ON MAXIMUM OF THE MODULUS OF KERNELS IN GAUSS-TURÁN QUADRATURES WITH CHEBYSHEV WEIGHTS: THE CASES S = 1, 2

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Abstract. We study the kernels $K_{n,s}(z)$ in the remainder terms $R_{n,s}(f)$ of Gauss-Turán quadrature formulae for analytic functions on elliptical contours with foci at $\pm 1$, when the weight $\omega$ is Chebyshev weight function of the first and third kind. It is shown that the modulus of the kernel attains its maximum on the real axis $\forall n \geq n_0$, $n_0 = n_0(\rho, s)$ in the case $s = 1$. Analogous results can be performed in the case $s = 2$.

1. Introduction

We consider the Gauss-Turán quadrature formula with multiple nodes

$$
\int_{-1}^{1} f(t)\omega(t)dt = \sum_{\nu=1}^{n} \sum_{i=0}^{2s} A_{i,\nu} f^{(i)}(\tau_{\nu}) + R_{n,s}(f) \ (n \in \mathbb{N}; \ s \in \mathbb{N}_0),
$$

where $\omega$ is nonnegative and integrable function on interval $(-1, 1)$, which is exact for all algebraic polynomials of degree at most $2(s + 1)n - 1$. The nodes $\tau_{\nu}$ in (1.1) must be zeros of the $s$-orthogonal polynomials with respect to the weight function $\omega(t)$. The $s$-orthogonal polynomials $\pi_n = \pi_{n,s}$ with respect to the weight function $\omega(t)$ are polynomials which satisfy the following orthogonality conditions

$$
\int_{-1}^{1} \pi_n(t)^{2s+1+k} \omega(t)dt = 0, \quad k = 0, 1, \ldots, n - 1.
$$

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Numerically stable methods for constructing nodes $\tau_\nu$ and coefficients $A_{i,\nu}$ can be found in [1, 4, 6]. For more details on quadrature formulae with multiple nodes see [2] and [3].

Let $\Gamma$ be a simple closed curve in the complex plane surrounding the interval $[-1, 1]$ and let $D$ be its interior. If integrand $f$ is analytic on $D$ and continuous on $\overline{D}$, then the remainder term $R_{n,s}$ in (1.1) admits the contour integral representation (see, for instance, [5] and reference therein)

$$R_{n,s}(f) = \frac{1}{2\pi i} \oint_{\Gamma} K_{n,s}(z) f(z) dz.$$  

(1.2)

The kernel is given by

$$K_{n,s}(z) = \frac{\rho_{n,s}(z)}{|\pi_{n,s}(z)|^{2s+1}}, \quad z \notin [-1, 1],$$

where

$$\rho_{n,s}(z) = \int_{-1}^{1} \frac{|\pi_{n,s}(t)|^{2s+1}}{z-t} \omega(t) dt.$$  

The modulus of the kernel is symmetric with respect to real axis, i.e., $|K_{n,s}(\overline{z})| = |K_{n,s}(z)|$. If the weight function in (1.1) is even the modulus of the kernel is symmetric with respect to both axes, i.e., $|K_{n,s}(-\overline{z})| = |K_{n,s}(\overline{z})|$ (see [5, Lemma 2.1]).

The integral representation (1.2) leads to the error estimate

$$|R_{n,s}| \leq \frac{l(\Gamma)}{2\pi} \left( \max_{z \in \Gamma} K_{n,s}(z) \right) \left( \max_{z \in \Gamma} |f(z)| \right),$$

where $l(\Gamma)$ denotes the length of the contour $\Gamma$. First maximum depends only on the quadrature rule (i.e., on $\omega$) and not on $f$.

2. The Maximum Modulus of the Kernel on Confocal Ellipses

In this section we take as contour $\Gamma$ an ellipse $\mathcal{E}_\rho$ with foci at points $\pm 1$, and a sum of semiaxes $\rho > 1$,

$$\mathcal{E}_\rho = \left\{ z \in \mathbb{C} : z = \frac{1}{2} \left( \rho e^{i\theta} + \rho^{-1} e^{-i\theta} \right), \ 0 \leq \theta \leq 2\pi \right\}.$$  

