CHRISTOFFEL-DARBOUX FORMULA FOR ORTHOGONAL TRIGONOMETRIC POLYNOMIALS OF SEMI-INTEGER DEGREE*

Gradimir V. Milovanović, Aleksandar S. Cvetković
and Marija P. Stanić

Abstract. In this paper we introduce orthonormal trigonometric polynomials of semi-integer degree with respect to a weight function on \([-\pi, \pi]\) and prove the Christoffel-Darboux formula for a such orthonormal trigonometric system.

1. Introduction

Let denote by \(T_{n}^{1/2}\) the linear span of the following trigonometric functions
\[
\cos x/2, \sin x/2, \cos(1 + 1/2)x, \sin(1 + 1/2)x, \ldots, \cos(n + 1/2)x, \sin(n + 1/2)x.
\]
Elements of \(T_{n}^{1/2}\), i.e., the trigonometric functions of the following form
\[
A_{n+1/2}(x) = \sum_{\nu=0}^{n} \left( c_{\nu} \cos \left( \nu + \frac{1}{2} \right) x + d_{\nu} \sin \left( \nu + \frac{1}{2} \right) x \right),
\]
where \(c_{\nu}, d_{\nu} \in \mathbb{R}, |c_{n}| + |d_{n}| \neq 0\), are called trigonometric polynomials of semi-integer degree.

For a given nonnegative weight function weight function \(w(x)\) on \([-\pi, \pi]\), which equals zero only on a set of the Lebesgue measure zero,
\[
(f, g) = \int_{0}^{2\pi} f(x)g(x)w(x) \, dx,
\]
denotes the corresponding inner product of the functions \(f\) and \(g\). For the given scalar product (1.1), the problem of finding \(A_{n+1/2} \in T_{n}^{1/2}\), such that
\[
\int_{-\pi}^{\pi} A_{n}(x)t(x)w(x) \, dt = 0, \quad t \in T_{n-1}^{1/2},
\]
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was considered at first in [3], and in detail in [1] and [2]. It turns out that this problem has unique solution if the leading coefficients $c_n$ and $d_n$ are fixed in advance (see [3, §3]).

Those orthogonal trigonometric systems have applications in construction of quadrature formulas with maximal trigonometric degree of exactness.

In cite [1] and [2] the two choices of leading coefficients were considered. For

the first choice $c_n = 1, d_n = 0$, we denote orthogonal trigonometric polynomial of semi-integer degree by $A_{n+1/2}^C$, and for the second choice $c_n = 0$ and $d_n = 1$ by $A_{n+1/2}^S$. For

the expanded forms of $A_{n+1/2}^C$ and $A_{n+1/2}^S$ we use the following notation

\begin{equation}
A_{n+1/2}^C(x) = \cos \left( n + \frac{1}{2} \right) x + \sum_{\nu=0}^{n-1} c^{(n)}_\nu \cos \left( \nu + \frac{1}{2} \right) x + d^{(n)}_\nu \sin \left( \nu + \frac{1}{2} \right) x,
\end{equation}

\begin{equation}
A_{n+1/2}^S(x) = \sin \left( n + \frac{1}{2} \right) x + \sum_{\nu=0}^{n-1} f^{(n)}_\nu \cos \left( \nu + \frac{1}{2} \right) x + g^{(n)}_\nu \sin \left( \nu + \frac{1}{2} \right) x.
\end{equation}

In [1] it was proved that orthogonal trigonometric polynomials of semi-integer degree $A_{k+1/2}^C(x)$ and $A_{k+1/2}^S(x)$, $k \in \mathbb{N}$, satisfy the following five-term recurrence relations:

\begin{equation}
A_{k+1/2}^C(x) = (2 \cos x - \alpha_k^{(1)}) A_{k-1/2}^C(x) - \beta_k^{(1)} A_{k-3/2}^C(x) - \alpha_k^{(2)} A_{k-3/2}^S(x) - \beta_k^{(2)} A_{k-5/2}^S(x),
\end{equation}

and

\begin{equation}
A_{k+1/2}^S(x) = (2 \cos x - \delta_k^{(1)}) A_{k-1/2}^S(x) - \gamma_k^{(1)} A_{k-3/2}^C(x) - \gamma_k^{(2)} A_{k-3/2}^S(x) - \delta_k^{(2)} A_{k-5/2}^S(x),
\end{equation}

