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A CLASS OF ORTHOGONAL POLYNOMIALS ON THE RADIAL RAYS IN THE COMPLEX PLANE, II*

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Abstract. In this paper we continue considerations on polynomials orthogonal on the radial rays in the complex plane started in [7–9]. We study a general case of an arbitrary number of the rays and consider the corresponding orthogonal polynomials and join them matrix orthogonal polynomials. For both of them we derive the recurrence relations and find a representation. In a special symmetric case we find connection with the standard orthogonal polynomials on the real line, locate the zeros and find differential equation. Finally, we consider some analogues of the classical Legendre and the generalized Laguerre polynomials.

1. Introduction

For a given lengths and angles

$$l_s \in \mathbb{R}^+, \quad \theta_s \in (-\pi, \pi], \quad s = 0, 1, \ldots, m-1,$$

we define the inner product,

$$(f, g) = \sum_{s=0}^{m-1} \varepsilon_s^{-1} \int_{L_s} f(z)\overline{g(z)}|w_s(z)|\, dz, \quad \varepsilon_s = e^{i\theta_s}, \tag{1.1}$$

with respect to the weight functions $w_s(z)$ on the radial rays L_s which connect the origin $z = 0$ and the points $z = l_s\varepsilon_s$, $0 \le s \le m-1$. This can be rewritten in the form

$$(f, g) = \sum_{s=0}^{m-1} \int_0^{l_s} f(x\varepsilon_s)\overline{g(x\varepsilon_s)}|w_s(x\varepsilon_s)|\, dx \tag{1.2}$$

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or

$$(f,g) = \int_0^1 \sum_{s=0}^{m-1} l_s f(l_s\varepsilon_s x)\overline{g(l_s\varepsilon_s x)}|w_s(l_s\varepsilon_s x)|\, dx. \tag{1.3}$$

Because of

$$\|f\|^2 = (f,f) = \sum_{s=0}^{m-1} \int_0^{l_s} |f(x\varepsilon_s)|^2 |w_s(x\varepsilon_s)|\, dx > 0,$$

except for $f(z) \equiv 0$, the corresponding *orthogonal polynomials* $\{\pi_N(z)\}_{N=0}^{+\infty}$ exist and they can be construct using Gram-Schmidt orthogonalizing process.

If we define the *moments*

$$\mu_{p,q} = (z^p, z^q)$$

and *moment-determinants*

$$\Delta_0 = 1, \quad \Delta_N = \begin{vmatrix} \mu_{00} & \mu_{10} & \cdots & \mu_{N-1,0} \\ \mu_{01} & \mu_{11} & & \mu_{N-1,1} \\ \vdots & & & \\ \mu_{0,N-1} & \mu_{1,N-1} & & \mu_{N-1,N-1} \end{vmatrix}, \quad N \geq 1,$$

then these polynomials can be expressed in the form

$$\pi_0(z) = 1,$$

$$\pi_N(z) = \frac{1}{\Delta_N} \begin{vmatrix} \mu_{00} & \mu_{10} & \cdots & \mu_{N-1,0} & 1 \\ \mu_{01} & \mu_{11} & & \mu_{N-1,1} & z \\ \vdots & & & & \\ \mu_{0,N-1} & \mu_{1,N-1} & & \mu_{N-1,N-1} & z^{N-1} \\ \mu_{0,N} & \mu_{1,N} & & \mu_{N-1,N} & z^N \end{vmatrix}, \ N \geq 1.$$

Like in [9], we can prove the following result:

Theorem 1.1. *If* $\Delta_n > 0$, $n \in \mathbb{N}$, *the monic polynomials* $\{\pi_N(z)\}_{N=0}^{+\infty}$, *orthogonal with respect to the inner product* (1.1) *exist uniquely and the norm is given by*

$$\|\pi_N\|^2 = \frac{\Delta_{N+1}}{\Delta_N}.$$

For $m = 1$ and $l_0 = 1$, we have the standard case of polynomials orthogonal on $(0, 1)$. Also, for $m = 2$, $l_0 = l_1 = 1$ and a weight $w(x)$ on $(-1, 1)$, we have $(f, g) = \int_{-1}^{1} f(x)\overline{g(x)}w(x)\, dx$, which is a standard case of polynomials orthogonal on $(-1, 1)$.

The case when m is an even number was considered by one of us in [4]. In this paper we study a general case. The paper is organized as follows. In Section 2 we find the recurrence relation for orthogonal polynomials on the radial rays and study the matrix polynomials joined them. In Section 3 we continue with an investigation of a symmetric case of equal lengths, angles and weights. In such a case, in Sections 4–7, we find a representation for polynomials, the joined matrix polynomials, distribution of zeros and a differential equation, respectively. Sections 8 and 9 are devoted to some analogues of the classical Legendre polynomials and the generalized Laguerre polynomials.

