appears in a study of spirals and defines the well-known Theodorus constant [3]. The first n terms in $T_1(2)$ give the following results:

$$\begin{aligned} n = 10^2 : &\quad \mathbf{1}.6611815943068811905, \\ n = 10^4 : &\quad \mathbf{1.8}400262457853669133, \\ n = 10^6 : &\quad \mathbf{1.8}580250803878559488, \end{aligned}$$

with maximal two exact decimal digits (given in bold). Here,

$$F(z) = 2 \arctan \sqrt{z} - \pi.$$

Relative errors

$$r_n(p) = \left| \frac{S_n(p) - T_1(p)}{T_1(p)} \right| \tag{3.10}$$

in the partial sums $S_n(p)$ of the series $T_1(p)$ for $p = 2, 6$, and 10, when we take the first $n = 10^k$ terms, $k = 1, \ldots, 6$, are presented in log-scale in Figure 1. For example, these relative errors for the partial sums of one million terms are 1.08×10^{-3}, 0.105, and 0.260, respectively. Thus, only for $p = 2$ two first digits are true, while for $p = 6$ and $p = 10$ all digits in the corresponding partial sums of one million terms are wrong. A rough calculation shows that in the case of $p = 10$, theoretically, 10^{30} first terms of this series should be taken in order to obtain two or three exact digits of the series $T_1(10)$!

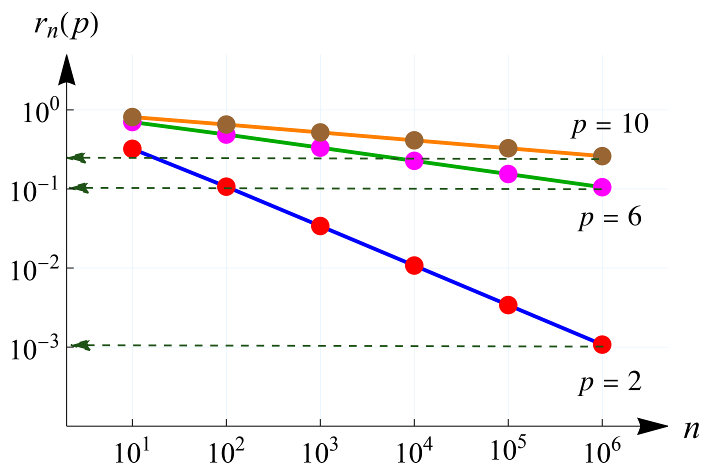

Figure 1: Relative errors (3.10) in log-scale of partial sums of series $T_1(p)$ for $p = 2, 6, 10$ and $n = 10^k$, $k = 1, \ldots, 6$

Let $T_1(p) = S_{m-1}(p) + T_m(p)$, i.e.,

$$\sum_{k=1}^{+\infty} \frac{1}{k^{1/p}(k+1)} = \sum_{k=1}^{m-1} \frac{1}{k^{1/p}(k+1)} + \sum_{k=m}^{+\infty} \frac{1}{k^{1/p}(k+1)}, \quad m \geq 1,$$

where for $m = 1$ the first (finite) sum on the right side is empty. Now, we apply our summation/integration method to $T_m(p)$, so that, according to Theorem 3.2, we have the following quadrature approximation

$$Q_{n,m}(f) = \sum_{k=1}^{m-1} f(k) - \frac{\pi}{4} \sum_{\nu=1}^{n} A_\nu^{(n)} \operatorname{Re}\left\{ F\left(m - \frac{1}{2} + \mathrm{i}\frac{1}{2}\sqrt{\xi_\nu^{(n)}} \right) \right\},$$

where f and F are given by (3.8) and (3.9), respectively.

Relative errors of the obtained results for $p = 10$,

$$\mathrm{err}_{n,m} = \left| \frac{Q_{n,m}(f) - T_1(10)}{T_1(10)} \right|,$$

are given in Table 1. Numbers in parentheses indicate decimal exponents. The results are given for $m = 1$ (a direct application to $T_1(10)$), $m = 2, 3, 4, 6$, 11, and 16.

Table 1: Relative errors $\mathrm{err}_{n,m}$ in of Gaussian approximations $Q_{n,m}(f)$ of $T_1(10)$ for $n = 5$ and $n = 10(10)60$ and for some selected values of m

n	$m = 1$	$m = 2$	$m = 3$	$m = 6$	$m = 11$	$m = 16$
5	1.17(−6)	2.23(−10)	5.16(−13)	4.21(−18)	5.89(−23)	4.47(−26)
10	1.92(−7)	2.03(−12)	2.76(−16)	1.20(−24)	1.77(−33)	2.03(−39)
20	3.40(−8)	2.09(−14)	1.71(−19)	1.99(−31)	1.33(−45)	5.53(−56)
30	1.26(−8)	1.49(−15)	2.35(−21)	2.11(−35)	5.60(−53)	1.61(−66)
40	6.32(−9)	2.30(−16)	1.13(−22)	3.20(−38)	2.95(−58)	3.64(−74)
50	3.71(−9)	5.44(−17)	1.09(−23)	2.08(−40)	2.31(−62)	3.77(−80)
60	2.41(−9)	1.68(−17)	1.61(−24)	3.42(−42)	1.01(−65)	4.63(−85)

