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CONSTRUCTION OF MULTIPLE ORTHOGONAL POLYNOMIALS BY DISCRETIZED STIELTJES-GAUTSCHI PROCEDURE AND CORRESPONDING GAUSSIAN QUADRATURES*

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Abstract. In this paper we consider multiple orthogonal polynomials, defined using orthogonality conditions spread out over r different measures. Such polynomials satisfy a linear recurrence relation of order $r+1$. This is a generalization of the second order linear recurrence relation for ordinary monic orthogonal polynomials (the case $r = 1$). Using the discretized Stieltjes-Gautschi procedure, we compute recurrence coefficients and also zeros of multiple orthogonal polynomials, as well as the weight coefficients for the corresponding quadrature formulae of Gaussian type. Some numerical examples are also included.

1. Introduction

Multiple orthogonal polynomials are a generalization of orthogonal polynomials in the sense that they satisfy r orthogonality conditions.

Let $r \geq 1$ be an integer and let $w_1, w_2, \ldots, w_r$ be r weight functions on the real line so that the support of each w_i is a subset of an interval E_i. Let $\vec{n} = (n_1, n_2, \ldots, n_r)$ be a vector of r nonnegative integers, which is called a *multi-index* with the length $|\vec{n}| = n_1 + n_2 + \cdots + n_r$.

There are two types of multiple orthogonal polynomials:

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1° *Type I multiple orthogonal polynomials.* Here we want to find a vector of polynomials $(A_{\vec{n},1}, A_{\vec{n},2}, \ldots, A_{\vec{n},r})$ such that each $A_{\vec{n},i}$ is a polynomial of degree $n_i - 1$ and the following orthogonality conditions hold:

$$(1.1) \qquad \sum_{j=1}^{r} \int_{E_j} A_{\vec{n},j}\, x^k w_j(x) dx = 0, \qquad k = 0, 1, \ldots, |\vec{n}| - 2.$$

Each $A_{\vec{n},i}$ has n_i coefficients and the type I vector is completely determined if we can find all the $|\vec{n}|$ unknown coefficients. The orthogonality relations (1.1) give $|\vec{n}| - 1$ homogenous linear equations for these $|\vec{n}|$ coefficients. If the matrix of coefficients has full rank, then we can determine the type I vector uniquely up to a multiplicative factor.

For $r = 1$ we have the case of of ordinary orthogonal polynomials.

2° *Type II multiple orthogonal polynomials.* The type II multiple orthogonal polynomial is a monic polynomial $\pi_{\vec{n}}$ of degree $|\vec{n}|$ such that the following orthogonality conditions:

$$(1.2) \qquad \int_{E_1} \pi_{\vec{n}}(x)\, x^k w_1(x) dx = 0, \qquad k = 0, 1, \ldots, n_1 - 1,$$

$$(1.3) \qquad \int_{E_2} \pi_{\vec{n}}(x)\, x^k w_2(x) dx = 0, \qquad k = 0, 1, \ldots, n_2 - 1,$$

$$\vdots$$

$$(1.4) \qquad \int_{E_r} \pi_{\vec{n}}(x)\, x^k w_r(x) dx = 0, \qquad k = 0, 1, \ldots, n_r - 1,$$

are satisfied.

Again, for $r = 1$ we have the ordinary orthogonal polynomials.

The conditions (1.2)–(1.4) give $|\vec{n}|$ linear equations for the $|\vec{n}|$ unknown coefficients $a_{k,\vec{n}}$ of the polynomial $\pi_{\vec{n}}(x) = \sum_{k=0}^{|\vec{n}|} a_{k,\vec{n}}\, x^k$, where $a_{|\vec{n}|,\vec{n}} = 1$. But the matrix of coefficients of this system can be singular and we need some additional conditions on the r weight functions to provide the uniqueness of the multiple orthogonal polynomial. If the polynomial $\pi_{\vec{n}}(x)$ is unique, then we say that $\vec{n}$ is a *normal multi-index* and if all multi-indices are normal then we have a *complete system*.

In this paper we consider only the type II multiple orthogonal polynomials. For each of the weight functions w_k, $k = 1, 2, \ldots, r$,

$$(1.5) \qquad (f, g)_k = \int_{E_k} f(x) g(x) w_k(x) dx$$

denotes the corresponding inner product of the functions f and g.

The paper is organized as follows. Some basic facts about the type II multiple orthogonal polynomials are given in Section 2. In Section 3 we consider type II multiple orthogonal polynomials with nearly diagonal multi-indices and the corresponding recurrence relations. In Section 4 we develop a numerical procedure for construction of type II multiple orthogonal polynomials based on the Stieltjes-Gautschi procedure [5]. An optimal set of quadrature formulas and the corresponding method for calculating the nodes and weight coefficients of such quadratures are considered in Section 5. Finally, a few numerical examples are presented in Section 6.

2. Type II Multiple Orthogonal Polynomials

Our interest is in systems of r weight functions for which all multi-indices are normal. There are two distinct cases for which the type II multiple orthogonal polynomials are given.

1. *Angelesco systems* for which the intervals E_i, on which the weight functions are supported, are disjoint, i.e., $E_i \cap E_j = \varnothing$ for $1 \leq i, j \leq r$, $i \neq j$.

Notice that it is sufficient that the open intervals $\overset{\circ}{E}_i$ are disjoint, so that the closed intervals E_i are allowed to touch.

2. *AT system* is such that all weight functions are supported on the same interval E and we also require that the $|\vec{n}|$ functions

$$w_1(x),\, xw_1(x),\, \ldots,\, x^{n_1-1}w_1(x),\;\; w_2(x),\, xw_2(x),\, \ldots,\, x^{n_2-1}w_2(x),\\ \ldots,\, w_r(x),\, xw_r(x),\, \ldots,\, x^{n_r-1}w_r(x)$$

form a Chebyshev system on E for each multi-index $\vec{n}$. This means that every linear combination

$$\sum_{j=1}^{r} Q_{n_j-1}(x)w_j(x),$$

where Q_{n_j-1} is a polynomial of degree at most $n_j - 1$, has at most $|\vec{n}| - 1$ zeros on E.

The following two theorems have been proved in [11]:

Theorem 2.1. *In an Angelesco system the type II multiple orthogonal polynomial* $\pi_{\vec{n}}(x)$ *factors into* r *polynomials* $\prod_{j=1}^{r} q_{n_j}(x)$, *where each* q_{n_j} *has exactly* n_j *zeros on* E_j.

Theorem 2.2. *In an AT system the type II multiple orthogonal polynomial* $\pi_{\vec{n}}(x)$ *has exactly* $|\vec{n}|$ *zeros on* E. *For the type I vector of multiple orthogonal polynomials, the linear combination* $\sum_{j=1}^{r} A_{\vec{n},j}(x)w_j(x)$ *has exactly* $|\vec{n}|-1$ *zeros on* E.