2. Recurrence Relation and Joined Matrix Polynomials

The properties of the introduced orthogonal polynomials essentially depend on lengths and angles of rays and their weights. Firstly, we prove the following result:

Lemma 2.1. *If there exists any $M \in \mathbb{N}$ such that*

$$\varepsilon_s^{2M} = 1, \quad s = 0, 1, \dots, m-1, \tag{2.1}$$

then the inner product $(\cdot, \cdot)$ has the property

$$(z^M f, g) = (f, z^M g).$$

Proof. From the condition $\varepsilon_s^M = -1$ or $\varepsilon_s^M = 1$, we have

$$\begin{aligned}
(z^M f, g) &= \sum_{s=0}^{m-1} l_s \int_0^1 (l_s\varepsilon_s x)^M f(l_s\varepsilon_s x)\overline{g(l_s\varepsilon_s x)}|w_s(l_s\varepsilon_s x)|\, dx \\
&= \sum_{s=0}^{m-1} l_s \int_0^1 f(l_s\varepsilon_s x)(l_s x)^M \varepsilon_s^M \overline{g(l_s\varepsilon_s x)}|w_s(l_s\varepsilon_s x)|\, dx \\
&= \sum_{s=0}^{m-1} l_s \int_0^1 f(l_s\varepsilon_s x)\overline{(l_s\varepsilon_s x)^M g(l_s\varepsilon_s x)}|w_s(l_s\varepsilon_s x)|\, dx \\
&= (f, z^M g). \quad \square
\end{aligned}$$

Theorem 2.1. *Let the conditions* (2.1) *be satisfied and let* M *be a minimal such integer. Then the monic polynomials* $\{\pi_N(z)\}_{N=0}^{+\infty}$ *satisfies* $(2M+1)$-*recurrence relation*

$$z^M \pi_N(z) = \pi_{N+M}(z) + \sum_{j=N-M}^{N+M-1} \alpha_j^{(N)} \pi_j(z), \quad N \geq M.$$

where $\pi_j(z)$, $0 \leq j < M$, *can be determined by the Gram-Schmidt orthogonalization and* $\alpha_j^{(N)} = 0$ *for* $j < 0$.

Proof. Writing

$$z^M \pi_N(z) = \pi_{N+M}(z) + \sum_{\nu=0}^{N+M-1} \alpha_\nu^{(N)} \pi_\nu(z), \quad N \geq 0,$$

for an arbitrary k, we have

$$(z^M \pi_N, \pi_k) = (\pi_{N+M}, \pi_k) + \sum_{\nu=0}^{N+M-1} \alpha_\nu^{(N)} (\pi_\nu, \pi_k).$$

If $0 \leq k \leq N+M-1$, then $(z^M \pi_N, \pi_k) = \alpha_k^{(N)} (\pi_k, \pi_k)$, i.e.,

$$\alpha_k^{(N)} = \frac{(z^M \pi_N, \pi_k)}{(\pi_k, \pi_k)}, \quad 0 \leq k \leq N+M-1.$$

If $N \geq M+1$ and $0 \leq k \leq N-M-1$, because of Lemma 2.1 and orthogonality, we have that $(z^M \pi_N, \pi_k) = (\pi_N, z^M \pi_k) = 0$, i.e.,

$$\alpha_k^{(N)} = 0, \quad k = 0, 1, \ldots, N-M-1, \quad N \geq M+1. \quad \square$$

Using a rotation, we can notice some interesting properties of our polynomials.

Theorem 2.2. *Let* $\alpha \in (-\pi, \pi]$ *be an angle and the ray* L_s, *after the rotation for the angle* α, *becomes* L_s^α. *Then, the sequence* $\{\pi_N^\alpha(z)\}_{N=0}^{+\infty}$ *orthogonal with respect to*

$$(f, g)_\alpha = \sum_{s=0}^{m-1} e^{-i\alpha} \varepsilon_s^{-1} \int_{L_s^\alpha} f(z) \overline{g(z)} |w_s(z e^{-i\alpha})| \, dz$$

can be expressed as

$$\pi_N^\alpha(z) = \pi_N(e^{-i\alpha} z),$$

where the polynomials $\{\pi_N(z)\}_{N=0}^{+\infty}$ *are orthogonal with respect to* (1.1).

Because of

$$\pi_{Mn+\nu}(z) = \sum_{j=0}^{Mn+\nu} \frac{\pi_{Mn+\nu}^{(j)}(0)}{j!} z^j = \sum_{k=0}^{M-1} \sum_j \frac{\pi_{Mn+\nu}^{(Mj+k)}(0)}{(Mj+k)!} z^{Mj+k}$$

and

$$\pi_{Mn+\nu}(z) = \sum_{k=0}^{M-1} z^k \sum_j \frac{\pi_{Mn+\nu}^{(Mj+k)}(0)}{(Mj+k)!} (z^M)^j,$$

we can define the polynomials

$$q_n^{(\nu)}(k;z) = \sum_j \frac{\pi_{Mn+\nu}^{(Mj+k)}(0)}{(Mj+k)!} z^j.$$

Obviously, it is

$$\text{degree } q_n^{(\nu)}(k;z) = \begin{cases} n, & \text{for } k \le \nu \\ n-1, & \text{for } k > \nu. \end{cases}$$

Then, we have

$$\pi_{Mn+\nu}(z) = q_n^{(\nu)}(0;z^M) + z q_n^{(\nu)}(1;z^M) + \cdots + z^{M-1} q_n^{(\nu)}(M-1;z^M).$$

Now, we can define the sequence of matrix polynomials $\{P_n(z)\}_{n=0}^{+\infty}$ by

$$P_n(z) = \begin{bmatrix} q_n^{(0)}(0;z) & \cdots & q_n^{(0)}(M-1;z) \\ \vdots & & \\ q_n^{(M-1)}(0;z) & \cdots & q_n^{(M-1)}(M-1;z) \end{bmatrix}, \quad N \ge 0.$$

Then, by [2–3], we conclude that there exists a positive definite matrix of measures, denote by $dM(z)$, such that

$$\int P_n(z) \cdot dM(z) \cdot P_m^*(z) = \delta_{mn} D_M,$$

where $D_M = \mathrm{diag}(d_{ii})$, for $d_{ii} > 0$, $i = 1, 2, \ldots, M$.

