TWO CONJECTURES FOR INTEGRALS WITH OSCILLATORY INTEGRANDS*

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Abstract. We present two conjectures connected with highly oscillatory integrals. A connection of validity of these conjectures with the existence of certain quadrature rules is also presented.

1.   Introduction

In [4] we considered the following quadrature rule

$$(1.1) \qquad \int_{-1}^{1} f(x)dx = \sum_{k=1}^{n} \sigma_k f(x_k) + R_n(f),$$

where the nodes x_k and the weights σ_k, $k = 1, \ldots, n$, are chosen such that this quadrature rule is exact on the linear span $\mathcal{F}_{2n}(\zeta)$ of the following functions $x^k \cos \zeta x$, $x^k \sin \zeta x$, $k = 0, 1, \ldots, n-1$, $\zeta \in \mathbb{R}$. Such quadrature rules were considered firstly by Ixaru [2] and Ixaru and Paternoster [3], but they have not proved the existence of such quadrature rules. Numerical method for constructing such quadrature rules is presented only with antisymmetric nodes in $(-1, 1)$ and symmetric weights for $n \le 6$ and $0 < \zeta < 50$.

In [4] the existence question was solved partially. Namely, we proved existence of such a quadrature rule with all positive (all negative) nodes. The existence question for the quadrature rule (1.1) which have the both positive and negative nodes is not solved, yet.

We give a brief survey of results presented in [4]. For a given $n \in \mathbb{N}$ and the set of nodes $\{x_1, \ldots, x_n\}$ we put $\mathbf{x} = (x_1, \ldots, x_n)$ and define the *node polynomial* $\omega(x) = \omega^{(n)}(x)$ by $\omega(x) = \prod_{k=1}^{n}(x - x_k)$. For the brevity we introduce, for $\nu, \mu = 1, \ldots, n$, the following notation

$$\omega_\nu(x) = \frac{\omega(x)}{x - x_\nu} = \prod_{k \neq \nu}(x - x_k), \quad \omega_{\nu,\mu}(x) = \frac{\omega(x)}{(x - x_\nu)(x - x_\mu)} = \prod_{k \neq \nu,\mu}(x - x_k),$$

and $\ell_\nu(x) = \omega_\nu(x)/\omega_\nu(x_\nu)$, as well as

$$(1.2) \qquad \Phi_\nu(\mathbf{x}) = \int_{-1}^{1} \omega_\nu(x) \sin \zeta(x - x_\nu) dx, \quad \nu = 1, \ldots, n.$$

Suppose we are given mutually different nodes x_k, $k = 1, \ldots, n$, of the quadrature rule (1.1). Then the weights can be expressed in the following form (see [4, Theorem 2.1])

$$(1.3) \qquad \sigma_k = \int_{-1}^{1} \ell_k(x) \cos \zeta(x - x_k) dx, \quad k = 1, \ldots, n.$$

Let x_k, $k = 1, \ldots, n$, be the nodes of the quadrature rule (1.1). Then they satisfy the following system of equations (see [4, Theorem 2.2])

$$(1.4) \qquad \int_{-1}^{1} \omega_\nu(x) \sin \zeta(x - x_\nu) dx = 0, \quad \nu = 1, \ldots, n.$$

Suppose that $\mathbf{x} = (x_1, \ldots, x_n)$ is a solution of the system of equations (1.4). Under the assumption $x_k \neq x_j$, $k \neq j$, $k, j = 1, \ldots, n$, we have that x_k, $k = 1, \ldots, n$, are the nodes of the quadrature rule (1.1).

Let x_ν, $\nu = 1, \ldots, n$, be the nodes of the quadrature rule (1.1). Then, provided $\sigma_\nu \neq 0$, $\nu = 1, \ldots, n$, the Jacobian at the solution x_ν, $\nu = 1, \ldots, n$, of the system (1.4) is non-singular (see [4, Theorem 2.3]).

The system of nonlinear equations (1.4) was the main topic in [4]. In the cases $\sin 2\zeta \geq 0$ and $\sin 2\zeta \leq 0$ we rewrote the system of equations (1.4) in the following forms

$$x_\nu = \Psi_\nu^C(\mathbf{x}) = \frac{1}{\zeta} \left(\arctan \frac{\int_{-1}^{1} \omega_\nu(x) \sin \zeta x\, dx}{\int_{-1}^{1} \omega_\nu(x) \cos \zeta x\, dx} + k_\nu \pi \right), \quad \nu = 1, \ldots, n, \ k_\nu \in \mathbb{Z},$$

and

$$x_\nu = \Psi_\nu^S(\mathbf{x}) = \frac{1}{\zeta} \left(\text{arccot} \frac{\int_{-1}^{1} \omega_\nu(x) \cos \zeta x\, dx}{\int_{-1}^{1} \omega_\nu(x) \sin \zeta x\, dx} + k_\nu \pi \right), \quad \nu = 1, \ldots, n, \ k_\nu \in \mathbb{Z},$$

respectively.