3. Recurrence Relations

Orthogonal polynomials on the real line always satisfy a three-term recurrence relation. There is an interesting recurrence relation of order $r+1$ for the type II multiple orthogonal polynomials with nearly diagonal multi-index. Let $n \in \mathbb{N}$ and write it as $n = \ell r + j$, with $\ell = [n/r]$ and $0 \le j < r$. The nearly diagonal multi-index $\vec{s}(n)$ corresponding to n is given by

$$\vec{s}(n) = (\underbrace{\ell+1, \ell+1, \ldots, \ell+1}_{j \text{ times}}, \underbrace{\ell, \ell, \ldots, \ell}_{r-j \text{ times}}).$$

Denote the corresponding type II multiple (monic) orthogonal polynomials by

$$\pi_n(x) = \pi_{\vec{s}(n)}(x).$$

Then, the following recurrence relation

$$x\pi_m(x) = \pi_{m+1}(x) + \sum_{i=0}^{r} \alpha_{m,r-i}\pi_{m-i}(x)\,, \quad m \ge 0, \tag{3.1}$$

holds, with initial conditions $\pi_0(x) = 1$ and $\pi_i(x) = 0$ for $i = -1, -2, \ldots, -r$ (see [10]).

Setting $m = 0, 1, \ldots, n-1$ in (3.1), we get

$$x \begin{bmatrix} \pi_0(x) \\ \pi_1(x) \\ \vdots \\ \pi_{n-1}(x) \end{bmatrix} = H_n \begin{bmatrix} \pi_0(x) \\ \pi_1(x) \\ \vdots \\ \pi_{n-1}(x) \end{bmatrix} + \pi_n(x) \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},$$

i.e.,

$$H_n \mathbf{p}_n(x) = x\,\mathbf{p}_n(x) - \pi_n(x)\mathbf{e}_n, \tag{3.2}$$

where

$$\mathbf{p}_n(x) = [\pi_0(x) \;\; \pi_1(x) \;\; \ldots \;\; \pi_{n-1}(x)]^T, \qquad \mathbf{e}_n = [0 \;\; 0 \;\; \ldots \;\; 0 \;\; 1]^T,$$

and H_n is the following lower (banded) Hessenberg matrix of order n

$$H_n = \begin{bmatrix} \alpha_{0,r} & 1 & & & & & & & \\ \alpha_{1,r-1} & \alpha_{1,r} & 1 & & & & & & \\ \vdots & \ddots & \ddots & \ddots & & & & & \\ \alpha_{r,0} & \cdots & \alpha_{r,r-1} & \alpha_{r,r} & 1 & & & & \\ & \alpha_{r+1,0} & \cdots & \alpha_{r+1,r-1} & \alpha_{r+1,r} & 1 & & & \\ & & \ddots & & \ddots & \ddots & \ddots & & \\ & & & \alpha_{n-2,0} & \cdots & \alpha_{n-2,r-1} & \alpha_{n-2,r} & 1 \\ & & & & \alpha_{n-1,0} & \cdots & \alpha_{n-1,r-1} & \alpha_{n-1,r} \end{bmatrix}.$$

This kind of matrix was obtained also in construction of orthogonal polynomials on the radial rays in the complex plane (see [7]).

Let $\xi_\nu \equiv \xi_\nu^{(n)}$ $(\nu = 1, \dots, n)$ be zeros of $\pi_n(x)$. Then (3.2) reduces to the eigenvalue problem

$$\xi_\nu \mathbf{p}_n(\xi_\nu) = H_n \mathbf{p}_n(\xi_\nu).$$

Thus, ξ_ν are eigenvalues of the matrix H_n and $\mathbf{p}_n(\xi_\nu)$ are the corresponding eigenvectors. According to (3.2) one can obtain the following determinant representation of the monic polynomials (see also [7])

$$\pi_n(x) = \det (xI_n - H_n),$$

where I_n is the identity matrix of the order n.

For computing zeros of $\pi_n(x)$ as the eigenvalues of the matrix H_n we use the EISPACK routine COMQR [8, pp. 277–284]. Notice that this routine needs an upper Hessenberg matrix, i.e., H_n^T. Also, the MATLAB can be used.

Our aim here is to compute the recurrence coefficients in (3.1), i.e., the elements of the Hessenberg matrix H_n. Only for the simplest case of multiple orthogonality, i.e., when $r = 2$, for some classical weight functions (Jacobi, Laguerre, Hermite) one can find explicit formulas for the recurrence coefficients (see [9], [11], [3]).

We calculate the elements of H_n using the discretized Stieltjes-Gautschi procedure [5]. A similar procedure was used in a numerical construction of orthogonal polynomials on the radial rays in the complex plane (see [7]).

4. Numerical Construction of Multiple Orthogonal Polynomials

In this section we present an effective numerical method for constructing the Hessenberg matrix H_n given in the previous section. We use some kind of the Stieltjes procedure (cf. [5]) and call it as the *discretized Stieltjes-Gautschi procedure*. At first, we express the elements of H_n in terms of the inner products[1] (1.5), and then we use the corresponding Gaussian formulas to discretize these inner products. Of course, we suppose that the type II multiple orthogonal polynomials exist with respect to the inner products $(\,\cdot\,,\,\cdot\,)_k$, $k = 1, 2, \ldots, r$, given by (1.5).

Firstly, we consider the simplest case $r = 2$. Then, we have the multi-indices $\vec{s}(n) = (n_1(n), n_2(n))$, where $n_1(n) = [(n+1)/2]$ and $n_2(n) = [n/2]$ ($[t]$ – integer part of t), $n_1(n)+n_2(n) = n$, and the corresponding Hessenberg matrix H_n is

$$H_n = \begin{bmatrix} \alpha_{02} & 1 & & & & & \\ \alpha_{11} & \alpha_{12} & 1 & & & & \\ \alpha_{20} & \alpha_{21} & \alpha_{22} & 1 & & & \\ & \alpha_{30} & \alpha_{31} & \alpha_{32} & 1 & & \\ & & & & & \ddots & \\ & & & \ddots & \ddots & \ddots & 1 \\ & & & & \alpha_{n-1,0} & \alpha_{n-1,1} & \alpha_{n-1,2} \end{bmatrix},$$

so that the recurrence equations for the type II multiple (monic) orthogonal polynomials are

$$\text{(4.1)} \qquad \begin{cases} \pi_1(x) &=& x\pi_0(x) - \alpha_{02}\pi_0(x), \\ \pi_2(x) &=& x\pi_1(x) - \alpha_{11}\pi_0(x) - \alpha_{12}\pi_1(x), \\ \pi_3(x) &=& x\pi_2(x) - \alpha_{20}\pi_0(x) - \alpha_{21}\pi_1(x) - \alpha_{22}\pi_2(x), \\ \pi_4(x) &=& x\pi_3(x) - \alpha_{30}\pi_1(x) - \alpha_{31}\pi_2(x) - \alpha_{32}\pi_3(x), \\ & \vdots & \end{cases}$$

i.e.,

$$\text{(4.2)} \quad \pi_{n+1}(x) = (x - \alpha_{n,2})\pi_n(x) - \alpha_{n,1}\pi_{n-1}(x) - \alpha_{n,0}\pi_{n-2}(x), \quad n \geq 0,$$

with initial conditions $\pi_0(x) = 1$, $\pi_{-1}(x) = \pi_{-2}(x) = 0$.

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[1] Such formulas for coefficients of the three-term recurrence relation for standard orthogonal polynomials on the real line are known as Darboux formulas.