where recurrence coefficients are given by

\begin{equation}
\alpha_k^{(1)} = \beta_k^{(1)} = \gamma_k^{(1)} = \delta_k^{(1)} = 0, \quad \alpha_k^{(2)} = \beta_k^{(2)} = \gamma_k^{(2)} = \delta_k^{(2)}.
\end{equation}

\begin{equation}
\alpha_k^{(1)} = \frac{I_{k-1}^C J_{k-1}^C - I_{k-1}^C J_{k-1}}{D_{k-1}}, \quad \alpha_k^{(2)} = \frac{I_{k-1}^C J_{k-2}^C - I_{k-1}^C J_{k-2}^C}{D_{k-2}},
\end{equation}

\begin{equation}
\beta_k^{(1)} = \frac{I_{k-1}^C J_{k-1} - I_{k-1}^C J_{k-1}}{D_{k-1}}, \quad \beta_k^{(2)} = \frac{I_{k-1}^C J_{k-2} - I_{k-1}^C J_{k-2}}{D_{k-2}},
\end{equation}

\begin{equation}
\gamma_k^{(1)} = \frac{I_{k-1}^S J_{k-1}^S - I_{k-1}^S J_{k-1}^S}{D_{k-1}}, \quad \gamma_k^{(2)} = \frac{I_{k-1}^S J_{k-2}^S - I_{k-1}^S J_{k-2}^S}{D_{k-2}},
\end{equation}

\begin{equation}
\delta_k^{(1)} = \frac{I_{k-1}^C J_{k-1}^C - I_{k-1}^C J_{k-1}}{D_{k-1}}, \quad \delta_k^{(2)} = \frac{I_{k-1}^S J_{k-2}^S - I_{k-1}^S J_{k-2}^S}{D_{k-2}},
\end{equation}

where \(D_{k-j} = I_{k-j}^C J_{k-j}^C - I_{k-j}^C J_{k-j}^C, \ j = 1, 2, \) and

\begin{equation}
I_\nu^C = (A_{\nu+1/2}^C, A_{\nu+1/2}^C), \quad J_\nu^C = (2 \cos x A_{\nu+1/2}^C, A_{\nu+1/2}^C),
\end{equation}

\begin{equation}
I_\nu^S = (A_{\nu+1/2}^S, A_{\nu+1/2}^S), \quad J_\nu^S = (2 \cos x A_{\nu+1/2}^S, A_{\nu+1/2}^S),
\end{equation}

\begin{equation}
I_\nu = (A_{\nu+1/2}, A_{\nu+1/2}), \quad J_\nu = (2 \cos x A_{\nu+1/2}, A_{\nu+1/2}).
\end{equation}
Christoffel-Darboux Formula for Orthogonal Trigonometric Polynomials . . .

For some special weight functions explicit formulas for five-term recurrence coefficients as well as explicit formulas for coefficients of expanded forms (1.2) were presented in [2].

In this paper, in Section 2, the orthonormal trigonometric polynomials of semi-integer degree are introduced and the Christoffel-Darboux formula for such orthonormal trigonometric system is proved.

2. Main Results

Let us denote
\[ m_n = \begin{bmatrix} I_n^C & I_n^S \\ I_n^S & I_n^S \end{bmatrix}. \]

Lemma 2.1. The matrix \( m_n \), \( n \in \mathbb{N}_0 \), given by (2.1) is positive definite.

Proof. Let \( a_1, a_2 \) be arbitrary real numbers such that at least one of them differs from zero. Let denote \( a = [a_1 \ a_2]^T \) and \( t(x) = a_1 A_{n+1/2}^C(x) + a_2 A_{n+1/2}^S(x) \). Then
\[ a^T m_n a = \int_{-\pi}^{\pi} t^2(x) w(x) \, dx > 0, \]
since \( t(x) \) is nonzero trigonometric polynomial of semi-integer degree \( n+1/2 \). Therefore, the matrix \( m_n \) is positive definite for all \( n \in \mathbb{N}_0 \). \( \Box \)