Using these transformed systems of nonlinear equations the existence of men-tioned quadrature rule (1.1) which has the both positive and negative nodes can be given under two conjectures, one for the case $\sin 2\zeta < 0$ and the second one for $\sin 2\zeta > 0$ (more details can be found in [6]). These two conjectures are given in Section 2.

2. Conjectures

Firstly, we consider the case $\sin 2\zeta < 0$. Let introduce the following notation

$$b_\nu = (N - \nu + 1)\frac{\pi}{\zeta}, \quad \nu = 1, \ldots, N, \quad N = [\zeta/\pi]$$

where $[t]$ denotes integer part of t.

We consider the integrals

$$(2.1) \qquad I_n = \mathrm{sgn}(\sin \zeta) \int_{-1}^{1} t \prod_{\nu=1}^{n} (t^2 - b_\nu^2) \sin \zeta t\, dt, \quad n = 0, 1, \ldots, N.$$

For $\zeta > 0$ and $\sin 2\zeta < 0$ it is easy to see that

$$I_0 = \mathrm{sgn}(\sin \zeta) \int_{-1}^{1} t \sin \zeta t\ dt = \frac{1}{|\sin \zeta|} \frac{-\zeta \sin 2\zeta + 2\sin^2 \zeta}{\zeta^2} > 0.$$

Based on numerous numerical experiments we can state the following conjecture:

Conjecture 2.1. *Suppose that $\zeta > 0$ and $\sin 2\zeta < 0$. Then $I_n > 0$ for each $n = 1, \ldots, N$.*

Using the well known formula (cf. [1, 1.431, p. 43])

$$(2.2) \qquad \sin x = x \prod_{k=1}^{+\infty} \left(1 - \frac{x^2}{k^2 \pi^2}\right),$$

for $x = \zeta$ we obtain a product where the first N factors are negative and all others are positive. Because of that, we have

$$\mathrm{sgn}(\sin \zeta) = (-1)^N,$$

and then

$$(2.3) \qquad I_n = (-1)^N \int_{-1}^{1} \prod_{\nu=1}^{n} (t^2 - b_\nu^2) t \sin \zeta t\, dt, \quad n = 1, \ldots, N.$$

Replacing (2.2) for $x = \zeta t$ in (2.3) we obtain

$$I_n = 2\zeta(-1)^{N+n} \int_0^1 t^2 \prod_{\nu=1}^n (t^2 - b_\nu^2)^2 \prod_{\ell=1}^N \frac{\zeta^2}{\ell^2 \pi^2} \prod_{\ell=1}^{N-n} \left(\frac{\ell^2 \pi^2}{\zeta^2} - t^2 \right) \prod_{\ell=N+1}^{+\infty} \left(1 - \frac{t^2 \zeta^2}{\ell^2 \pi^2} \right) dt$$

(empty product is equal to 1). From the previous formula it is easy to see that $I_N > 0$.

In these integrals for $n = 1, \ldots, N - 1$, all factors are positive except for the product $\prod_{\ell=1}^{N-n} \left(\frac{\ell^2 \pi^2}{\zeta^2} - t^2 \right)$. This means that integrand changes its sign $N - n$ times in the interval $(0, 1)$. Let denote integrand by $f_n(t; \zeta)$, i.e.,

$$f_n(t; \zeta) = t^2 \prod_{\nu=1}^n (t^2 - b_\nu^2)^2 \prod_{\ell=1}^N \frac{\zeta^2}{\ell^2 \pi^2} \prod_{\ell=1}^{N-n} \left(\frac{\ell^2 \pi^2}{\zeta^2} - t^2 \right) \prod_{\ell=N+1}^{+\infty} \left(1 - \frac{t^2 \zeta^2}{\ell^2 \pi^2} \right).$$

In Figure 2.1 the graph of function $f_n(t; \zeta)$, $t \in (0, 1)$, for $\zeta = 100$ and $n = 1$ (left) and $n = 2$ (right) is given.
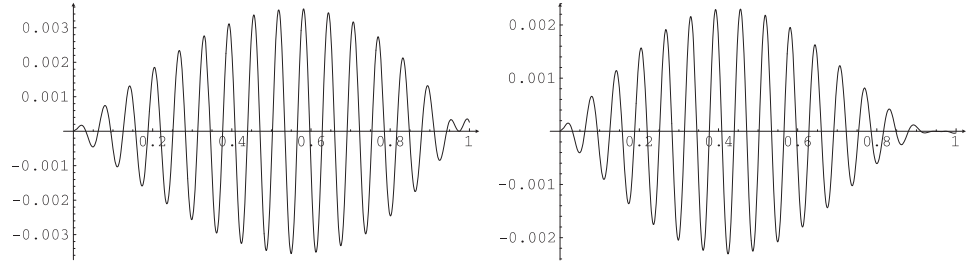

FIG. 2.1: Graph of function $f_n(t; \zeta)$, $t \in (0, 1)$, for $\zeta = 100$, $n = 1$ (left) and $n = 2$ (right)

In the numerical experiments we saw that I_n has the same sign as the function $f_n(t; \zeta)$ after the last sign change, i.e., that $I_n > 0$, $n = 1, \ldots, N - 1$.