In order to determine the recursion coefficients we use (4.1), i.e., (4.2), and the orthogonality

$$(\pi_n, \pi_i)_1 = 0 \quad \text{for} \quad i \le \left[\frac{n-1}{2}\right] \quad \text{and} \quad (\pi_n, \pi_i)_2 = 0 \quad \text{for} \quad i \le \left[\frac{n-2}{2}\right].$$

Since $(\pi_1, \pi_0)_1 = 0$, from the first equation in (4.1), we find

$$\alpha_{02} = \frac{(x\pi_0, \pi_0)_1}{(\pi_0, \pi_0)_1}. \tag{4.3}$$

In the following step, we use the second equation in (4.1), as well as the fact that $(\pi_2, \pi_0)_1 = 0$ and $(\pi_2, \pi_0)_2 = 0$. Thus, from

$$(\pi_2, \pi_0)_1 = (x\pi_1, \pi_0)_1 - \alpha_{11}(\pi_0, \pi_0)_1 - \alpha_{12}(\pi_1, \pi_0)_1,$$

because of $(\pi_1, \pi_0)_1 = 0$, we obtain

$$\alpha_{11} = \frac{(x\pi_1, \pi_0)_1}{(\pi_0, \pi_0)_1}. \tag{4.4}$$

In a similar way, from $(\pi_2, \pi_0)_2 = (x\pi_1 - \alpha_{11}\pi_0, \pi_0)_2 - \alpha_{12}(\pi_1, \pi_0)_2$ we get

$$\alpha_{12} = \frac{(x\pi_1 - \alpha_{11}\pi_0, \pi_0)_2}{(\pi_1, \pi_0)_2}. \tag{4.5}$$

Since $(\pi_3, \pi_0)_1 = 0$, $(\pi_3, \pi_0)_2 = 0$, and $(\pi_3, \pi_1)_1 = 0$, using the third equation in (4.1), we obtain successively

$$\alpha_{20} = \frac{(x\pi_2, \pi_0)_1}{(\pi_0, \pi_0)_1} \qquad \text{(because of } (\pi_1, \pi_0)_1 = 0,\ (\pi_2, \pi_0)_1 = 0), \tag{4.6}$$

$$\alpha_{21} = \frac{(x\pi_2 - \alpha_{20}\pi_0, \pi_0)_2}{(\pi_1, \pi_0)_2} \qquad \text{(because of } (\pi_2, \pi_0)_2 = 0), \tag{4.7}$$

$$\alpha_{22} = \frac{(x\pi_2 - \alpha_{20}\pi_0 - \alpha_{21}\pi_1, \pi_1)_1}{(\pi_2, \pi_1)_1}. \tag{4.8}$$

In a similar way, the forth equation in (4.1) and orthogonality conditions $(\pi_4, \pi_0)_1 = 0$, $(\pi_4, \pi_0)_2 = 0$, $(\pi_4, \pi_1)_1 = 0$, and $(\pi_4, \pi_1)_2 = 0$, give

$$\alpha_{30} = \frac{(x\pi_3, \pi_0)_2}{(\pi_1, \pi_0)_2} \qquad \text{(because of } (\pi_2, \pi_0)_2 = 0,\ (\pi_3, \pi_0)_2 = 0), \tag{4.9}$$

$$\alpha_{31} = \frac{(x\pi_3 - \alpha_{30}\pi_1, \pi_1)_1}{(\pi_2, \pi_1)_1} \qquad \text{(because of } (\pi_3, \pi_1)_1 = 0\text{)}, \tag{4.10}$$

$$\alpha_{32} = \frac{(x\pi_3 - \alpha_{30}\pi_1 - \alpha_{31}\pi_2, \pi_1)_2}{(\pi_3, \pi_1)_2}. \tag{4.11}$$

In general, continuing this procedure we can prove the following result:

Theorem 4.1. *Let $n = 2\ell + \nu$, where $\ell = [n/2]$ and $\nu \in \{0, 1\}$. The recursion coefficients in (4.2) can be expressed in the form*

$$\alpha_{n,0} = \frac{\big(x\pi_n, \pi_{[(n-2)/2]}\big)_{\nu+1}}{\big(\pi_{n-2}, \pi_{[(n-2)/2]}\big)_{\nu+1}}, \tag{4.12}$$

$$\alpha_{n,1} = \frac{\big(x\pi_n - \alpha_{n,0}\pi_{n-2}, \pi_{[(n-1)/2]}\big)_{\nu}}{\big(\pi_{n-1}, \pi_{[(n-1)/2]}\big)_{\nu}}, \tag{4.13}$$

$$\alpha_{n,2} = \frac{\big(x\pi_n - \alpha_{n,0}\pi_{n-2} - \alpha_{n,1}\pi_{n-1}, \pi_{[n/2]}\big)_{\nu-1}}{\big(\pi_n, \pi_{[n/2]}\big)_{\nu-1}}, \tag{4.14}$$

where we put $(f, g)_{j+2m} = (f, g)_j$ for each $m \in \mathbb{Z}$.

Proof. The formulae (4.12)–(4.14) were proved before for $n \le 3$. For a general n we start from the recurrence relation (4.2) and take the inner product $(\,\cdot\,,\,\cdot\,)_j$ with respect to the weight functions w_j $(j = 1, 2)$. Also, we put $(\,\cdot\,,\,\cdot\,)_{j+2m} = (\,\cdot\,,\,\cdot\,)_j$, where $m \in \mathbb{Z}$.

Let $n = 2\ell + \nu$, where $\ell = [n/2]$ and $\nu \in \{0, 1\}$. The orthogonality conditions, in this case, are

$$(\pi_n, \pi_i)_1 = 0 \quad \left(i \le \left[\frac{n-1}{2}\right]\right) \quad \text{and} \quad (\pi_n, \pi_i)_2 = 0 \quad \left(i \le \left[\frac{n-2}{2}\right]\right).$$

From

$$(\pi_{n+1}, \pi_i)_j = (x\pi_n - \alpha_{n,0}\pi_{n-2} - \alpha_{n,1}\pi_{n-1} - \alpha_{n,2}\pi_n, \pi_i)_j, \tag{4.15}$$

i.e.,

$$(\pi_{n+1}, \pi_i)_j = (x\pi_n, \pi_i)_j - \alpha_{n,0}(\pi_{n-2}, \pi_i)_j - \alpha_{n,1}(\pi_{n-1}, \pi_i)_j - \alpha_{n,2}(\pi_n, \pi_i)_j,$$

for $i = [(n-2)/2]$ and $j = \nu + 1$, we conclude that (4.12) holds, because of $(\pi_m, \pi_{[(n-2)/2]})_{\nu+1} = 0$ for $m = n-1, n, n+1$.

Further, for $i = [(n-1)/2]$ and $j = \nu$, the same equation reduces to

$$\begin{aligned}\left(\pi_{n+1}, \pi_{[(n-1)/2]}\right)_\nu &= \left(x\pi_n - \alpha_{n,0}\pi_{n-2}, \pi_{[(n-1)/2]}\right)_\nu - \alpha_{n,1}\left(\pi_{n-1}, \pi_{[(n-1)/2]}\right)_\nu \\ &\quad -\alpha_{n,2}\left(\pi_n, \pi_{[(n-1)/2]}\right)_\nu,\end{aligned}$$

wherefrom, because of $(\pi_m, \pi_{[(n-1)/2]})_\nu = 0$ for $m = n, n+1$, we obtain (4.13).