By \( \tilde{m}_n \), \( n \in \mathbb{N}_0 \), we denote the positive definite square root of \( m_n \), i.e., the unique positive definite matrix such that \( m_n = \tilde{m}_n \tilde{m}_n \) (see [4]). Since \( m_n \) is a symmetric matrix, the matrix \( \tilde{m}_n \) is also symmetric, i.e., it has the following form
\[ \tilde{m}_n = \begin{bmatrix} a_n & b_n \\ b_n & c_n \end{bmatrix}, \]
where
\[ a_n^2 + b_n^2 = I_n^C, \quad a_n b_n + b_n c_n = I_n \quad \text{and} \quad b_n^2 + c_n^2 = I_n^S. \]

Let introduce the following trigonometric polynomials of semi-integer degree \( n + 1/2 \):
\[ \tilde{A}_{n+1/2}^C(x) = \frac{c_n}{a_n c_n - b_n^2} A_{n+1/2}^C(x) - \frac{b_n}{a_n c_n - b_n^2} A_{n+1/2}^S(x), \]
\[ \tilde{A}_{n+1/2}^S(x) = \frac{a_n}{a_n c_n - b_n^2} A_{n+1/2}^S(x) - \frac{b_n}{a_n c_n - b_n^2} A_{n+1/2}^C(x). \]

We call the trigonometric polynomials \( \tilde{A}_{n+1/2}^C \) and \( \tilde{A}_{n+1/2}^C \), \( n \in \mathbb{N}_0 \), orthonormal trigonometric polynomials of semi-integer degree. The reason for that name lies in the following simple property.
Theorem 2.1. If $\tilde{A}_{n+1/2}^C(x)$ and $\tilde{A}_{n+1/2}^C(x)$, $n \in \mathbb{N}_0$, are given by (2.3), then the following equalities

$$(\tilde{A}_{n+1/2}^C(x), \tilde{A}_{n+1/2}^C(x)) = 1, \quad (\tilde{A}_{n+1/2}^S(x), \tilde{A}_{n+1/2}^S(x)) = 0 \quad \text{and} \quad (\tilde{A}_{n+1/2}^S(x), \tilde{A}_{n+1/2}^S(x)) = 1$$

hold.

Proof. By direct calculation we have

$$(\tilde{A}_{n+1/2}^C(x), \tilde{A}_{n+1/2}^C(x)) = \left( \frac{c_n A_{n+1/2}^C - b_n A_{n+1/2}^S}{a_n c_n - b_n^2}, \frac{c_n A_{n+1/2}^C - b_n A_{n+1/2}^S}{a_n c_n - b_n^2} \right)$$

so we get the first equality. The second and the third equalities can be proved analogously. \(\square\)

It is easy to see that the following equalities

$$(2.4) \quad A_{n+1/2}^C(x) = a_n \tilde{A}_{n+1/2}^C(x) + b_n \tilde{A}_{n+1/2}^S(x),$$

$$A_{n+1/2}^S(x) = b_n \tilde{A}_{n+1/2}^C(x) + c_n \tilde{A}_{n+1/2}^S(x)$$

hold.

Theorem 2.2. The orthonormal trigonometric polynomials of semi-integer degree $\tilde{A}_{n+1/2}^C(x)$ and $\tilde{A}_{n+1/2}^S(x)$, $n \in \mathbb{N}$, satisfy the following recurrence relations:

$$(2.5) \quad 2 \cos x \tilde{A}_{n-1/2}^C(x) = \frac{a_n c_{n-1} - b_n b_{n-1}}{a_n c_{n-1} - b_n^2} \tilde{A}_{n+1/2}^C(x) + \frac{b_n c_{n-1} - b_{n-1} c_n}{a_n c_{n-1} - b_n^2} \tilde{A}_{n+1/2}^S(x) + \frac{c_n^2 J_{n-1}^C + b_n^2 J_{n-1}^S}{a_n c_{n-1} - b_n^2} \tilde{A}_{n-3/2}^C(x) + \frac{b_n^2 - b_{n-1}^2}{a_n c_{n-1} - b_n^2} \tilde{A}_{n-3/2}^S(x)$$