Under condition that Conjecture 2.1 is true it is possible to prove that in the case when $\zeta > 0$ and $\sin 2\zeta < 0$ for all $\mathbf{x} = (x_1, \ldots, x_{2n+1}) \in B_n$, where

$$(2.4) \qquad B_n = \underset{\nu=1}{\overset{n}{\times}} ([-b_\nu, 0] \times [0, b_\nu]) \times [-b_{n+1}, b_{n+1}], \quad n < N,$$

the following inequality

$$(2.5) \qquad \text{sgn}(\sin \zeta) \int_{-1}^1 \prod_{\nu=1}^{2n+1} (t - x_\nu) \sin \zeta t \, dt > 0$$

holds. This inequality is of great importance for solving the existence of the quadrature rule (1.1) in general for the case $\sin 2\zeta < 0$.

In order to prove the inequality (2.5) we need some auxiliary results.

Let $\mathcal{A} : \mathcal{P} \mapsto \mathbb{R}$ be linear functional on $\mathcal{P}$ ($\mathcal{P}-$ set of all polynomials). Denote with $\sigma_{n,k}$ elementary symmetric functions

$$\sigma_{n,k} = (-1)^k \sum_{(i_1,\ldots,i_k)} x_{i_1} \ldots x_{i_k}, \quad k = 0, 1, \ldots, n,$$

where summation is performed over all combinations, without repetition, of length k of numbers $1, \ldots, n$ (see [5]). Every polynomial p of degree n with real zeros $x_1, \ldots, x_n$ can be represented in the following form

$$p(x) = \sum_{k=0}^{n} \sigma_{n,n-k} x^k.$$

We denote the vector of zeros of a polynomial p with $\mathbf{x} = (x_1, \ldots, x_n)$ and consider the following problem: *Find*

$$\min_{\mathbf{x} \in C} \mathcal{A}p = \min_{\mathbf{x} \in C} \sum_{k=0}^{n} \sigma_{n,n-k} \mathcal{A} x^k,$$

for a given compact and connected set $C \in \mathbb{R}^n$. For that purpose we define the function Φ in the following way

$$\Phi(\mathbf{x}) = \mathcal{A}p, \quad \mathbf{x} \in C.$$

Now, our problem is to determine $\min\limits_{\mathbf{x} \in C} \Phi(\mathbf{x})$.

Lemma 2.1. *The function Φ is harmonic on $C \setminus \partial C$.*

Proof. Every elementary symmetric function is obviously harmonic, i.e., satisfies the Laplace's equation

$$\Delta \sigma_{n,k} = 0, \quad k = 0, 1, \ldots, n,$$

where Δ is the Laplacian. Because Φ is a linear combination of harmonic functions it is also harmonic. $\square$

According to the strong maximum (minimum) principle for the harmonic functions we have the following lemma.

Lemma 2.2. *The maximum (minimum) value of Φ on C must occur on ∂C, i.e.,*

$$\max_{\mathbf{x} \in C} (\min_{\mathbf{x} \in C}) \Phi(\mathbf{x}) = \max_{\mathbf{x} \in \partial C} (\min_{\mathbf{x} \in \partial C}) \Phi(\mathbf{x}).$$

Because of that, our problem is to determine $\min\limits_{\mathbf{x} \in \partial C} \Phi(\mathbf{x})$.

Lemma 2.3. *If the set B_n is defined with (2.4) then*

$$\max_{\mathbf{x}\in B_n}(\min_{\mathbf{x}\in B_n})\Phi(\mathbf{x})$$

(2.6)
$$= \max_{k_i\in\{0,1\},i=1,\ldots,2n+1}(\min) \; \mathcal{A}\prod_{\nu=1}^{n}(x+b_\nu)^{1-k_{2\nu-1}}x^{k_{2\nu-1}}(x-b_\nu)^{1-k_{2\nu}}x^{k_{2\nu}}\times$$
$$\times(x+b_{n+1})^{1-k_{2n+1}}(x-b_{n+1})^{k_{2n+1}}.$$

Proof. According to Lemma 2.2 we have that

$$\max_{\mathbf{x}\in B_n}(\min_{\mathbf{x}\in B_n})\Phi(\mathbf{x}) = \max_{\mathbf{x}\in\partial B_n}(\min_{\mathbf{x}\in\partial B_n})\Phi(\mathbf{x}),$$

and all we need to do is to describe ∂B_n. Every point $\mathbf{x}=(x_1,\ldots,x_{2n+1})$, which satisfies the following conditions

$$-b_\nu < x_{2\nu-1} < 0, \quad 0 < x_{2\nu} < b_\nu, \quad \nu = 1,\ldots,n, \quad -b_{n+1} < x_{2n+1} < b_{n+1},$$

is an interior point for the set B_n.