Finally, for $i = [n/2]$ and $j = \nu - 1$, we have $(\pi_{n+1}, \pi_{[n/2]})_{\nu-1} = 0$, and Eq. (4.15) gives (4.14). □

This theorem can be extended to the case of r weight functions w_j ($j = 1, 2, \ldots, r$). Taking that for inner products $(\,\cdot\,,\,\cdot\,)_{j+mr} = (\,\cdot\,,\,\cdot\,)_j$ ($m \in \mathbb{Z}$), the following result holds:

Theorem 4.2. *The type II multiple monic orthogonal polynomials* $\{\pi_n\}$, *with nearly diagonal multi-index, satisfy the recurrence relation*

$$\pi_{n+1}(x) = (x - \alpha_{n,r})\pi_n(x) - \sum_{k=0}^{r-1} \alpha_{n,k}\pi_{n-r+k}(x), \quad n \geq 0, \tag{4.16}$$

where

$$\alpha_{n,0} = \frac{\left(x\pi_n, \pi_{[(n-r)/r]}\right)_{\nu+1}}{\left(\pi_{n-r}, \pi_{[(n-r)/r]}\right)_{\nu+1}}$$

and

$$\alpha_{n,k} = \frac{\left(x\pi_n - \sum_{i=0}^{k-1} \alpha_{n,i}\pi_{n-r+i}, \pi_{[(n-r+k)/r]}\right)_{\nu+k+1}}{\left(\pi_{n-r+k}, \pi_{[(n-r+k)/r]}\right)_{\nu+k+1}}$$

for $k = 1, 2, \ldots, r$.

The proof of this theorem is quite similar to the previous one (for $r = 2$). Here, we put $n = \ell r + \nu$, where $\ell = [n/r]$ and $\nu \in \{0, 1, \ldots, r-1\}$, and use the corresponding orthogonality conditions.

Remark 4.1. Coefficients of the recurrence relation (4.16) can be given also in the following way: For $0 \leq k < r$

$$\alpha_{k,r-k} = \frac{(x\pi_k, \pi_0)_1}{(\pi_0, \pi_0)_1},$$

and for $m = k-1, k-2, \ldots, 0$

$$\alpha_{k,r-m} = \frac{\left(x\pi_k - \sum_{i=r-k}^{r-m-1} \alpha_{r,i}\pi_{k-r+i}, \pi_0\right)_{k-m+1}}{(\pi_{k-m}, \pi_0)_{k-m+1}}.$$

For $n \geq r$ ($n = \ell r + \nu$, $\ell = [n/r]$ and $\nu \in \{0, 1, \ldots, r-1\}$) we have

$$\alpha_{n,0} = \frac{(x\pi_n, \pi_{\ell-1})_{\nu+1}}{(\pi_{n-r}, \pi_{\ell-1})_{\nu+1}}$$

and for $1 \leq k \leq r$

$$\alpha_{n,k} = \begin{cases} \dfrac{\left(x\pi_n - \sum\limits_{i=0}^{k-1} \alpha_{n,k}\pi_{n-r+i}, \pi_{\ell-1}\right)_{\nu+k+1}}{(\pi_{n-r+k}, \pi_{\ell-1})_{\nu+k+1}}, & \nu + k + 1 \leq r, \\[2ex] \dfrac{\left(x\pi_n - \sum\limits_{i=0}^{k-1} \alpha_{n,k}\pi_{n-r+i}, \pi_{\ell}\right)_{\nu+k+1-r}}{(\pi_{n-r+k}, \pi_{\ell})_{\nu+k+1-r}}, & \nu + k + 1 > r. \end{cases}$$

All of the necessary inner products in the previous formulas can be computed exactly, except for rounding errors, by using the Gauss-Christoffel quadrature formulae with respect to the corresponding weight functions

$$\int_{E_i} g(t)w_i(t)dt = \sum_{\nu=1}^{N} A_{i,\nu}^{(N)} g(\tau_{i,\nu}^{(N)}) + R_{i,N}(g), \qquad i = 1, 2, \ldots, r. \tag{4.17}$$

Thus, for all calculations we use only the recurrence relation (4.16) for the type II multiple orthogonal polynomials and the Gauss-Christoffel quadrature formulae (4.17).

We consider now some examples with Jacobi and generalized Laguerre weights.

a) Multiple Jacobi polynomials. As an example we will take the type II multiple Jacobi polynomials. These are type II multiple orthogonal polynomials with respect to an AT system consisting of Jacobi weight functions on $[-1, 1]$ with different singularities at -1 and the same singularity at 1.

The weight functions are

$$w_m(x) = (1-x)^{\alpha}(1+x)^{\beta_m}, \quad m = 1, 2, \ldots, r,$$

where $\alpha, \beta_m > -1$, $m = 1, 2, \ldots, r$ and $\beta_i - \beta_j \notin \mathbb{Z}$ whenever $i \neq j$.

In Table 4.1 the recurrence coefficients for multiple Jacobi polynomials for $r = 3$ and $\alpha = 1$, $\beta_1 = 1/2$, $\beta_2 = 1/4$, $\beta_3 = -1/4$ are given. (Numbers in parentheses denote decimal exponents.)

b) Multiple Laguerre polynomials. These polynomials are type II multiple orthogonal polynomials with respect to the r generalized Laguerre weight functions on $[0, +\infty)$ with different singularities at 0.

Table 4.1: Recursion coefficients $\alpha_{n,k}$, $k = 0, 1, \ldots, r$, for the type II multiple Jacobi polynomials, when $r = 3$, $\alpha = 1$, $\beta_1 = 1/2$, $\beta_2 = 1/4$, $\beta_3 = -1/4$, and $n \leq 20$