and

$$(2.6) \quad 2 \cos x \tilde{A}_{n-1/2}^S(x) = \frac{a_{n-1} b_n - a_n b_{n-1}}{a_{n-1} c_{n-1} - b_{n-1}^2} \tilde{A}_{n+1/2}^C(x) + \frac{a_n c_{n-1} - b_{n-1} c_n}{a_{n-1} c_{n-1} - b_{n-1}^2} \tilde{A}_{n+1/2}^S(x) + \frac{b_{n-1}^2 + a_n c_{n-1}}{a_{n-1} c_{n-1} - b_{n-1}^2} \tilde{A}_{n-3/2}^C(x) + \frac{a_{n-1} b_n - a_n b_{n-1}}{a_{n-1} c_{n-1} - b_{n-1}^2} \tilde{A}_{n-3/2}^S(x)$$
\[
\frac{b_n c_n - b_{n-1} c_{n-1}}{a_n c_n - b_n^2} + \frac{b_n c_n - b_{n-2} c_{n-2}}{a_n - 2 c_n - b_{n-2}^2} \tilde{A}_{n-3/2}(x) + \frac{a_n c_n - b_n^2}{a_n - 2 c_n - b_{n-2}^2} \tilde{A}_{n-1/2}(x).
\]

Specially for \(n = 1\) coefficients multiplying \(\tilde{A}_{1/2}(x)\) and \(\tilde{A}_{3/2}(x)\) in both recurrence relation are equal to zero.

**Proof.** Using connections (2.4) from (1.3) and (1.4) as well as (2.2), (1.5) and (1.6), solving obtained linear system for \(\cos x \tilde{A}_{n-1/2}^C\) and \(\cos x \tilde{A}_{n-1/2}^S\) we get what is stated. The statement for \(n = 1\) follows directly from the fact that \(\alpha_1^{(2)} = \beta_1^{(2)} = \gamma_1^{(2)} = 0\) in (1.3) and (1.4). \(\square\)

**Theorem 2.3.** (Christoffel-Darboux formula) For the orthonormal trigonometric polynomials of semi-integer degree the following formula

\[
(2.7) \quad 2(\cos x - \cos y) \sum_{k=0}^{n} \left( \tilde{A}_{k+1/2}^C(x) \tilde{A}_{k+1/2}^C(y) + \tilde{A}_{k+1/2}^S(x) \tilde{A}_{k+1/2}^S(y) \right)
\]

\[
= \frac{(a_n c_n - b_n^2) (a_n c_n - b_{n+1}^2) (a_n c_n - b_{n+2}^2) (a_n c_n - b_{n+3}^2)}{(a_n c_n - b_n^2) (a_n c_n - b_{n+1}^2) (a_n c_n - b_{n+2}^2) (a_n c_n - b_{n+3}^2)} \left( \tilde{A}_{n+3/2}^C(x) \tilde{A}_{n+3/2}^C(y) - \tilde{A}_{n+3/2}^S(x) \tilde{A}_{n+3/2}^S(y) \right)
\]

holds.

**Proof.** Let us introduce some notation

\[
e_n = c_n^2 J_n^C + b_n^2 J_n^S - 2 b_n c_n J_n, \quad f_n = (b_n^2 + a_n c_n) J_n - b_n c_n J_n^C - a_n b_n J_n^S,
\]

\[
g_n = b_n^2 J_n^C + a_n c_n J_n^S - 2 a_n b_n J_n, \quad h_n = a_n c_n - b_n^2,
\]

\[
\Delta_n^0 = a_n b_n + a_{n+1} b_{n-1}, \quad \Delta_n^0 = c_n b_n + c_{n-1} b_{n+1},
\]

\[
\Delta_n^1 = a_n c_n - b_{n+1} b_n, \quad \Delta_n^1 = a_n c_n + b_{n+1} b_n.
\]

We prove theorem using mathematical induction. For \(n = 0\), using Theorem 2.2, we have

\[
2(\cos x - \cos y) \left( \tilde{A}_{1/2}^C(x) \tilde{A}_{1/2}^C(y) + \tilde{A}_{1/2}^S(x) \tilde{A}_{1/2}^S(y) \right)
\]

\[
= \left[ \Delta_0^0 \tilde{A}_{1/2}^C + \Delta_0^0 \tilde{A}_{1/2}^S + \sum_{k=1}^{n} \frac{\Delta_k^0 \tilde{A}_{k+1/2}^C + \Delta_k^0 \tilde{A}_{k+1/2}^S}{h_k} + \sum_{k=1}^{n} \frac{\Delta_k^0 \tilde{A}_{k+1/2}^C + \Delta_k^0 \tilde{A}_{k+1/2}^S}{h_k} \right] (x) \tilde{A}_{1/2}^C(y)
\]
Suppose it is true for some \( n = m \), then we have