Thus, $\{P_n(z)\}_{n=0}^{+\infty}$ is a sequence of matrix polynomials orthogonal on the real line and it satisfies the matrix three-term recurrence relation

$$zP_n(z) = P_{n+1}(z) + A_n P_n(z) + B_n P_{n-1}(z), \qquad P_{-1} = 0, \quad P_0 = I.$$

3. Case of Equal Lengths, Angles and Weights

In this section and further we suppose that

$$l_s = l, \quad \varepsilon_s = e^{i2\pi s/M}, \quad 0 \le s \le M-1$$

and

$$|w_s(x\varepsilon_s)| = w(x), \quad x \in (0,l), \quad s = 0,1,\dots,M-1.$$

Then the inner product (1.1) becomes

$$(f,g) = \int_0^l \left(\sum_{s=0}^{M-1} f(x\varepsilon_s)\overline{g(x\varepsilon_s)} \right) w(x)\,dx. \tag{3.1}$$

Like in [9] we can prove:

Lemma 3.1. *For $p \in \mathbb{N}$ let $n = [p/M]$ and $\nu = p - Mn$. Then*

$$\sum_{s=0}^{M-1} \varepsilon_s^p = \sum_{s=0}^{M-1} \varepsilon_s^\nu = \begin{cases} M, & \textit{if} \ \ \nu = 0 \\ 0, & \textit{if} \ \ 1 \le \nu \le M-1. \end{cases}$$

Lemma 3.2. *The polynomials $\pi_N(z)$, $N = 0,1,\dots$, satisfy*

$$\pi_N(z\varepsilon_s) = \varepsilon_s^N \pi_N(z), \qquad s = 0,1,\dots,M-1.$$

Lemma 3.3. *For $0 \le \nu < N \le M-1$, we have $(z^N, \pi_\nu) = 0$.*

Lemma 3.4. *For $N = 0,1,\dots,M-1$ we have $\pi_N(z) = z^N$.*

Now, we can prove

Theorem 3.5. *The monic polynomials $\{\pi_N(z)\}_{N=0}^{+\infty}$ satisfy the recurrence relation*

$$\pi_{N+M}(z) = \left(z^M - \alpha_N\right)\pi_N(z) - \beta_N \pi_{N-M}(z), \quad N \ge 0, \tag{3.2}$$

$$\pi_N(z) = z^N, \quad N = 0,1,\dots,M-1,$$

where

$$\alpha_N = \frac{(z^M\pi_N, \pi_N)}{(\pi_N, \pi_N)}, \ N \ge 0, \quad \beta_N = \begin{cases} \dfrac{\|\pi_N\|^2}{\|\pi_{N-M}\|^2}, & N \ge M, \\ 0, & N \le M-1. \end{cases}$$

Proof. According to Theorem 2.1 we have $(2M+1)$-recurrence relation with the coefficients

$$\alpha_k^{(N)} = \frac{(z^M \pi_N, \pi_k)}{(\pi_k, \pi_k)}, \qquad N - M \leq k \leq N + M - 1.$$

Now, we yield

$$\begin{aligned}
(z^M \pi_N, \pi_k) &= \int_0^l \left(\sum_{s=0}^{M-1} (x\varepsilon_s)^M \pi_N(x\varepsilon_s) \overline{\pi_k(x\varepsilon_s)} \right) w(x)\, dx \\
&= \int_0^l \left(\sum_{s=0}^{M-1} x^M \varepsilon_s^N \pi_N(x) \overline{\varepsilon_s^k \pi_k(x)} \right) w(x)\, dx \\
&= \sum_{s=0}^{M-1} \varepsilon_s^{N-k} \int_0^l x^M \pi_N(x) \overline{\pi_k(x)} w(x)\, dx.
\end{aligned}$$

According to Lemma 3.1, the value of $(z^M \pi_N, \pi_k)$ will be different of zero only for $N-k=0$ and $N-k=M$. Thus,

$$\pi_{N+M}(z) = z^M \pi_N(z) - \alpha_N^{(N)} \pi_N(z) - \alpha_{N-M}^{(N)} \pi_{N-M}(z).$$

Since

$$\alpha_{N-M}^{(N)} = \frac{(z^M \pi_N, \pi_{N-M})}{(\pi_{N-M}, \pi_{N-M})} = \frac{(\pi_N, z^M \pi_{N-M})}{(\pi_{N-M}, \pi_{N-M})},$$

we get coefficients in (3.2) as $\alpha_N = \alpha_N^{(N)}$ and $\beta_N = \alpha_{N-M}^{(N)}$.

Remark 3.1. In the case of even number of rays there exist a few simpler relations (see [9]).

4. Representation of Polynomials $\pi_N(z)$

In this section we conclude that $\pi_N(z)$ are incomplete polynomials with the following representation:

Lemma 4.1. *The polynomials $\pi_N(z)$ can be expressed in the form*

$$\pi_N(z) = \sum_{j=0}^{[N/M]} \gamma_{N-Mj} z^{N-Mj}, \tag{4.1}$$

where $\gamma_N = 1$ and $\gamma_{N-Mj} \in \mathbb{R}$, $j = 1, 2, \ldots, [N/M]$.