i	$\alpha_{i,3}$	$\alpha_{i,2}$	$\alpha_{i,1}$	$\alpha_{i,0}$
0	−1.4285714285714286(−1)			
1	−2.8851540616246499(−1)	2.1768707482993197(−1)		
2	−3.8544221516357739(−1)	2.4885533536052567(−1)	5.4324760207113148(−2)	
3	−9.4489583642901721(−2)	2.5558194822003397(−1)	8.6774057165275854(−2)	1.6315423013987607(−2)
4	−1.6673090667975578(−1)	2.5701587367507144(−1)	1.6443964001352355(−2)	2.6675531012774156(−3)
5	−2.3917669428505342(−1)	2.6044916232539520(−1)	3.7021425677769084(−2)	9.3611909096357025(−5)
6	−1.1675493268393866(−1)	2.6354196522803514(−1)	5.6926131661188442(−2)	5.3653620722313774(−3)
7	−1.5805969659825489(−1)	2.6335336044273302(−1)	2.5309273618079248(−2)	2.2709691393999234(−3)
8	−2.0585232191457832(−1)	2.6413791455531671(−1)	3.6650017572954548(−2)	5.6560604717520359(−4)
9	−1.2741821376445422(−1)	2.6533854996293367(−1)	4.9582124305304638(−2)	3.8361194660048999(−3)
10	−1.5612993377662020(−1)	2.6512203005415770(−1)	2.8981366184597451(−2)	2.1639728364239452(−3)
11	−1.9145697954087840(−1)	2.6540589436380828(−1)	3.6765551681749009(−2)	8.8783317461228555(−4)
12	−1.3357792695166176(−1)	2.6601855379084442(−1)	4.6270196508266970(−2)	3.2511017686051238(−3)
13	−1.5553744375047290(−1)	2.6585132520725863(−1)	3.0966883005005431(−2)	2.1143133877612638(−3)
14	−1.8348871061253406(−1)	2.6598104740033892(−1)	3.6886585572864122(−2)	1.0966845714025530(−3)
15	−1.3757717226728043(−1)	2.6634705046296363(−1)	4.4386333038230584(−2)	2.9447307424077395(−3)
16	−1.5534192543920438(−1)	2.6622059700938253(−1)	3.2207381758124402(−2)	2.0856866464893746(−3)
17	−1.7844389078361894(−1)	2.6628881061328641(−1)	3.6981744226277791(−2)	1.2402019680140608(−3)
18	−1.4038002152177360(−1)	2.6653039931011929(−1)	4.3171017016114228(−2)	2.7567838441883074(−3)
19	−1.5529015667141288(−1)	2.6643295689567229(−1)	3.3055076847041242(−2)	2.0670810214041445(−3)
20	−1.7496809628326628(−1)	2.6647232016548233(−1)	3.7055003346066915(−2)	1.3442305416000771(−3)

The weight functions are

$$w_m(x) = x^{s_m} e^{-x}, \quad m = 1, 2, \ldots, r,$$

where $s_m > -1$, $m = 1, 2, \ldots, r$, with conditions $s_i - s_j \notin \mathbb{Z}$ whenever $i \neq j$ in order to have an AT system.

In the case $r = 2$ some explicit formulas for the recurrence coefficients can be done (see [11]). For example, when $s_1, s_2 \in \mathbb{Q}$, the recurrence coefficients are rational numbers:

$$\begin{aligned}
\alpha_{2k,2} &= 3k + s_1 + 1, \\
\alpha_{2k+1,2} &= 3k + s_2 + 2, \\
\alpha_{2k,1} &= k(3k + s_1 + s_2), \\
\alpha_{2k+1,1} &= 3k^2 + (s_1 + s_2 + 3)n + s_1 + 1, \\
\alpha_{2k,0} &= k(k + s_1)(k + s_1 - s_2), \\
\alpha_{2k+1,0} &= k(k + s_2)(k + s_2 - s_1).
\end{aligned}$$

Using the previous described numerical procedure we get exact values (up to rounding errors) for these coefficients.

5. Quadrature Formulae of Gaussian Type

Borges [4] has considered a problem that arises in evaluation of computer graphics illumination models. Starting with that problem, he has examined the problem of numerically evaluating a set of r definite integrals taken with respect to distinct weight functions but related by a common integrand and interval of integration.

It is shown that it is not efficient to use a set of r Gauss-Christoffel quadrature formulas because valuable information is wasted.

Borges has introduced a performance ratio, defined as:

$$R = \frac{\text{Overall degree of precision} + 1}{\text{Number of integrand evaluation}}\,.$$

If we use the set of r Gauss-Christoffel quadrature formulas we have $R = 2/r$ and hence $R < 1$ for all $r > 2$.

If we select a set of n distinct nodes, common for all quadrature formulas, weight coefficients for each of r quadrature formulas can be chosen in that way that a performance ratio is $R = 1$. Because the selection of nodes is arbitrary, the quadrature formulas may not be the best possible.

The aim is to find an optimal set of nodes, by mimicking the development of the Gauss-Christoffel quadrature formulas.

Denote with $W = \{w_1, w_2, \ldots, w_r\}$ an AT system.

Following [4, Definition 3] we introduce the following definition:

Definition 5.1. Let W be an AT system (the weight functions w_i, $i = 1, 2, \ldots, r$ are supported on the interval E), $\vec{n} = (n_1, n_2, \ldots, n_r)$ be a multi-index, and $n = |\vec{n}|$. A set of quadrature formulas of the form:

$$\int_E f(x)\, w_m(x)dx \approx \sum_{i=1}^{n} A_{m,i} f(x_i), \qquad m = 1, 2, \ldots, r \tag{5.1}$$

will be called an optimal set with respect to $(W, \vec{n})$ if and only if the weight coefficients, $A_{m,i}$, and the nodes, x_i, satisfy the following equations:

$$\begin{aligned} \sum_{i=1}^{n} A_{m,i} &= \int_E w_m(x)dx \\ \sum_{i=1}^{n} A_{m,i}\, x_i &= \int_E x\, w_m(x)dx \\ &\vdots \\ \sum_{i=1}^{n} A_{m,i}\, x_i^{n+n_m-1} &= \int_E x^{n+n_m-1}\, w_m(x)dx \end{aligned} \tag{5.2}$$

for $m = 1, 2, \ldots, r$.

Now, we can prove a generalization of fundamental theorem of Gauss-Christoffel quadrature formulas:

Theorem 5.1. *Let W be an AT system, $\vec{n} = (n_1, n_2, \ldots, n_r)$, $n = |\vec{n}|$. Consider the quadrature formulas:*

$$\int_E f(x)\, w_m(x)dx \approx \sum_{i=1}^{n} A_{m,i} f(x_i) \tag{5.3}$$

where $m = 1, 2, \ldots, r$.

These formulas form an optimal set with respect to $(W, \vec{n})$ if and only if:

1° *They are exact for all polynomials of degree $\leq n - 1$.*

2° *The polynomial $q(x) = \prod_{i=1}^{n}(x - x_i)$ is the type II multiple orthogonal polynomial $\pi_{\vec{n}}$ with respect to W.*

Proof . Suppose first that the quadrature formulas (5.3) form the optimal set with respect to $(W, \vec{n})$.

In order to prove 1° we note that for each m the corresponding quadrature formula (5.3) is exact for all polynomials of degree $\le n + n_m - 1$ and then it is exact for those of degree $\le n - 1$.

For 2° and $m = 1, 2, \ldots r$, assume that $p_m(x)$ is a polynomial of degree $\le n_m - 1$. Then the polynomial $q(x)p_m(x)$ has degree $\le n + n_m - 1$. Since the corresponding quadrature formula is exact for all such polynomials, it follows that

$$\int_E q(x)p_m(x)\, w_m(x)dx = \sum_{i=1}^{n} A_{m,i}q(x_i)p_m(x_i). \tag{5.4}$$

Since $q(x_i) = 0$ for $i = 1, 2, \ldots, n$, the sum on the right hand side in (5.4) is identically zero. Thus, we have

$$\int_E q(x)p_m(x)\, w_m(x)dx = 0, \quad m = 0, 1, \ldots, r$$

and 2° follows.

Suppose now that for quadrature formulas (5.3) 1° and 2° hold.