As an exact value of $T_1(10)$ we use the value obtained by our summation process for $m = 21$ and $n = 80$ (given by 108 exact decimal digits),

$$\begin{aligned} T_1(10) =& 9.65517164385061458223654143981787260923387635195307915 00 \\ & 853233328254979267218936762153258496737740516646703 8\ldots \end{aligned}$$

As we can see the sequence of quadrature sums $\{Q_{n,m}(f)\}_n$ has faster convergence to $T_1(10)$ when m increases (see also Remark 2.3). For example, the quadrature formula (3.6), with $n = 60$ nodes and $m = 16$ (the first fifteen terms of the series $T_1(10)$ is extracted), gives about 85 exact decimal digits!

The advantage of the exposed summation/quadrature process in relation to the previous procedures of this type is the possibility of constructing such a quadrature formula, without previously generating recurrence coefficients for the corresponding orthogonal polynomials, and this formula is on the semiaxis $\mathbb{R}_+$ (see Remark 2.4).

4. *Calculation of the values of the Riemann zeta function*

Thanks to the mentioned advantage, this summation/quadrature process will be applied to calculation the values of the Riemann zeta function $s \mapsto \zeta(s)$, defined by

$$\zeta(s) = \sum_{k=1}^{+\infty} k^{-s}, \quad \text{when } \mathrm{Re}\, s > 1. \tag{4.1}$$

The series (4.1) converges for any s with $\mathrm{Re}\, s > 1$, uniformly, for any fixed $\sigma > 1$, in any subset of $\mathrm{Re}\, s \geq \sigma$, which establishes that $\zeta(s)$ is an analytic function in $\operatorname{Re} s > 1$. The function $\zeta(s)$ admits analytic continuation to $\mathbb{C}$, where it satisfies the functional equation (cf. [8, p. 8], [9, p. 3])

$$\pi^{-s/2}\zeta(s)\Gamma\left(\tfrac{1}{2}s\right) = \pi^{-(1-s)/2}\zeta(1-s)\Gamma\left(\tfrac{1}{2}(1-s)\right),$$

i.e.,

$$\zeta(s) = 2^s\pi^{s-1}\sin\frac{\pi s}{2}\,\Gamma(1-s)\zeta(1-s). \tag{4.2}$$

Thus, by means of analytic continuation, $\zeta(s)$ is analytic function for any complex s, except for $s = 1$, which is a simple pole of $\zeta(s)$ with residue 1. The Laurent's expansion of $\zeta(s)$ in the neighborhood of its pole $s = 1$ is

$$\zeta(s) = \frac{1}{s-1} + \gamma_0 + \gamma_1(s-1) + \gamma_(s-1)^2 + \cdots,$$

where the coefficients γ_k are given by (cf. [8, p. 4]

$$\gamma_k = \frac{(-1)^k}{k!}\lim_{N\to+\infty}\left\{\sum_{m\le N}\frac{1}{m}\log^k m - \frac{\log^{k+1} N}{k+1}\right\},$$

and $\gamma_0 = \gamma = 0.5772157\ldots$ is Euler's constant.

Using the approach from the previous section we get the an integral representation of $\zeta(s+1)$. Since $f(z) = 1/z^{s+1}$ and $F(z) = -1/(sz^s)$, according to (3.4), we have

$$\begin{aligned}\zeta(s+1) &= \sum_{k=1}^{m-1}\frac{1}{k^{s+1}} + \sum_{k=m}^{+\infty}\frac{1}{k^{s+1}}\\ &= \sum_{k=1}^{m-1}\frac{1}{k^{s+1}} + \frac{\pi}{4s\left(m-\frac{1}{2}\right)^s}\int_0^{+\infty} g\big(c_m\sqrt{x};s\big)w(x)\,\mathrm{d}x,\end{aligned} \tag{4.3}$$

where $w(x)$ is given by (3.1) and

$$t \mapsto g(t;s) = \exp\Big(-\frac{s}{2}\log(1+t^2)\Big)\cos\big(s\arctan t\big), \quad c_m = \frac{1}{2m-1}.$$

Now, applying the quadrature formula (3.6) to the integral on the right hand side in (4.3) we obtain

$$\zeta(s+1) = \sum_{k=1}^{m-1}\frac{1}{k^{s+1}} + \frac{\pi}{4s\left(m-\frac{1}{2}\right)^s}\sum_{\nu=1}^{n}A_\nu^{(n)}g\big(\xi_\nu^{(n)}\big) + E_{n,m}(s), \tag{4.4}$$

where $E_{n,m}(s)$ is the corresponding error term.