Proof. It is obvious from the recurrence relation (3.2) and Lemma 3.4. $\square$

Lemma 4.2. *For the polynomials $\pi_N(z)$ we have the following representation*

$$\pi_{Mn+\nu}(z) = z^\nu q_n^{(\nu)}(z^M),\ \nu = 0, \dots , M-1;\ n = 0, 1, \dots , \tag{4.2}$$

where $q_n^{(\nu)}(t)$, $\nu = 0, 1, \dots , M-1$ are monic polynomials of the degree n.

Proof. ¿From Lemma 4.1, we have

$$\pi_{Mn+\nu}(z) = \sum_{j=0}^{n} \gamma_{Mj+\nu} z^{Mj+\nu},$$

i.e.,

$$\pi_{Mn+\nu}(z) = z^\nu \sum_{j=0}^{n} \gamma_{Mj+\nu} \left(z^M\right)^j ,$$

wherefrom we get (4.2). □

For the introduced polynomials $q_n^{(\nu)}(z)$ we have the following two results:

Theorem 4.3. *The monic polynomials $\{q_n^{(\nu)}(t)\}_{n=0}^{+\infty}$, $0 \le \nu \le M-1$, satisfy the three-term recurrence relation*

$$q_{n+1}^{(\nu)}(t) = (t - a_n^{(\nu)})q_n^{(\nu)}(t) - b_n^{(\nu)} q_{n-1}^{(\nu)}(t), \quad n = 0, 1, \dots , \tag{4.3}$$

$$q_0^{(\nu)}(t) = 1, \quad q_{-1}^{(\nu)}(t) = 0,$$

where $a_n^{(\nu)} = \alpha_N$ and $b_n^{(\nu)} = \beta_N$ for $N = Mn + \nu$, $n \in \mathbb{N}_0$.

Proof. For a given N, using the recurrence relation (3.2), we have

$$\pi_{M(n+1)+\nu}(z) = \left(z^M - \alpha_N\right) \pi_{Mn+\nu}(z) - \beta_N \pi_{M(n-1)+\nu}(z).$$

By (4.2), we yield

$$z^\nu q_{n+1}^{(\nu)}(z^M) = (z^M - \alpha_N) z^\nu q_n^{(\nu)}(z^M) - \beta_N z^\nu q_{n-1}^{(\nu)}(z^M),$$

wherefrom, introducing $t = z^M$, we obtain

$$q_{n+1}^{(\nu)}(t) = (t - \alpha_N) q_n^{(\nu)}(t) - \beta_N q_{n-1}^{(\nu)}(t), \quad n = 0, 1, \dots . \quad \square$$

The three-term recurrence relation (4.3) suggests the orthogonality of the sequences $\{q_n^{(\nu)}(t)\}_{n=0}^{+\infty}$, $\nu \in \{0, 1, \dots , M-1\}$ on the real line.

Theorem 4.4. *Let $x \mapsto w(x)$ be a weight function which enables the existence of the polynomials $\{\pi_N(z)\}_{N=0}^{+\infty}$. Then, for any $\nu \in \{0,1,\dots,M-1\}$, the sequence of polynomials $\{q_n^{(\nu)}(t)\}_{n=0}^{+\infty}$ is orthogonal on $(0,l^M)$ with respect to the weight function*

$$t \mapsto w_\nu(t) = t^{(2\nu+1)/M-1} w(t^{1/M}).$$

Proof. For $N = Mn+\nu$ and $K = Mk+\nu$, where $n,k \in \mathbb{N}_0$, we have

$$\begin{aligned}
(\pi_N, \pi_K) &= \int_0^l \left(\sum_{s=0}^{M-1} \pi_N(x\varepsilon_s)\overline{\pi_K(x\varepsilon_s)}\right) w(x)\,dx \\
&= \int_0^l \left(\sum_{s=0}^{M-1} \varepsilon_s^N \pi_N(x)\overline{\varepsilon_s^K \pi_K(x)}\right) w(x)\,dx \\
&= \int_0^l \left(\sum_{s=0}^{M-1} \varepsilon_s^\nu \pi_N(x)\overline{\varepsilon_s^\nu \pi_K(x)}\right) w(x)\,dx \\
&= \int_0^l \left(\sum_{s=0}^{M-1} \pi_N(x)\overline{\pi_K(x)}\right) w(x)\,dx \\
&= M\int_0^l \pi_N(x)\overline{\pi_K(x)} w(x)\,dx \\
&= M\int_0^l x^\nu q_n^{(\nu)}(x^M) x^\nu q_k^{(\nu)}(x^M) w(x)\,dx.
\end{aligned}$$

Taking $t = x^M$ we get

$$(\pi_N, \pi_K) = \int_0^{l^M} q_n^{(\nu)}(t) q_k^{(\nu)}(t) t^{(2\nu+1)/M-1} w(t^{1/M})\,dt.$$

Because of $(\pi_N, \pi_K) = \delta_{NK}\|\pi_N\|^2 = \delta_{nk}\|\pi_N\|^2$, we conclude that

$$\int_0^l q_n^{(\nu)}(t) q_k^{(\nu)}(t) t^{(2\nu+1)/M-1} w(t^{1/M})\,dt = \delta_{nk}\|\pi_N\|^2. \quad \square$$

5. The Joined Orthogonal Matrix Polynomials

The results of previous sections can be rewritten in a simpler form, using the matrices defined in Section 2.