For $m = 1, 2, \ldots, r$ let $p_m(x)$ be a polynomial of degree $\le n+n_m-1$. We can write $p_m(x) = q(x)s_m(x) + r(x)$, where $s_m(x)$ is a polynomial of degree $\le n_m - 1$ and $r(x)$ is also a polynomial of degree $\le n - 1$. Then

$$\begin{aligned}\int_E p_m(x)\, w_m(x)dx &= \int_E [q(x)s_m(x) + r(x)]\, w_m(x)dx \\ &= \int_E q(x)s_m(x)\, w_m(x)dx + \int_E r(x)\, w_m(x)dx\,.\end{aligned}$$

From 2° we have $\int_E q(x)s_m(x)\, w_m(x)dx = 0$ and therefore we obtain

$$\int_E p_m(x)\, w_m(x)dx = \int_E r(x)\, w_m(x)dx\,.$$

Since $r(x)$ is a polynomial of degree $\le n - 1$ it follows from 1° that

$$\int_E r(x)\, w_m(x)dx = \sum_{i=1}^{n} A_{m,i}r(x_i)$$

and hence

$$\int_E p_m(x)\, w_m(x)dx = \sum_{i=1}^{n} A_{m,i}r(x_i).$$

Finally, since $q(x_i) = 0$ for $i = 1, 2, \ldots, n$, it follows that

$$\begin{aligned}\int_E p_m(x)\, w_m(x)dx &= \sum_{i=1}^{n} A_{m,i}[q(x_i)s_m(x_i) + r(x_i)] \\ &= \sum_{i=1}^{n} A_{m,i} p_m(x_i)\end{aligned}$$

and quadrature formula is exact for all polynomial of degree $\leq n+n_m-1$. $\square$

Notice that all of the zeros of the type II multiple orthogonal polynomial $\pi_{\vec{n}}$ are distinct and located in the interval E (Theorem 2.2).

For $r = 1$ in Definition 5.1 we get the standard Gauss-Christoffel quadrature formulae.

According to Theorem 5.1 the nodes of optimal set of quadrature formulas (of Gaussian type) with respect to $(W, \vec{n})$ are the zeros of the type II multiple orthogonal polynomial $\pi_{\vec{n}}$ with respect to the given AT system W. When we find the nodes, then we can choose the weight coefficients $A_{m,i}$, $m = 1, 2, \ldots, r$, $i = 1, 2, \ldots, n$, so that they satisfy the following Vandermonde systems of equations

$$V(x_1, x_2, \ldots, x_n) \begin{bmatrix} A_{m,1} \\ A_{m,2} \\ \vdots \\ A_{m,n} \end{bmatrix} = \begin{bmatrix} \mu_0^{(m)} \\ \mu_1^{(m)} \\ \vdots \\ \mu_{n-1}^{(m)} \end{bmatrix}, \quad m = 1, 2, \ldots, r,$$

where

$$\mu_i^{(m)} = \int_E x^i\, w_m(x)dx\,, \quad m = 1, 2, \ldots, r, \quad i = 0, 1, \ldots, n-1.$$

Each of these Vandermonde systems always has unique solution since the zeros of the type II multiple orthogonal polynomial $\pi_{\vec{n}}$ are distinct.

For the case of the nearly diagonal multi-indices $\vec{s}(n)$ we can compute the nodes x_i, $i = 1, 2, \ldots, n$, of the Gaussian type quadrature formulas as eigenvalues of the corresponding banded Hessenberg matrix H_n. Then from (3.2) it follows that the eigenvector associated with x_i is given by $\mathbf{p}_n(x_i)$, where $\mathbf{p}_n(x) = [\pi_0(x)\ \ \pi_1(x)\ \ldots\ \pi_{n-1}(x)]^T$. We can use this fact to compute the weight coefficients $A_{m,i}$ by requiring that each rule correctly generate the first n modified moments.

Denote by $V_n = [\mathbf{p}_n(x_1)\, \mathbf{p}_n(x_2)\ \ldots\ \mathbf{p}_n(x_n)]$ the matrix of the eigenvectors of matrix H_n, each normalized so that the first component is equal to 1.

Then, the weight coefficients $A_{m,i}$ can be found by solving

$$V_n \cdot \begin{bmatrix} A_{m,1} \\ A_{m,2} \\ \vdots \\ A_{m,n} \end{bmatrix} = \begin{bmatrix} \mu_0^{*(m)} \\ \mu_1^{*(m)} \\ \vdots \\ \mu_{n-1}^{*(m)} \end{bmatrix}, \quad m = 1, 2, \dots, r, \tag{5.5}$$

where

$$\mu_i^{*(m)} = \int_E \pi_i(x)\, w_m(x) dx\,, \quad m = 1, 2, \dots, r, \quad i = 0, 1, \dots, n-1\,,$$

are modified moments and $\pi_i = \pi_{\vec{s}(i)}$.

All of modified moments can be computed exactly, except for rounding errors, by using the Gauss-Christoffel quadrature formulas with respect to the corresponding weight function w_m, $m = 1, 2, \dots r$.

6. Numerical Examples

In this section we give the numerical values for parameters of quadrature formulas for some special cases with multiple Jacobi and generalized Laguerre measures. The corresponding multiple polynomials were treated in Section 4.

a) Quadrature with multiple Jacobi polynomials. In Tables 6.1, 6.2 and 6.3 the nodes and weights for quadrature formulas of Gaussian type with respect to an AT system of Jacobi weights and nearly diagonal multi-index are given.

b) Quadratures with multiple Laguerre polynomials. The nodes and weights for quadrature formulas of Gaussian type with respect to an AT system of the generalized Laguerre weights and nearly diagonal multi-index are given in Table 6.4.

Table 6.1: Quadrature formulas of Gaussian type, $r = 2$, $\alpha = -1/4$, $\beta_1 = 1$, $\beta_2 = -1/2$, when $n = 5, 6, 8, 16$