The point $\mathbf{x}=(x_1,\ldots,x_{2n+1})\in\partial B_n$ if exists an index ν ($\in\{1,\ldots,n\}$) such that $x_{2\nu-1}=-b_\nu$ or $x_{2\nu-1}=0$ or if exists ν ($\in\{1,\ldots,n\}$) such that $x_{2\nu}=b_\nu$ or $x_{2\nu}=0$ or if $x_{2n+1}=\pm b_{n+1}$. Therefore, our max(min) must occur in some of such boundary point. The function Φ is harmonic. We can choose $x_1=-b_1$ or $x_1=0$, and define the following two functions

$$\Phi_2^{-b_1}(x_2,\ldots,x_{2n+1}) = \Phi(-b_1,x_2,\ldots,x_{2n+1}),$$
$$\Phi_2^{0}(x_2,\ldots,x_{2n+1}) = \Phi(0,x_2,\ldots,x_{2n+1}).$$

Obviously, the function $\Phi_2^{-b_1}$, as well as the function Φ_2^{0}, is also harmonic on

$$B_n^1 = [0,b_1] \times \mathop{\times}_{\nu=2}^{n}([-b_\nu,0]\times[0,b_\nu]) \times [-b_{n+1},b_{n+1}].$$

Thus, any of these two functions must achieve max(min) on ∂B_n^1. Now, we fix $x_2=0$ or $x_2=b_1$. Continuing in this manner we get the equation (2.6). □

We rewrite the equation (2.6) in the following way

(2.7)
$$\max_{\mathbf{x}\in B_n}(\min_{\mathbf{x}\in B_n})\Phi(\mathbf{x}) = \max_{\mathbf{x}\in Q_n}(\min_{\mathbf{x}\in Q_n})\Phi(\mathbf{x}),$$

where $Q_n\subset\partial B_n$, such that $\mathbf{x}=(x_1,\ldots,x_{2n+1})\in Q_n$ if $x_{2\nu-1}\in\{-b_\nu,0\}$, $x_{2\nu}\in\{0,b_\nu\}$, $\nu=1,\ldots,n$, and $x_{2n+1}\in\{-b_{n+1},b_{n+1}\}$.

In the sequel we consider only polynomials p_{2n+1} with zeros $\mathbf{x}\in Q_n$ for some nonnegative integer n and we will not explicitly mark that.

According to Lemma 2.3 we have

$$\operatorname{sgn}(\sin\zeta)\int_{-1}^{1}\prod_{\nu=1}^{2n+1}(t-x_\nu)\sin\zeta t\,dt \geq \min_{\mathbf{x}\in Q_n}\operatorname{sgn}(\sin\zeta)\int_{-1}^{1}p_{2n+1}(t)\sin\zeta t\,dt.$$

Therefore, the inequality (2.5) will be true if

$$\text{(2.8)} \qquad \min_{\mathbf{x} \in Q_n} \operatorname{sgn}(\sin \zeta) \int_{-1}^{1} p_{2n+1}(t) \sin \zeta t \, dt > 0.$$

Under the condition that Conjecture 2.1 is true, we can prove the following result.

Theorem 2.1. *Let $\zeta > 0$, $\sin 2\zeta < 0$, $N = [\zeta/\pi]$, and let all integrals I_n in (2.1) be positive. Then, for $n \leq N$,*

$$\min_{\mathbf{x} \in Q_n} \operatorname{sgn}(\sin \zeta) \int_{-1}^{1} p_{2n+1}(t) \sin \zeta t \, dt = I_n.$$

Proof. First of all we note the following trivial fact

$$\operatorname{sgn}(\sin \zeta) \int_{-1}^{1} \prod_{\nu=1}^{n} (t^2 - b_\nu^2)(t \pm b_{n+1}) \sin \zeta t \, dt = I_n.$$

At the beginning, we prove the assertion for $n = 1$. Since

$$I_1 = \operatorname{sgn}(\sin \zeta) \int_{-1}^{1} (t^2 - b_1^2) t \sin \zeta t \, dt,$$

we consider all other possible cases with respect to values of zeros x_1, x_2, x_3 as follows. At first

$$I_1^{1,2} = \operatorname{sgn}(\sin \zeta) \int_{-1}^{1} (t + b_1) t (t \pm b_2) \sin \zeta t \, dt = \operatorname{sgn}(\sin \zeta) \int_{-1}^{1} (t^3 \pm b_1 b_2 t) \sin \zeta t \, dt,$$

and it is easy to see that $I_1^{1,2} - I_1 = b_1 (b_1 \pm b_2) I_0 > 0$ since $b_1 > b_2 > 0$ and $I_0 > 0$. On a similar way, for

$$I_1^{3,4} = \operatorname{sgn}(\sin \zeta) \int_{-1}^{1} (t - b_1) t (t \pm b_2) \sin \zeta t \, dt = \operatorname{sgn}(\sin \zeta) \int_{-1}^{1} (t^3 \mp b_1 b_2 t) \sin \zeta t \, dt,$$

we have $I_1^{3,4} - I_1 = b_1 (b_1 \mp b_2) I_0 > 0$.