To every polynomial $\pi_N(z)$, where $N = Mn + \nu$, $0 \le \nu \le M-1$, we can join the matrix polynomial

$$P_n(x) = \mathrm{diag}\big(q_n^{(0)}(x), q_n^{(1)}(x), \dots, q_n^{(M-1)}(x)\big),$$

where $q_n^{(k)}(x) = q_n^{(k)}(k;x)$, $0 \le k \le M-1$, and a matrix of the weight functions

$$W(x) = \mathrm{diag}\big(w_0(x), w_1(x), \dots, w_{M-1}(x)\big).$$

Then we have

$$\int_0^{l^M} P_m(x) \cdot W(x) \cdot P_n^*(x)\, dx = \delta_{mn} D_n,$$

where

$$D_n = \mathrm{diag}\big(\|q_n^{(0)}\|^2, \dots, \|q_n^{(M-1)}\|^2\big)$$

and $P_n^*(x)$ denotes the conjugate and transpose matrix of $P_n(x)$.

Therefore, $\{P_N(z)\}_{N=0}^{+\infty}$ is a sequence of the matrix polynomials orthogonal on $(0, l)$ with respect to the positive definite matrix of weights $W(x)$. This sequence satisfies the matrix three-term recurrence relation

$$xP_n(x) = P_{n+1}(x) + A_n P_n(x) + B_n P_{n-1}(x), \quad P_{-1} = 0, \quad P_0 = I,$$

where

$$A_n = \mathrm{diag}\big(a_n^{(0)}, a_n^{(1)}, \dots, a_n^{(M-1)}\big), \quad B_n = \mathrm{diag}\big(b_n^{(0)}, b_n^{(1)}, \dots, b_n^{(M-1)}\big).$$

6. Zeros of $\pi_N(z)$ and Christoffel-Darboux Identity

For zeros of $\pi_N(z)$ we can prove:

Theorem 6.1. *All zeros of the polynomial $\pi_N(z)$ are simple and located on the radial rays, with possible exception of a multiple zero in origin $z = 0$ of the order ν if $N \equiv \nu \pmod M$.*

Proof. The proof is very close to one in [9]. □

Let $\tau_k^{(n,\nu)}$, $k = 1, \ldots, n$, denote the zeros of $q_n^{(\nu)}(t)$ in an increasing order

$$\tau_1^{(n,\nu)} < \tau_2^{(n,\nu)} < \cdots < \tau_n^{(n,\nu)}.$$

Each zero $\tau_k^{(n,\nu)}$ generates M zeros

$$z_{k,s}^{(n,\nu)} = \sqrt[M]{\tau_k^{(n,\nu)}}\, e^{i2\pi s/M}, \quad s = 0, 1, \ldots, M-1,$$

of $\pi_N(z)$. On every ray we have

$$|z_{1,s}^{(n,\nu)}| < |z_{2,s}^{(n,\nu)}| < \cdots < |z_{n,s}^{(n,\nu)}|, \quad s = 0, 1, \ldots, M-1.$$

Using the same notation, from Theorem 2.2 it follows:

Theorem 6.2. *The zeros of $\pi_N^\alpha(z)$ are the zeros of $\pi_N(z)$ rotated for the angle α.*

The properties of the zeros of $\pi_N(z)$ were completely discussed in our paper [10]. Here we mention the main conclusion. By our computational investigations about zeros, we can state the following conjecture:

Conjecture 6.1. *All zeros of the polynomial $\pi_N(z)$ orthogonal with respect to (1.1) are located in the convex hull of the endpoints of the rays.*

It is easy to prove:

Theorem 6.3. *The monic polynomials $\{\pi_N(z)\}_{N=0}^{+\infty}$ satisfy the identity of Christoffel-Darboux type*

$$\sum_{k=0}^{n} \frac{\pi_{kM+\nu}(z)\pi_{kM+\nu}(t)}{\|\pi_{kM+\nu}\|^2} = \frac{1}{\|\pi_{nM+\nu}\|^2} \times \\ \times \frac{\pi_{(n+1)M+\nu}(z)\pi_{nM+\nu}(t) - \pi_{(n+1)M+\nu}(t)\pi_{nM+\nu}(z)}{z^M - t^M}$$

for $\nu = 0, 1, \ldots, M-1$ and $n = 0, 1, \ldots$.

7. Differential Equation

Using relations from Section 4, we can prove:

Theorem 7.1. *If a monic polynomial* $q_n^{(\nu)}(t)$, $\nu = 0, 1, \ldots, M-1$, *where* M *is a number of the rays, satisfies the differential equation*

$$c_0^{(\nu)}(t)y'' + c_1^{(\nu)}(t)y' + c_2^{(\nu)}(t,n)y = 0, \tag{7.1}$$

then the monic polynomial $\pi_{Mn+\nu}(z)$ *satisfies the following differential equation*

$$d_0^{(\nu)}(z)Y'' + d_1^{(\nu)}(z)Y' + d_2^{(\nu)}(z,n)Y = 0, \tag{7.2}$$

where

$$\begin{aligned}
d_0^{(\nu)}(z) &= z^2 c_0^{(\nu)}(z^M),\\
d_1^{(\nu)}(z) &= (1-2\nu-M)zc_0^{(\nu)}(z^M) + Mz^{M+1}c_1^{(\nu)}(z^M),\\
d_2^{(\nu)}(z,n) &= \nu(\nu+M)c_0^{(\nu)}(z^M) - \nu M z^M c_1^{(\nu)}(z^M) + M^2 z^{2M} c_2^{(\nu)}(z^M).
\end{aligned}$$