i	x_i	$A_{1,i}$	$A_{2,i}$
1	−9.595739732963885(−1)	6.085528646609526(−3)	6.792157651535534(−1)
2	−6.429557235013292(−1)	1.448828264237293(−1)	6.808231614855028(−1)
3	−8.563321535083310(−2)	5.546715987145651(−1)	6.341969901657482(−1)
4	5.139342213733857(−1)	9.766942509866085(−1)	5.243590871208606(−1)
5	9.193715479180223(−1)	8.803977274302843(−1)	3.310787799115283(−1)
1	−9.763650757517414(−1)	2.049797491247068(−3)	5.231693814830747(−1)
2	−7.779228623312244(−1)	5.835107309355270(−2)	5.582115897381238(−1)
3	−3.801432801669256(−1)	2.731946113853040(−1)	5.597593346684589(−1)
4	1.361177426328452(−1)	6.241401620411635(−1)	5.154089057680225(−1)
5	6.258082404991945(−1)	8.808613543314381(−1)	4.249153713748993(−1)
6	9.389388015514181(−1)	7.241349338590913(−1)	2.682092008046140(−1)
1	−9.886995595675056(−1)	4.657060697401689(−4)	3.636423493025722(−1)
2	−8.890000823095323(−1)	1.495796500576839(−2)	4.047480255190821(−1)
3	−6.692705951078319(−1)	8.133867893732568(−2)	4.276372272958223(−1)
4	−3.397509595583518(−1)	2.282259252625604(−1)	4.254067482828985(−1)
5	5.519134932224639(−2)	4.353613810197344(−1)	4.016547542013336(−1)
6	4.498518834434292(−1)	6.235260087583095(−1)	3.571652132341037(−1)
7	7.729307134228531(−1)	6.817427718957693(−1)	2.887904888087898(−1)
8	9.638670760627538(−1)	4.971134952525889(−1)	1.806289771925911(−1)
1	−9.982593521223466(−1)	1.098552045504686(−5)	1.441066772880576(−1)
2	−9.817291574247172(−1)	4.226450510599256(−4)	1.711788751055894(−1)
3	−9.407756176104369(−1)	2.780997620514126(−3)	1.929514233662454(−1)
4	−8.695291944228105(−1)	9.759829836089312(−3)	2.070962723424741(−1)
5	−7.655502123735464(−1)	2.448418898468910(−2)	2.156805365209539(−1)
6	−6.294503342599388(−1)	4.960732841302910(−2)	2.199258100411888(−1)
7	−4.645800212281584(−1)	8.640162903919979(−2)	2.205362939405117(−1)
8	−2.766686612890329(−1)	1.340638010684390(−1)	2.179242732460876(−1)
9	−7.338464711498739(−2)	1.893823545890457(−1)	2.123196914278861(−1)
10	1.361863364296501(−1)	2.468423207685339(−1)	2.038192868004388(−1)
11	3.421285501631106(−1)	2.991521205655611(−1)	1.923981595665098(−1)
12	5.343218013065390(−1)	3.380778041835591(−1)	1.778860841359874(−1)
13	7.030721540670085(−1)	3.553567251522042(−1)	1.598876674382878(−1)
14	8.397122251225786(−1)	3.432706621803739(−1)	1.375662333963814(−1)
15	9.371372223537221(−1)	2.937586824517584(−1)	1.089557444264535(−1)
16	9.902623590387524(−1)	1.893598567772850(−1)	6.744075479414001(−2)

Table 6.2: Quadrature formulas of Gaussian type for $r = 2$ and $\alpha = 1$, $\beta_1 = 1/2$, $\beta_2 = 1/4$, when $n = 8, 16, 20$

i	x_i	$A_{1,i}$	$A_{2,i}$
1	−9.807736962905802(−1)	1.325904689017911(−2)	3.587388796878142(−2)
2	−8.876182343170376(−1)	8.993704645232481(−2)	1.553022696100243(−1)
3	−6.936308793740333(−1)	2.295086020893152(−1)	3.084910553923076(−1)
4	−4.044677498567003(−1)	3.557671050559271(−1)	4.049850264214527(−1)
5	−5.085190374678864(−2)	3.796403494806395(−1)	3.846262824920836(−1)
6	3.192334275974310(−1)	2.814356508999960(−1)	2.626023899339839(−1)
7	6.503798123969747(−1)	1.318293702509232(−1)	1.163097742511322(−1)
8	8.904965191882185(−1)	2.711729541199648(−2)	2.312609971188207(−2)
1	−9.967548537764203(−1)	9.571585241731793(−4)	4.031747449071050(−3)
2	−9.795318390039872(−1)	7.766425149231034(−3)	2.053152048928658(−2)
3	−9.396786824125953(−1)	2.531111328265616(−2)	5.107333208301494(−2)
4	−8.720737165846898(−1)	5.520108488960052(−2)	9.230126343675327(−2)
5	−7.745128722854601(−1)	9.488967509296785(−2)	1.377014527332767(−1)
6	−6.473891617925948(−1)	1.381068532201385(−1)	1.792219984017873(−1)
7	−4.934188362486365(−1)	1.764094860063975(−1)	2.091027166880968(−1)
8	−3.173466546442108(−1)	2.014702594071720(−1)	2.216465061887285(−1)
9	−1.255985677983854(−1)	2.074591770034887(−1)	2.145383497352685(−1)
10	7.412472926200013(−2)	1.927938882876051(−1)	1.893780251966169(−1)
11	2.733053271928383(−1)	1.606915590066285(−1)	1.512724794180531(−1)
12	4.630996337572934(−1)	1.183079493368857(−1)	1.075710607729705(−1)
13	6.348413954801026(−1)	7.468921452913027(−2)	6.605245832617015(−2)
14	7.805309103213726(−1)	3.814550835762502(−2)	3.302220184457527(−2)
15	8.932842079741912(−1)	1.385392337032966(−2)	1.181051479537479(−2)
16	9.677130399651226(−1)	2.441191067271689(−3)	2.061158222603309(−3)
1	−9.982288278660783(−1)	3.889716609793118(−4)	1.905456134188979(−3)
2	−9.886386626719993(−1)	3.265598826328111(−3)	1.000188652665892(−2)
3	−9.659739987497579(−1)	1.106728185934343(−2)	2.576847013301273(−2)
4	−9.266485635472170(−1)	2.530651619544536(−2)	4.862730454694643(−2)
5	−8.684551170401689(−1)	4.611534787923296(−2)	7.657323581180498(−2)
6	−7.904178223676872(−1)	7.212856742354534(−2)	1.066028263921650(−1)
7	−6.926855921082041(−1)	1.006552949043125(−1)	1.351888566765899(−1)
8	−5.764377040285674(−1)	1.281177978289150(−1)	1.588108395621707(−1)
9	−4.437840060752403(−1)	1.506866430234970(−1)	1.744873269865763(−1)
10	−2.976510800275687(−1)	1.649957834108218(−1)	1.802331262684936(−1)
11	−1.416511021178208(−1)	1.687964514392549(−1)	1.753667885304791(−1)
12	2.006639866445671(−2)	1.614137925633005(−1)	1.606140492565033(−1)
13	1.829773548202771(−1)	1.439055859943552(−1)	1.379855295136471(−1)
14	3.423564271674032(−1)	1.188814661906727(−1)	1.104452877589257(−1)
15	4.934555482718982(−1)	9.001142762433863(−2)	8.142346401480130(−2)
16	6.316819336591816(−1)	6.131880499981948(−2)	5.425438018450921(−2)
17	7.527694731196839(−1)	3.640059955758444(−2)	3.163569186501606(−2)
18	8.529374177090320(−1)	1.773603342592148(−2)	1.520167035575239(−2)
19	9.290303254075266(−1)	6.227991495789366(−3)	5.284613709793606(−3)
20	9.786309311141836(−1)	1.074510227843854(−3)	9.059815536125253(−4)