As in Example 3.1 we made many experiments in calculating the Riemann zeta function $\zeta(s)$ for different values of s and the summation/quadrature formula (4.4) shows rapidly increasing speed of convergence especially as m increases. Here, we give only a few graphics obtained by using formula (4.4). In Figure we displayed graphics $s \mapsto |\zeta(s)|$ for real values of s. As we can see the trivial zeros of $\zeta(s)$ at $s = -2k$, $k = 1, 2, \ldots$.

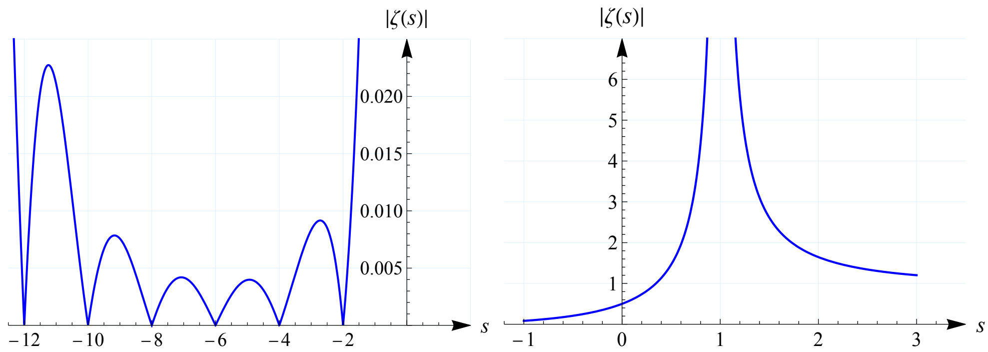

Figure 2: Graphics $s \mapsto |\zeta(s)|$ for $s \in (-12, -1)$ (left) and $s \in (-1, 3)$ (right)

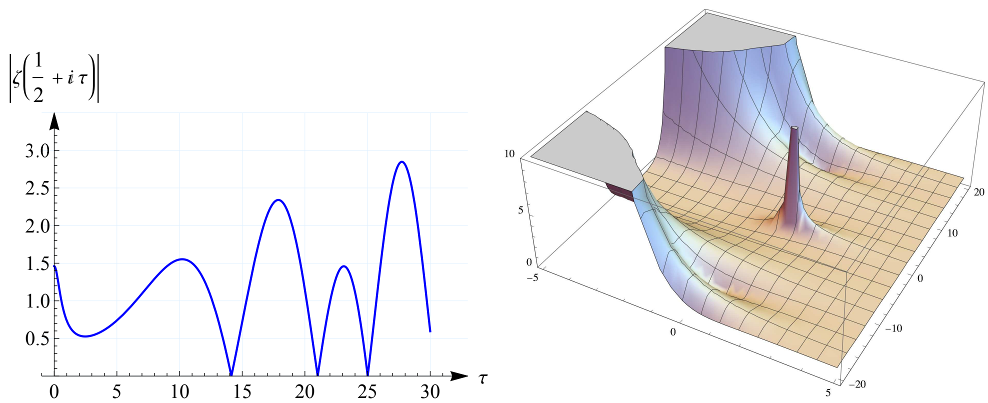

Figure 3: Graphics $y \mapsto |\zeta(1/2 + \mathrm{i}\tau)|$ on the critical line for $\tau \in (0, 30)$ (left) and $3D$-graphics $(\sigma, \tau) \mapsto |\zeta(\sigma + \mathrm{i}\tau)|$, when $\sigma \in (-5, 5)$ and $\tau \in (-20, 20)$ (right)

Figure 3 (left) shows absolute value of the Riemann zeta function on the critical line $s = 1/2 + i\tau$ for $0 \le \tau \le 30$, while other the second one (right) shows $3D$-graphics $(\sigma, \tau) \mapsto |\zeta(\sigma + \mathrm{i}\tau)|$, when $\sigma \in (-5, 5)$ and $\tau \in (-20, 20)$.

5. *Final comments*

The International Conference on Algebra, Logic and Discrete Mathematics took place in Niš, April 14–16, 1995, and was organized by the Faculty of Economics Niš, Faculty of Philosophy Niš, and Mathematical Institute SANU Belgrade (papers were published in Filomat **9**:3 (1995) [Eds. S. Bogdanović, M.Ćirić and Ž. Perović]). Just before this conference, I was preparing an article dedicated to the Italian mathematician Luigi Gatteschi on the occasion of his 70th birthday, and the paper was about summation of series and Gaussian quadrature rules. During the conference in Niška Banja, I presented the main idea to Sanja Ivić and he immediately suggested me to include the calculation of the value of Riemann's zeta function as an interesting example. So, in my paper [13] there is a short section on the Riemann zeta function.

Figure 4: G.V. Milovanović, S. Simić and A. Ivić (from left to the right in Niška Banja, April 1995)

Otherwise, very often Ivić advised colleagues and older researchers to occasionally return to consider their older papers and that, given the constant progress in the science, they could sometimes improve something in their work from those ancient times. This paper dedicated to my friend Aleksandar Sanja Ivić is also such a product!

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