Proof. From Lemma 4.2, we have $\pi_{Mn+\nu}(z) = z^\nu q_n^{(\nu)}(z^M)$. Introducing using a new variable $t = z^M$, we obtain

$$\begin{aligned}
q_n^{(\nu)}(t) &= t^{-\nu/M}\pi_{Mn+\nu}(z),\\
\frac{\partial q_n^{(\nu)}(t)}{\partial t} &= -\frac{\nu}{M}t^{-(\nu+M)/M}\pi_{Mn+\nu}(z) + \frac{1}{M}t^{(1-\nu-M)/M}\frac{\partial \pi_{Mn+\nu}(z)}{\partial z}\\
\frac{\partial^2 q_n^{(\nu)}(t)}{\partial t^2} &= \frac{\nu(\nu+M)}{M^2}t^{-(\nu+2M)/M}\pi_{Mn+\nu}(z) + \frac{1-2\nu-M}{M^2}t^{(1-\nu-2M)/M}\times\\
&\qquad\times\frac{\partial \pi_{Mn+\nu}(z)}{\partial z} + \frac{1}{M^2}t^{(2-\nu-2M)/M}\frac{\partial^2 \pi_{Mn+\nu}(z)}{\partial z^2}.
\end{aligned}$$

Putting these expressions in (7.1) we find (7.2). $\square$

Theorem 7.2. *Under conditions of Theorem* 7.1 *the joined matrix polynomial satisfies the differential matrix equation*

$$C_0(t,n)P_n''(t) + C_1(t,n)P_n'(t) + C_2(t,n)P_n(t) = 0,$$

where

$$C_k = \operatorname{diag}(c_k^{(0)}, c_k^{(1)}, \ldots, c_k^{(M-1)}), \qquad k = 0, 1, 2.$$

8. Polynomials Orthogonal With the Weight $w(z)$=1

We consider M radial rays which connect $z=0$ and the points

$$\varepsilon_s = e^{i2\pi s/M}, \quad s = 0, 1, \ldots, M-1.$$

The corresponding inner product is given by

$$(f, g) = \int_0^1 \sum_{s=0}^{M-1} f(x\varepsilon_s)\overline{g(x\varepsilon_s)}\, dx. \tag{8.1}$$

Using the moments

$$\mu_{p,q} = (z^p, z^q) = \begin{cases} \dfrac{M}{p+q+1}, & p \equiv q (\operatorname{mod} M), \\ 0, & \text{otherwise}, \end{cases}$$

i.e.,

$$\mu_{Mi+\nu, Mj+\nu} = \frac{M}{M(i+j)+2\nu+1}, \quad 0 \le \nu \le M-1, \quad i, j \ge 0.$$

we can evaluate the moment-determinants

$$\Delta_0 = 1, \quad \Delta_N = \begin{vmatrix} \mu_{00} & \mu_{10} & \cdots & \mu_{N-1,0} \\ \mu_{01} & \mu_{11} & & \mu_{N-1,1} \\ \vdots & & & \\ \mu_{0,N-1} & \mu_{1,N-1} & & \mu_{N-1,N-1} \end{vmatrix}, \quad N \ge 1,$$

in the following form

$$\Delta_{Mn} = E_n^{(0)} E_n^{(1)} \cdots E_n^{(M-1)},$$

$$\Delta_{Mn+\nu} = \prod_{i=0}^{\nu-1} E_{n+1}^{(i)} \prod_{j=\nu}^{M-1} E_n^{(j)}, \quad 0 < \nu < M,$$

where $E_0^{(\nu)} = 1$ and

$$E_n^{(\nu)} = \begin{vmatrix} \mu_{\nu,\nu} & \mu_{M+\nu,\nu} & \cdots & \mu_{M(n-1)+\nu,\nu} \\ \mu_{\nu,M+\nu} & \mu_{M+\nu,M+\nu} & & \mu_{M(n-1)+\nu,M+\nu} \\ \vdots & & & \\ \mu_{\nu,M(n-1)+\nu} & \mu_{M+\nu,M(n-1)+\nu} & & \mu_{M(n-1)+\nu,M(n-1)+\nu} \end{vmatrix}.$$

The value of $E_n^{(\nu)}$ is

$$E_n^{(\nu)} = M^{n^2} \frac{[0!1!\cdots(n-1)!]^2}{\prod\limits_{i,j=0}^{n-1}[M(i+j)+2\nu+1]}$$

and

$$\|\pi_{Mn+\nu}\|^2 = \frac{E_{n+1}^{(\nu)}}{E_n^{(\nu)}} = \begin{cases} \dfrac{M}{2\nu+1}, & n=0, \\ \dfrac{M}{2nM+2\nu+1}\left(\prod\limits_{k=n}^{2n-1} \dfrac{M(k-n+1)}{Mk+2\nu+1}\right)^2, & n \geq 1, \end{cases}$$

where $0 \leq \nu \leq M-1$.