Finally,

$$I_1^{5,6} = \operatorname{sgn}(\sin \zeta) \int_{-1}^{1} t^2 (t \pm b_2) \sin \zeta t \, dt = \operatorname{sgn} \sin \zeta \int_{-1}^{1} t^3 \sin \zeta t \, dt,$$

and $I_1^{5,6} - I_1 = b_1^2 I_0 > 0$.

Therefore, we have already proved that

$$\min_{\mathbf{x} \in Q_1} \operatorname{sgn}(\sin \zeta) \int_{-1}^{1} p_3(t) \sin \zeta t \, dt = I_1.$$

Next, we prove that if

$$\min_{\mathbf{x}\in Q_{n-1}} \operatorname{sgn}(\sin\zeta) \int_{-1}^{1} p_{2n-1}(t)\sin\zeta t\, dt = I_{n-1}$$

for some integer $2 \le n \le N-1$, then

$$\min_{\mathbf{x}\in Q_n} \operatorname{sgn}(\sin\zeta) \int_{-1}^{1} p_{2n+1}(t)\sin\zeta t\, dt = I_n.$$

We have three crucial steps in proof. First one is to prove that in case when we change one symmetric factor $t^2 - b_\ell^2$, $1 \le \ell \le n$, in $\prod_{\nu=1}^{n}(t^2 - b_\nu^2)$ with any possible nonsymmetric case, as well as the another symmetric case (t^2), the obtained integrals are greater than I_n. According to possible values of zeros $x_{2\ell-1}, x_{2\ell}$ and x_{2n+1} it is easy to see that there are six such cases. Let denote the corresponding integrals with I_n^i, $i = 1,\dots,6$, and consider all of them. So, we have

$$
\begin{aligned}
I_n^{1,2} &= \operatorname{sgn}(\sin\zeta) \int_{-1}^{1} \prod_{\substack{\nu=1\\ \nu\neq\ell}}^{n}(t^2 - b_\nu^2)(t+b_\ell)t(t\pm b_{n+1})\sin\zeta t\, dt \\
&= \operatorname{sgn}(\sin\zeta) \int_{-1}^{1} \prod_{\substack{\nu=1\\ \nu\neq\ell}}^{n}(t^2 - b_\nu^2)(t^3 \pm b_\ell b_{n+1}t)\sin\zeta t\, dt,
\end{aligned}
$$

and

$$I_n^{1,2} - I_n = b_\ell(b_\ell \pm b_{n+1})\operatorname{sgn}(\sin\zeta) \int_{-1}^{1} \prod_{\substack{\nu=1\\ \nu\neq\ell}}^{n}(t^2 - b_\nu^2)t\sin\zeta t\, dt > 0,$$

because $b_\ell > b_{n+1} > 0$ and

$$\operatorname{sgn}(\sin\zeta) \int_{-1}^{1} \prod_{\substack{\nu=1\\ \nu\neq\ell}}^{n}(t^2 - b_\nu^2)t\sin\zeta t\, dt \ge I_{n-1} > 0$$

(the equality in previous inequality holds for $\ell = n$).

Similarly, one has

$$
\begin{aligned}
I_n^{3,4} &= \operatorname{sgn}(\sin\zeta) \int_{-1}^{1} \prod_{\substack{\nu=1\\ \nu\neq\ell}}^{n}(t^2 - b_\nu^2)t(t-b_\ell)(t\pm b_{n+1})\sin\zeta t\, dt \\
&= \operatorname{sgn}(\sin\zeta) \int_{-1}^{1} \prod_{\substack{\nu=1\\ \nu\neq\ell}}^{n}(t^2 - b_\nu^2)(t^3 \mp b_\ell b_{n+1}t)\sin\zeta t\, dt,
\end{aligned}
$$

and, again,

$$I_n^{3,4} - I_n = b_\ell(b_\ell \mp b_{n+1})\operatorname{sgn}(\sin\zeta) \int_{-1}^{1} \prod_{\substack{\nu=1\\ \nu\neq\ell}}^{n}(t^2 - b_\nu^2)t\sin\zeta t\, dt > 0.$$

Finally,

$$
\begin{aligned}
I_n^{5,6} &= \operatorname{sgn}(\sin \zeta) \int_{-1}^{1} \prod_{\substack{\nu=1 \\ \nu \neq \ell}}^{n} (t^2 - b_\nu^2) t^2 (t \pm b_{n+1}) \sin \zeta t \, dt \\
&= \operatorname{sgn}(\sin \zeta) \int_{-1}^{1} \prod_{\substack{\nu=1 \\ \nu \neq \ell}}^{n} (t^2 - b_\nu^2) t^3 \sin \zeta t \, dt
\end{aligned}
$$

and

$$
I_n^{5,6} - I_n = b_\ell^2 \operatorname{sgn}(\sin \zeta) \int_{-1}^{1} \prod_{\substack{\nu=1 \\ \nu \neq \ell}}^{n} (t^2 - b_\nu^2) t \sin \zeta t \, dt \geq b_\ell^2 I_{n-1} > 0.
$$

The second step in this proof is to prove the following assertion: *If $p_{2n+1,k}(t)$, $1 < k \leq n$, is a polynomial with k pairs of nonsymmetric zeros and*

$$
I_{2n+1}^{(k)} = \operatorname{sgn}(\sin \zeta) \int_{-1}^{1} p_{2n+1,k}(t) \sin \zeta t \, dt
$$

then there exists $p_{2n+1,k-1}$ such that $I_{2n+1}^{(k)} > I_{2n+1}^{(k-1)}$.