Table 6.3: Quadrature formulas of Gaussian type for $r = 3$, $\alpha = -1/2$, $\beta_1 = -1/4$, $\beta_2 = 1/4$, $\beta_3 = 1$, when $n = 12$ and $n = 16$

i	x_i	$A_{1,i}$	$A_{2,i}$	$A_{3,i}$
1	−9.9748115240107660(−1)	2.8100087585819025(−2)	1.3886176725129215(−3)	1.5720468735029964(−5)
2	−9.7398597869356525(−1)	7.4439011788389378(−2)	1.2005928418128381(−2)	7.7773763811012880(−4)
3	−9.0719614253642898(−1)	1.2320670528724125(−1)	3.7533452871867679(−2)	6.3109146308700732(−3)
4	−7.8324598960152212(−1)	1.6962091258346735(−1)	7.8970143145002854(−2)	2.5086396839678953(−2)
5	−5.9872638558249099(−1)	2.1196465652695978(−1)	1.3427147343991359(−1)	6.7696187150173298(−2)
6	−3.6057359410274506(−1)	2.4953886623508269(−1)	1.9954161419710746(−1)	1.4268437137313251(−1)
7	−8.4838748348476527(−2)	2.8202639121677911(−1)	2.6979789816888778(−1)	2.5244209736558800(−1)
8	2.0550873751958384(−1)	3.0926252255799465(−1)	3.3955683370131352(−1)	3.9065217539936507(−1)
9	4.8356261417963050(−1)	3.3114808077848299(−1)	4.0334361153557648(−1)	5.4219385738339921(−1)
10	7.2192990523487745(−1)	3.4761623569198341(−1)	4.5615011283086265(−1)	6.8567669390461140(−1)
11	8.9623854228762318(−1)	3.5862034038797524(−1)	4.9383447288425145(−1)	7.9799676449688561(−1)
12	9.8826245362999969(−1)	3.6412997319701835(−1)	5.1344423845879597(−1)	8.5970324967770417(−1)
1	−9.9906459054687916(−1)	1.3599575971337402(−2)	4.1077201090853676(−4)	2.2073500251065646(−6)
2	−9.8972523369040094(−1)	3.8220983178118081(−2)	3.8741015690243335(−3)	1.2503237065737780(−4)
3	−9.6138275121546238(−1)	6.6012783053220409(−2)	1.2972364177529990(−2)	1.1300691113072383(−3)
4	−9.0536073793600138(−1)	9.4016541848022849(−2)	2.8922774034644522(−2)	4.9350714888495819(−3)
5	−8.1617163556465465(−1)	1.2094730160925280(−1)	5.1856409680448855(−2)	1.4558339318391224(−2)
6	−6.9183652994371045(−1)	1.4620128742246357(−1)	8.1159950764161202(−2)	3.3568168131011862(−2)
7	−5.3387370237733501(−1)	1.6947834840285167(−1)	1.1570870306180972(−1)	6.5274526447350550(−2)
8	−3.4707627628238060(−1)	1.9062006869237470(−1)	1.5402806023047204(−1)	1.1187851229881149(−1)
9	−1.3909762096861382(−1)	2.0953651796686239(−1)	1.9441809195868022(−1)	1.7376092008902707(−1)
10	8.0141880239502588(−2)	2.2617191984994137(−1)	2.3506019233251793(−1)	2.4905180356121826(−1)
11	2.9919816153596727(−1)	2.4048827439156841(−1)	2.7411424468793739(−1)	3.3357089698204777(−1)
12	5.0595219925967548(−1)	2.5245756434445782(−1)	3.0980896541450529(−1)	4.2116509583262061(−1)
13	6.8850714313464215(−1)	2.6205815362765708(−1)	3.4052509904415083(−1)	5.0440142471613475(−1)
14	8.3607004055806621(−1)	2.6927324050027395(−1)	3.6486971620726020(−1)	5.7551248833920016(−1)
15	9.3975338163739056(−1)	2.7409029435156335(−1)	3.8173933806039613(−1)	6.2744664919558565(−1)
16	9.9323645823966934(−1)	2.7650092862722739(−1)	3.9036961408977354(−1)	6.5485496109601476(−1)

Table 6.4: Quadrature formulas of Gaussian type, $r = 2$, $s_1 = -1/2$, $s_2 = -1/4$, when $n = 6, 8, 10, 14$

i	x_i	$A_{1,i}$	$A_{2,i}$
1	1.221185842900368(1)	9.858095463414438(−6)	1.843100435350308(−5)
2	6.905310670462018	1.563569554670318(−3)	2.534438809694791(−3)
3	3.607225605397539	3.719978319819824(−2)	5.127172367743109(−2)
4	1.555796061047994	2.616823832489270(−1)	2.921560902816410(−1)
5	4.370925795315986(−1)	7.068628435926800(−1)	5.763948219949018(−1)
6	3.271665455716685(−2)	7.651354132155770(−1)	3.030411966971555(−1)
1	1.789194218928257(1)	3.193875186492825(−8)	6.568831380045222(−8)
2	1.162324061287780(1)	1.319532098932192(−5)	2.436407133309024(−5)
3	7.406393364002694	7.868009006936770(−4)	1.297980363259742(−3)
4	4.422678545320278	1.422101928893957(−2)	2.062288509168762(−2)
5	2.348979881582482	1.048667333973918(−1)	1.298273199962789(−1)
6	1.009338591486776	3.685487493262911(−1)	3.693596621291925(−1)
7	2.776163582753400(−1)	6.733489587467163(−1)	4.896454916262365(−1)
8	1.981045717206063(−2)	6.106683619857424(−1)	2.146389334988754(−1)
1	2.374640867015977(1)	8.795230932886089(−11)	1.941544428023490(−10)
2	1.669744109177428(1)	7.901670810441663(−8)	1.597280805402878(−7)
3	1.175507935365535(1)	9.715595498207142(−6)	1.798978180453034(−5)
4	8.051691947679313	3.610222289099737(−4)	6.081429065694955(−4)
5	5.246321006744356	5.573313571204766(−3)	8.434849771004707(−3)
6	3.158685164349243	4.230364001315984(−2)	5.639671708628531(−2)
7	1.675573856630426	1.749916107879895(−1)	1.990952264994221(−1)
8	7.130658414963164(−1)	4.223858744391781(−1)	3.881185109685965(−1)
9	1.924563246846599(−1)	6.193806385028605(−1)	4.107800791244563(−1)
10	1.327674282628529(−2)	5.074479566620547(−1)	1.619650264048037(−1)
1	3.577621460223470(1)	4.938285007123488(−16)	1.207743629179033(−15)
2	2.748162705583546(1)	1.534759928488584(−12)	3.513994366645536(−12)
3	2.140866187151855(1)	5.826491343489010(−10)	1.253298404042186(−9)
4	1.660843936134304(1)	6.473553611833618(−8)	1.306847631603544(−7)
5	1.270874455888727(1)	2.988803634154539(−6)	5.643168855451219(−6)
6	9.513909324041849	6.907399354967256(−5)	1.213121549323418(−4)
7	6.904524446265654	8.955693150985517(−4)	1.451719292539717(−3)
8	4.800298056283760	7.031017607299445(−3)	1.040723196995920(−2)
9	3.142842229190352	3.530344004029820(−2)	4.700539200856037(−2)
10	1.886099900785945	1.181631265299382(−1)	1.384755478708790(−1)
11	9.897696742876845(−1)	2.724468351053578(−1)	2.717477154375032(−1)
12	4.133127038654632(−1)	4.439458842476815(−1)	3.559497043197113(−1)
13	1.084025628498585(−1)	5.158367555060495(−1)	2.962453188848733(−1)
14	7.153652610414782(−3)	3.787590944368885(−1)	1.040069854157870(−1)

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