Remark 8.1. In the case $M = 2m$, the monic polynomials $\{\pi_N(z)\}_{N=0}^{+\infty}$, satisfy the recurrence relation (see [9])

$$\pi_{N+m}(z) = z^m \pi_N(z) - b_N \pi_{N-m}(z), \quad N \geq m, \tag{8.2}$$

$$\pi_N(z) = z^N, \quad N = 0, 1, \ldots, M-1,$$

where

$$b_N = \begin{cases} \dfrac{\|\pi_N\|^2}{\|\pi_{N-m}\|^2}, & N \geq m, \\ 0, & N \leq m-1. \end{cases}$$

Hence, we have

$$b_N = \frac{\Delta_{N+1}}{\Delta_N} \bigg/ \frac{\Delta_{N-m+1}}{\Delta_{N-m}}$$

and

$$b_{Mn+\nu} = \frac{E_{n+1}^{(\nu)}}{E_n^{(\nu)}} \bigg/ \frac{\Delta_{Mn+\nu-m+1}}{\Delta_{Mn+\nu-m}}.$$

Now, it gives

$$b_{Mn+\nu} = \begin{cases} \dfrac{E_{n+1}^{(\nu)}}{E_n^{(\nu)}} \bigg/ \dfrac{E_n^{(\nu+m)}}{E_{n-1}^{(\nu+m)}} & 0 \leq \nu \leq m-1, \\ \dfrac{E_{n+1}^{(\nu)}}{E_n^{(\nu)}} \bigg/ \dfrac{E_{n+1}^{(\nu-m)}}{E_n^{(\nu-m)}} & m \leq \nu \leq M-1, \end{cases}$$

i.e.,

$$b_{Mn+\nu} = \begin{cases} \dfrac{(Mn)^2}{(2nM+2\nu+1)(2nM+2\nu+1-M)}, & 0 \le \nu \le m-1, \\ \dfrac{(Mn+2\nu+1-M)^2}{(2nM+2\nu+1)(2nM+2\nu+1-M)}, & m \le \nu \le M-1. \end{cases}$$

For every M, the monic polynomials $\{\pi_N(z)\}_{N=0}^{+\infty}$ satisfy the recurrence relation

$$\pi_{N+M}(z) = (z^M - \alpha_N)\pi_N(z) - \beta_N \pi_{N-M}(z), \quad N \ge M,$$

$$\pi_N(z) = z^N, \quad N = 0, 1, \ldots, M-1,$$

where, for $N = Mn + \nu$ $\big(n = [N/M],\ \nu \in \{0, 1, \ldots, M-1\}\big)$, we have

$$\alpha_N = \frac{1}{2}\left[1 + \frac{q^2}{(2n+q)(2n+q+2)}\right], \quad \beta_N = \frac{n^2(n+q)^2}{(2n+q)^2[(2n+q)^2-1]},$$

with $q = (2\nu+1)/M - 1$. In an explicit form, we have

$$\alpha_N = \frac{2(Mn)^2 + (2\nu+1)[(2n-1)M + 2\nu + 1]}{[(2n-1)M + 2\nu + 1][(2n+1)M + 2\nu + 1]},$$

$$\beta_N = \frac{(Mn)^2[(n-1)M + 2\nu + 1]^2}{[(2n-2)M + 2\nu + 1][(2n-1)M + 2\nu + 1]^2[2nM + 2\nu + 1]}.$$

Remark 8.2. In an even case $M = 2m$, using (8.2) we find

$$\pi_{N+M}(z) = (z^M - \alpha_N)\pi_N(z) - \beta_N \pi_{N-M}(z),$$

where

$$\alpha_N = b_N + b_{N+m}, \qquad \beta_N = b_N b_{N-m}.$$

Now, we denote by $G_n(p, x)$ the monic Jacobi polynomial orthogonal with respect to the weight $w_p(x) = x^{p-1}$, $p > 0$, on the interval $(0, 1)$. On the other hand, by Lemma 4.2, we conclude that is

$$\pi_{Mn+\nu}(z) = z^\nu q_n^{(\nu)}(z^M), \quad \nu = 0, 1, \ldots, M-1,$$

where $\{q_n^{(\nu)}(t)\}_{n=0}^{+\infty}$ are orthogonal on $(0, 1)$ with respect to the weight function

$$t \mapsto w_\nu(t) = t^{(2\nu+1)/M-1}, \quad \nu = 0, 1, \ldots, M-1.$$

Theorem 8.1. *The monic polynomials* $\{\pi_N(z)\}_{N=0}^{+\infty}$ *can be expressed in the form*

$$\pi_{Mn+\nu}(z) = \frac{\Gamma\left(n+\frac{2\nu+1}{M}\right)}{\Gamma\left(2n+\frac{2\nu+1}{M}\right)} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{\Gamma\left(2n-k+\frac{2\nu+1}{M}\right)}{\Gamma\left(n-k+\frac{2\nu+1}{M}\right)} z^{M(n-k)+\nu}.$$

Proof. According to [5], the monic Jacobi polynomials $G_n(p,x)$ can be expressed in the form

$$G_n(p,x) = \frac{\Gamma(p+n)}{\Gamma(p+2n)} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{\Gamma(p+2n-k)}{\Gamma(p+n-k)} x^{n-k}.$$

In order to prove our result we put $p = (2\nu+1)/M$. $\square$

Theorem 8.2. *The monic polynomials* $\{\pi_N(z)\}_{N=0}^{+\infty}$ *have the generating function*

$$\sum_{n=0}^{+\infty} \sum_{\nu=0}^{M-1} 2^{1-(2\nu+1)/M} \frac{\Gamma\left(2n+\frac{2\nu+1}{M}\right)}{n!\Gamma\left(n+\frac{2\nu+1}{M}\right)} \pi_{Mn+\nu}(z) w^{Mn+\nu}$$
$$= R^{-1}(1+w^M+R)^{1/M-1} \frac{(zw)^M - (1+w^M+R)^2}{zw-(1+w^M+R)^{2/M}},$$

where $R = \sqrt{1-2(2z^M-1)w^M+w^{2M}}$.