When $\rho \to 1$ the ellipse shrinks to the interval $[-1, 1]$, while with increasing $\rho$ it becomes more and more circle-like.
We study the magnitude of $|K_{n,s}(z)|$ on the contour $E_\rho$ for the generalized Chebyshev weight functions of first and third kind, respectively, (cf. [5])

$$\omega_1(t) = (1-t^2)^{-1/2} \quad \text{and} \quad \omega_3(t) = \frac{(1+t)^{1/2+s}}{(1-t)^{1/2}}.$$ 

2.1. The weight function $\omega_1(t) = (1-t^2)^{-1/2}$. Explicit representation of the kernel $K^{(1)}_{n,s}(z)$ on the ellipse $E_\rho$ for the weight function $\omega_1(t)$ was given by Milovanović and Spalević in [5], as well

$$|K^{(1)}_{n,s}(z)| = \frac{2^{1-s}\pi}{\rho^n} \frac{|Z^{(1)}_{n,s}(\rho e^{i\theta})|}{(a_2 - \cos 2\theta)^{1/2}(a_{2n} + \cos 2n\theta)^{1/2+s}}, \quad z \in E_\rho,$$

where

$$a_j = a_j(\rho) = \frac{1}{2}(\rho^j + \rho^{-j}), \quad j \in \mathbb{N},$$

and

$$Z^{(1)}_{n,s}(\rho e^{i\theta}) = \sum_{k=0}^{s} \left( \begin{array}{c} 2s + 1 \\ s + k + 1 \end{array} \right) (\rho e^{i\theta})^{-2nk}.$$ 

The weight function $\omega_1(t)$ is even, so we can take $\theta \in [0, \pi/2]$.

Using the representation (2.1) Milovanović and Spalević stated the following conjecture:

**Conjecture 2.1.** For each fixed $\rho > 1$ and $s \in \mathbb{N}_0$ there exists $n_0 = n_0(\rho, s)$ such that

$$\max_{z \in E_\rho} |K^{(1)}_{n,s}(z)| = K^{(1)}_{n_0,s}\left(\frac{1}{2}(\rho + \rho^{-1})\right)$$

for each $n \geq n_0$.

**Theorem 2.1.** Conjecture 2.1 holds for $s = 1$.

**Proof.** Because (2.1) it is sufficiently to prove

$$\leq (9 + 6\rho^{-2n} \cos 2n\theta + \rho^{-4n})(a_2 - 1)(a_{2n} + 1)^3,$$

(2.4)

for sufficiently large $n (n \geq n_0(\rho))$ and $\theta \in (0, \pi/2]$, where $a_j$ are given by (2.2). Introducing half-angles, this is equivalent to

$$[(3 + \rho^{-2n})^2 - 12\rho^{-2n} \sin^2 n\theta](a_2 - 1)(a_{2n} + 1)^3$$

$$\leq (3 + \rho^{-2n})^2[(a_2 - 1) + 2 \sin^2 \theta][(a_{2n} + 1)^3 - 6a_{2n}^2 \sin^2 n\theta$$

$$- 12a_{2n} \sin^2 n\theta \cos^2 n\theta - 6 \sin^2 n\theta + 12 \sin^4 n\theta - 8 \sin^6 n\theta].$$
Now, it is sufficiently to prove
\[
(a_{2n} + 1)^3 - \frac{\sin^2 n\theta}{\sin^2 n\theta} (a_2 - 1)(3a_{2n}^2 + 6a_{2n} \cos^2 n\theta)
\]
\[
+ 3 - 6 \sin^2 n\theta + 4 \sin^4 n\theta) - 2 \sin^2 n\theta(3a_{2n}^2
\]
\[
+ 6a_{2n} \cos^2 n\theta + 3 - 6 \sin^2 n\theta + 4 \sin^4 n\theta) \geq 0,
\]
if \( n \geq n_0(\rho) \) and \( \theta \in (0, \pi/2] \). Since
\[
\left| \frac{\sin n\theta}{\sin \theta} \right| \leq n, \quad (a_2 - 1) > 0,
\]
and
\[
(\forall n \in \mathbb{N}) \quad 3a_{2n}^2 + 6a_{2n} \cos^2 n\theta + 3 - 6 \sin^2 n\theta + 4 \sin^4 n\theta \geq 0,
\]
the left-hand side of (2.5) is larger or equal to
\[
(a_{2n} + 1)^3 - n^2(a_2 - 1)(3a_{2n}^2 + 6a_{2n} + 7) - 2(3a_{2n}^2 + 6a_{2n} + 7) := F(n).
\]
Using (2.2) we get
\[
F(n) = \frac{1}{8} \left[ \rho^{6n} - (3An^2 + 6)\rho^{4n} - (12An^2 + 33)\rho^{2n}
\]
\[
-(34An^2 + 116) - (12An^2 + 33)\rho^{-2n} - (3An^2 + 6)\rho^{-4n} + \rho^{-6n} \right],
\]
where \( A = (a_2 - 1) = (\rho - \rho^{-1})^2 = \text{const}. \) Since \( F(n) \) is continuous on \( \mathbb{R} \)
and \( \lim_{n \to +\infty} = +\infty \), it follows that \( F(n) > 0 \), for all \( n > t \), where \( t \) is the
largest zero of \( F(n) \). For \( n_0 \) we can take \( [t] + 1 \).