Let $(i_1, i_2, \ldots, i_n)$ be an arbitrary permutation without repetition of the following set $\{1, 2, \ldots, n\}$. The possible values of $I_{2n+1}^{(k)}$ are

$$
I_{2n+1}^{(k)} = \operatorname{sgn}(\sin \zeta) \int_{-1}^{1} \prod_{\ell=1}^{k-j} (t + b_{i_\ell}) \prod_{\nu=k-j+1}^{k} (t - b_{i_\nu}) t^k \prod_{\mu=k+1}^{n} (t^2 - b_{i_\mu}^2)(t \pm b_{n+1}) \sin \zeta t \, dt,
$$

for $0 \leq j \leq k$ (empty product equals to 1 by definition).

Let suppose that $0 < j < k$, choose arbitrary indices i_s, $1 \leq s \leq k - j$ and i_r, $k - j + 1 \leq r \leq k$ and consider the following integrals:

$$
\begin{aligned}
I_{2n+1}^{(k-1),1} = \operatorname{sgn}(\sin \zeta) \int_{-1}^{1} \prod_{\substack{\ell=1 \\ \ell \neq s}}^{k-j} (t + b_{i_\ell}) \prod_{\nu=k-j+1}^{k} (t - b_{i_\nu}) t^{k-1} (t^2 - b_{i_s}^2) \times \\
\times \prod_{\mu=k+1}^{n} (t^2 - b_{i_\mu}^2)(t \pm b_{n+1}) \sin \zeta t \, dt
\end{aligned}
$$

and

$$
\begin{aligned}
I_{2n+1}^{(k-1),2} = \operatorname{sgn}(\sin \zeta) \int_{-1}^{1} \prod_{\ell=1}^{k-j} (t + b_{i_\ell}) \prod_{\substack{\nu=k-j+1 \\ \nu \neq r}}^{k} (t - b_{i_\nu}) t^{k-1} (t^2 - b_{i_r}^2) \times \\
\times \prod_{\mu=k+1}^{n} (t^2 - b_{i_\mu}^2)(t \pm b_{n+1}) \sin \zeta t \, dt.
\end{aligned}
$$

Our aim is to prove that one of the differences $I_{2n+1}^{(k)} - I_{2n+1}^{(k-1),1}$ or $I_{2n+1}^{(k)} - I_{2n+1}^{(k-1),2}$ is positive. It is easy to see that these differences are equal to the following expressions

$$
\begin{aligned}
I_{2n+1}^{(k)} - I_{2n+1}^{(k-1),1} &= \operatorname{sgn}(\sin \zeta) \int_{-1}^{1} \prod_{\substack{\ell=1 \\ l \neq s}}^{k-j} (t + b_{i_\ell}) \prod_{\nu=k-j+1}^{k} (t - b_{i_\nu}) t^{k-1} \times \\
&\quad \times \prod_{\mu=k+1}^{n} (t^2 - b_{i_\mu}^2)(t \pm b_{n+1})(t^2 + t b_{i_s} - t^2 + b_{i_s}^2) \sin \zeta t \, dt \\
&= b_{i_s} \operatorname{sgn}(\sin \zeta) \int_{-1}^{1} \prod_{\ell=1}^{k-j} (t + b_{i_\ell}) \prod_{\nu=k-j+1}^{k} (t - b_{i_\nu}) t^{k-1} \times \\
&\quad \times \prod_{\mu=k+1}^{n} (t^2 - b_{i_\mu}^2)(t \pm b_{n+1}) \sin \zeta t \, dt
\end{aligned}
$$

and

$$
\begin{aligned}
I_{2n+1}^{(k)} - I_{2n+1}^{(k-1),2} &= \operatorname{sgn}(\sin \zeta) \int_{-1}^{1} \prod_{\ell=1}^{k-j} (t + b_{i_\ell}) \prod_{\substack{\nu=k-j+1 \\ \nu \neq r}}^{k} (t - b_{i_\nu}) t^{k-1} \times \\
&\quad \times \prod_{\mu=k+1}^{n} (t^2 - b_{i_\mu}^2)(t \pm b_{n+1})(t^2 - t b_{i_r} - t^2 + b_{i_r}^2) \sin \zeta t \, dt \\
&= -b_{i_r} \operatorname{sgn}(\sin \zeta) \int_{-1}^{1} \prod_{\ell=1}^{k-j} (t + b_{i_\ell}) \prod_{\nu=k-j+1}^{k} (t - b_{i_\nu}) t^{k-1} \times \\
&\quad \times \prod_{\mu=k+1}^{n} (t^2 - b_{i_\mu}^2)(t \pm b_{n+1}) \sin \zeta t \, dt.
\end{aligned}
$$

Since the obtained integrals on the right hand sides of these differences are the same and $b_{i_s}, b_{i_r} > 0$, we can conclude that one of considered differences is positive.