Proof. The monic polynomials $G_n(p,x)$ have the generating function

$$\sum_{n=0}^{+\infty} 2^{1-p} \frac{\Gamma(p+2n)}{n!\Gamma(p+n)} G_n(p,x) u^n = R_G^{-1}(1+u+R_G)^{1-p},$$

where $R_G = \sqrt{1-2(2x-1)u+u^2}$. Using $p = (2\nu+1)/M$ and two new variables t and u, defined by $t = z^M$ and $u = w^M$, for $\nu = 0, 1, \ldots, M-1$, we yield

$$\sum_{n=0}^{+\infty} 2^{1-(2\nu+1)/M} \frac{\Gamma\left(2n+\frac{2\nu+1}{M}\right)}{n!\Gamma\left(n+\frac{2\nu+1}{M}\right)} \pi_{Mn+\nu}(z) w^{Mn+\nu}$$
$$= (zw)^\nu R^{-1}(1+w^M+R)^{1-(2\nu+1)/M}.$$

After making the sum by ν, we end the proof. $\square$

Theorem 8.3. *The monic polynomials* $\{\pi_N(z)\}_{N=0}^{+\infty}$ *satisfy the Rodrigues type formula*

$$\pi_{Mn+\nu}(\sqrt[M]{x}) = (-1)^n \frac{\Gamma(\frac{2\nu+1}{M}+n)}{\Gamma(\frac{2\nu+1}{M}+2n)} x^{1-\frac{2\nu+1}{M}} \frac{\partial^n}{\partial x^n}\left[(1-x)^n x^{n+\frac{2\nu+1}{M}-1}\right],$$

where $\nu = 0, 1, \ldots, M-1$ *and* $n = 0, 1, \ldots$.

Proof. This follows from the Rodrigues formula

$$G_n(p,x) = (-1)^n \frac{\Gamma(p+n)}{\Gamma(p+2n)} x^{1-p} \frac{\partial^n}{\partial x^n}\left[(1-x)^n x^{n+p-1}\right]. \quad \square$$

9. Orthogonal Polynomials of the Generalized Laguerre Type

In the same way like in the previous section, we consider

$$(f,g) = \int_0^{+\infty} \sum_{s=0}^{M-1} f(x\varepsilon_s)\overline{g(x\varepsilon_s)}\,dx, \quad \varepsilon_s = e^{i2\pi s/M}, \tag{9.1}$$

where

$$w(x) = x^{M\gamma} e^{-x^M}, \quad \gamma M + 1 > 0.$$

We denote by $\hat{L}_n^{(s)}(x)$ the monic generalized Laguerre polynomials orthogonal with respect to the weight $w_s(x) = x^s e^{-x}$ on the interval $(0, +\infty)$. On the other hand, by Lemma 4.2, we conclude that is

$$\pi_{Mn+\nu}(z) = z^\nu \hat{L}_n^{(\alpha_\nu)}(z^M), \quad \nu = 0, 1, \ldots, M-1,$$

where

$$\alpha_\nu = \gamma - 1 + \frac{2\nu+1}{M}.$$

Thus, using results from Section 4 and the previous facts, we prove:

Theorem 9.1. *The monic polynomials* $\{\pi_N(z)\}_{N=0}^{+\infty}$ *can be expressed in the form*

$$\pi_{Mn+\nu}(z) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \frac{\Gamma(\alpha_\nu+n+1)}{\Gamma(\alpha_\nu+k+1)} z^{Mk+\nu}.$$

Proof. According to [5], the monic generalized Laguerre polynomials can be expressed by

$$\hat{L}_n^{(s)}(x) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \frac{\Gamma(s+n+1)}{\Gamma(s+k+1)} x^k.$$

Now, we use it for $s = \alpha_\nu$. $\square$

Also, we can prove:

Theorem 9.2. *The monic polynomials* $\{\pi_N(z)\}_{N=0}^{+\infty}$ *have the generating function*

$$\sum_{n=0}^{+\infty}\sum_{\nu=0}^{M-1}\pi_{Mn+\nu}(z)\frac{w^{Mn+\nu}}{n!}=\frac{(zw)^M-(1+w^M)^2}{zw-(1+w^M)^{2/M}}\cdot\frac{e^{(zw)^M/(1+w^M)}}{(1+w^M)^{2+\gamma-1/M}}.$$

Theorem 9.3. *The monic polynomials* $\{\pi_N(z)\}_{N=0}^{+\infty}$ *satisfy the Rodrigues type formula*

$$\pi_{Mn+\nu}(\sqrt[M]{x})=(-1)^n x^{\nu/M-\alpha_\nu}e^x\frac{\partial^n}{\partial x^n}\left[x^{n+\alpha_\nu}e^{-x}\right],$$

where $\nu=0,1,\dots,M-1$ *and* $n=0,1,\dots$.

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