We can use the function \( F(n) \) from the proof to estimate \( n_0 \). Numerical
values of \( [t] + 1 \ (t \ is \ the \ largest \ zero \ of \ F) \) for some values of \( \rho \) are presented
in Table 1. The least possible values of \( n_0 \) are also presented. We can see
that the least possible \( n_0 \) is estimated by \( [t] + 1 \) very well.

<table>
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<th>( \rho )</th>
<th>( [t] + 1 )</th>
<th>( \text{l.p. } n_0 )</th>
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Analogous results can be derived in the case \( s = 2 \), in a similar way. But when \( s \) increases the derivation becomes drastically complex.

### 2.2. The weight function \( \omega_3(t) = (1 + t)^{1/2+s}(1 - t)^{-1/2} \)

Explicit representation of the kernel \( K_{n,s}^{(3)}(z) \) on the ellipse \( \mathcal{E}_{\rho} \) for the generalized Chebyshev weight function of third kind \( \omega_3(t) \) was given by Milovanović and Spalević in [5], as well

\[
(2.6) \quad \left| K_{n,s}^{(3)}(z) \right| = \frac{2^{1-s}\pi}{\rho^{n+1/2}} \frac{(a_1 + \cos \theta) \left| Z_{n,s}^{(3)}(\rho e^{i\theta}) \right|}{(a_2 - \cos 2\theta)^{1/2}(a_{2n+1} + \cos (2n + 1)\theta)^{1/2+s}},
\]

where

\[
(2.7) \quad Z_{n,s}^{(3)}(\rho e^{i\theta}) = \sum_{k=0}^{s} \binom{2s + 1}{s + k + 1} \left( \rho e^{i\theta} \right)^{-(2n+1)k}.
\]

Using representation (2.6) in [5] was been stated the following conjecture:

**Conjecture 2.** For each fixed \( \rho > 1 \) and \( s \in \mathbb{N}_0 \) there exists \( n_0 = n_0(\rho, s) \) such that

\[
\max_{z \in \mathcal{E}_{\rho}} \left| K_{n,s}^{(3)}(z) \right| = K_{n_0,s}^{(3)} \left( \frac{1}{2}(\rho + \rho^{-1}) \right)
\]

for each \( n \geq n_0 \).

**Theorem 2.2.** The conjecture 2 holds for \( s = 1 \).

**Proof.** Because (2.6) it is sufficiently to prove

\[
(9 + 6\rho^{-2n-1} \cos (2n + 1)\theta + \rho^{-4n-2})(a_2 - 1)(a_{2n+1} + 1)^3
\]

\[
\leq (9 + 6\rho^{-2n-1} + \rho^{-4n-2})(a_2 - \cos 2\theta)(a_{2n+1} + \cos (2n + 1)\theta)^3,
\]

for enough large \( n \) \((n \geq n_0(\rho))\) and \( \theta \in (0, \pi] \), where \( a_j \) are given by (2.2). Introducing the new variable \( k \) with \( n = (2k - 1)/2 \) inequality (2.8) becomes inequality (2.4), which holds \( \forall k, \ k > t \), where \( t \) is the largest zero of the function \( F(k) \) from the proof of Theorem 2.1. Furthermore, we can conclude that inequality (2.8) holds for every \( n \), such that \( n > (2t - 1)/2 \). For \( n_0 \) we can take \( [(2t - 1)/2] + 1 \). \( \square \)


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