In the cases $j = 0$ and $j = k$ we have the following integrals

$$
I_{2n+1}^{(k),+} = \operatorname{sgn}(\sin \zeta) \int_{-1}^{1} \prod_{\ell=1}^{k} (t + b_{i_\ell}) t^k \prod_{\mu=k+1}^{n} (t^2 - b_{i_\mu}^2)(t \pm b_{n+1}) \sin \zeta t \, dt
$$

and

$$
I_{2n+1}^{(k),-} = \operatorname{sgn}(\sin \zeta) \int_{-1}^{1} \prod_{\ell=1}^{k} (t - b_{i_\ell}) t^k \prod_{\mu=k+1}^{n} (t^2 - b_{i_\mu}^2)(t \pm b_{n+1}) \sin \zeta t \, dt.
$$

Changing variable t with $-t$ in $I_{2n+1}^{(k),-}$ we obtain

$$
I_{2n+1}^{(k),-} = \operatorname{sgn}(\sin \zeta) \int_{-1}^{1} \prod_{\ell=1}^{k} (t + b_{i_\ell}) t^k \prod_{\mu=k+1}^{n} (t^2 - b_{i_\mu}^2)(t \mp b_{n+1}) \sin \zeta t \, dt.
$$

Therefore, we need to prove our assertion for $I_{2n+1}^{(k),+}$. Now, we prove that obtained integral $I_{2n+1}^{(k),+}$ for $j = 0$ is greater than the corresponding integral $I_{2n+1}^{(k)}$ for $j = 1$. For this purpose we consider the following difference

$$\text{sgn}(\sin \zeta) \int_{-1}^{1} \prod_{\ell=1}^{k}(t + b_{i_\ell})t^k \prod_{\mu=k+1}^{n}(t^2 - b_{i_\mu}^2)(t \pm b_{n+1}) \sin \zeta t \, dt$$

$$- \text{sgn}(\sin \zeta) \int_{-1}^{1} \prod_{\ell=2}^{k}(t + b_{i_\ell})(t - b_{i_1})t^k \prod_{\mu=k+1}^{n}(t^2 - b_{i_\mu}^2)(t \pm b_{n+1}) \sin \zeta t \, dt$$

$$= 2b_{i_1} \text{sgn}(\sin \zeta) \int_{-1}^{1} \prod_{\ell=2}^{k}(t + b_{i_\ell})(t \pm b_{n+1})t^k \prod_{\mu=k+1}^{n}(t^2 - b_{i_\mu}^2) \sin \zeta t \, dt.$$

All we need to prove is that for all $1 < k \leq n$ the following inequality

$$J_k = \text{sgn}(\sin \zeta) \int_{-1}^{1} \prod_{\ell=2}^{k}(t + b_{i_\ell})(t \pm b_{n+1})t^k \prod_{\mu=k+1}^{n}(t^2 - b_{i_\mu}^2) \sin \zeta t \, dt > 0$$

holds.

Firstly, we prove that $J_2 > 0$, and then that $J_{k+1} > J_k$ holds for all $k < n$. Since

$$\begin{aligned}
J_2 &= \text{sgn}(\sin \zeta) \int_{-1}^{1}(t + b_{i_2})(t \pm b_{n+1})t^2 \prod_{\mu=3}^{n}(t^2 - b_{i_\mu}^2) \sin \zeta t \, dt \\
&= \text{sgn}(\sin \zeta) \int_{-1}^{1}(t^2 + (b_{i_2} \pm b_{n+1})t \pm b_{i_2}b_{n+1})t^2 \prod_{\mu=3}^{n}(t^2 - b_{i_\mu}^2) \sin \zeta t \, dt \\
&= (b_{i_2} \pm b_{n+1}) \text{sgn}(\sin \zeta) \int_{-1}^{1} \prod_{\mu=3}^{n}(t^2 - b_{i_\mu}^2)t^3 \sin \zeta t \, dt > 0,
\end{aligned}$$

because of $b_{i_2} \pm b_{n+1} > 0$ and

$$\text{sgn}(\sin \zeta) \int_{-1}^{1} \prod_{\mu=3}^{n}(t^2 - b_{i_\mu}^2)t^3 \sin \zeta t \, dt > I_{n-1} > 0,$$

we get

$$J_{k+1} - J_k = \text{sgn}(\sin \zeta) \int_{-1}^{1} \prod_{\ell=2}^{k+1}(t + b_{i_\ell})(t \pm b_{n+1})t^{k+1} \prod_{\mu=k+2}^{n}(t^2 - b_{i_\mu}^2) \sin \zeta t \, dt$$

$$- \text{sgn}(\sin \zeta) \int_{-1}^{1} \prod_{\ell=2}^{k}(t + b_{i_\ell})(t \pm b_{n+1})t^k \prod_{\mu=k+1}^{n}(t^2 - b_{i_\mu}^2) \sin \zeta t \, dt$$

$$= b_{i_{k+1}} \text{sgn}(\sin \zeta) \int_{-1}^{1} \prod_{\ell=2}^{k+1}(t + b_{i_\ell})(t \pm b_{n+1})t^k \prod_{\mu=k+2}^{n}(t^2 - b_{i_\mu}^2) \sin \zeta t \, dt > 0,$$

because of $b_{i_{k+1}} > 0$ and

$$\text{sgn}(\sin\zeta)\int_{-1}^{1}\prod_{\ell=2}^{k+1}(t+b_{i_\ell})(t\pm b_{n+1})t^k\prod_{\mu=k+2}^{n}(t^2-b_{i_\mu}^2)\sin\zeta t\,dt > I_{n-1} > 0.$$

Therefore, our assertion has been proved completely.

The third step is to prove that the integrals obtained when some of k symmetric factors, $2 < k \le n$, in the product $\prod_{\nu=1}^{n}(t^2-b_\nu^2)$ are changed with the another symmetric factors (i.e., with t^2), and some of them are changed with nonsymmetric factors, are greater than I_n. So, we have to consider the following integrals

$$I_{2n+1}^{(k),0} = \text{sgn}(\sin\zeta)\int_{-1}^{1}\prod_{\ell=p+1}^{k-j}(t+b_{i_\ell})\prod_{\nu=k-j+q+1}^{k}(t-b_{i_\nu})t^{k-p-q}\prod_{\mu=k+1}^{n}(t^2-b_{i_\mu}^2)\times$$
$$\times t^{2(p+q)}(t\pm b_{n+1})\sin\zeta t\,dt$$

for all $0 \le j \le k$ and for all $0 \le p \le k-j$ and $0 \le q \le j$ such that $p^2+q^2 \ne 0$.

Let s be an arbitrary number satisfying $1 \le s \le p$ or $k-j+1 \le s \le k-j+q$. If we change t^2 in the previous integrals with $t^2-b_{i_s}^2$ we get

$$I_{2n+1}^{(k-1),0} = \text{sgn}(\sin\zeta)\int_{-1}^{1}\prod_{\ell=p+1}^{k-j}(t+b_{i_\ell})\prod_{\nu=k-j+q+1}^{k}(t-b_{i_\nu})t^{k-p-q}\prod_{\mu=k+1}^{n}(t^2-b_{i_\mu}^2)\times$$
$$\times t^{2(p+q)-2}(t^2-b_{i_s}^2)(t\pm b_{n+1})\sin\zeta t\,dt.$$

It is easy to see that

$$I_{2n+1}^{(k),0} - I_{2n+1}^{(k-1),0} = b_{i_s}^2\,\text{sgn}(\sin\zeta)\int_{-1}^{1}\prod_{\ell=p+1}^{k-j}(t+b_{i_\ell})\prod_{\nu=k-j+q+1}^{k}(t-b_{i_\nu})t^{k-p-q}\times$$
$$\times\prod_{\mu=k+1}^{n}(t^2-b_{i_\mu}^2)t^{2(p+q)-2}(t\pm b_{n+1})\sin\zeta t\,dt > 0,$$

because the integral on the right hand side is greater than I_{n-1}. According to the first step (integrals $I_n^{5,6}$ for $k=1$) the inequality $I_{2n+1}^{(k),0} > I_n$ holds.

With these three steps the statement of this theorem is proved. $\square$

Thus, the inequality (2.5) is a direct corollary of Theorem 2.1 and Conjecture 2.1.

Now, we can consider the case $\sin 2\zeta > 0$. Introducing the notation

$$a_\nu = \left(N-\nu+\frac{1}{2}\right)\frac{\pi}{\zeta}, \quad \nu = 1,\dots,N, \quad N = [\zeta/\pi],$$

and

$$I_n^C = \text{sgn}(\sin \zeta) \int_{-1}^{1} \prod_{\nu=1}^{n} (t^2 - a_\nu^2) \cos \zeta t \, dt, \quad n = 0, 1, \ldots, N,$$

it is easy to see that

$$I_0^C = \text{sgn}(\sin \zeta) \int_{-1}^{1} \cos \zeta t \, dt = \frac{2|\sin \zeta|}{\zeta} > 0.$$

According to the numerous numerical experiments we can state the second conjecture.

Conjecture 2.2. *Suppose that $\zeta > 0$ and $\sin 2\zeta > 0$. Then $I_n^C > 0$ for each $n = 1, \ldots, N$.*

Analogously as in the case $\sin 2\zeta < 0$, under condition that Conjecture 2.2 is true it is possible to prove that in case when $\zeta > 0$ and $\sin 2\zeta > 0$ for all $\mathbf{x} = (x_1, \ldots, x_{2n}) \in A_n$, where

$$A_n = \underset{\nu=1}{\overset{n}{\times}} \left([-a_\nu, 0] \times [0, a_\nu]\right), \quad n = 1, \ldots, N,$$

the following inequality

$$\text{sgn}(\sin \zeta) \int_{-1}^{1} \prod_{\nu=1}^{2n} (t - x_\nu) \cos \zeta t \, dt > 0$$

holds.

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