\[
2(\cos x - \cos y) \left( \tilde{A}_{m+3/2}^C(x)\tilde{A}_{m+3/2}^C(y) + \tilde{A}_{m+3/2}^S(x)\tilde{A}_{m+3/2}^S(y) \right)
\]

\[
= \frac{\Delta_m^{1+1} \tilde{A}_{m+5/2}^C + \frac{\Delta_m^{1+1} \tilde{A}_{m+5/2}^S}{h_m} + \frac{\Delta_m^{1+1} \tilde{A}_{m+5/2}^C}{f_m + 1} + \frac{\Delta_m^{1+1} \tilde{A}_{m+5/2}^S}{g_m + 1}}{h_m} \left( x \tilde{A}_{m+3/2}^c(y) + \frac{\Delta_m^{1+1} \tilde{A}_{m+5/2}^C}{h_m} + \frac{\Delta_m^{1+1} \tilde{A}_{m+5/2}^S}{f_m + 1} + \frac{\Delta_m^{1+1} \tilde{A}_{m+5/2}^C}{g_m + 1} \right) (y)
\]

\[
+ \frac{\Delta_m^{1+1} \tilde{A}_{m+3/2}^C}{f_m + 1} + \frac{\Delta_m^{1+1} \tilde{A}_{m+3/2}^S}{g_m + 1} \left( x \tilde{A}_{m+3/2}^S(y) + \frac{\Delta_m^{1+1} \tilde{A}_{m+3/2}^C}{h_m} + \frac{\Delta_m^{1+1} \tilde{A}_{m+3/2}^S}{f_m + 1} + \frac{\Delta_m^{1+1} \tilde{A}_{m+3/2}^C}{g_m + 1} \right) (y)
\]

\[
- \frac{\Delta_m^{1+1} \tilde{A}_{m+3/2}^C}{h_m^2} + \frac{\Delta_m^{1+1} \tilde{A}_{m+3/2}^S}{h_m^2} + \frac{\Delta_m^{1+1} \tilde{A}_{m+3/2}^C}{(f_m + 1)^2} + \frac{\Delta_m^{1+1} \tilde{A}_{m+3/2}^S}{(g_m + 1)^2} \left( x \tilde{A}_{m+3/2}^C(y) + \frac{\Delta_m^{1+1} \tilde{A}_{m+5/2}^C}{h_m} + \frac{\Delta_m^{1+1} \tilde{A}_{m+5/2}^S}{f_m + 1} + \frac{\Delta_m^{1+1} \tilde{A}_{m+5/2}^C}{g_m + 1} \right) (y)
\]

\[
+ \frac{\Delta_m^{1+1} \tilde{A}_{m+3/2}^C}{f_m + 1} + \frac{\Delta_m^{1+1} \tilde{A}_{m+3/2}^S}{g_m + 1} \left( x \tilde{A}_{m+3/2}^S(y) + \frac{\Delta_m^{1+1} \tilde{A}_{m+3/2}^C}{h_m} + \frac{\Delta_m^{1+1} \tilde{A}_{m+3/2}^S}{f_m + 1} + \frac{\Delta_m^{1+1} \tilde{A}_{m+3/2}^C}{g_m + 1} \right) (y)
\]

\[
= \frac{\Delta_m^{1+1} \tilde{A}_{m+3/2}^C(x)\tilde{A}_{m+3/2}^C(y) - \tilde{A}_{m+3/2}^C(x)\tilde{A}_{m+3/2}^C(y)}{h_m}
\]

\[
+ \frac{\Delta_m^{1+1} \tilde{A}_{m+3/2}^S(x)\tilde{A}_{m+3/2}^S(y) - \tilde{A}_{m+3/2}^S(x)\tilde{A}_{m+3/2}^S(y)}{h_m^2}
\]
Now statement readily follows, since we have

\[
2(\cos x - \cos y) \sum_{k=0}^{m+1} \left( \tilde{A}^C_{m+3/2}(x) \tilde{A}^C_{m+1/2}(y) + \tilde{A}^S_{m+3/2}(x) \tilde{A}^S_{m+1/2}(y) \right)
= 2(\cos x - \cos y) \sum_{k=0}^{m} \left( \tilde{A}^C_{m+3/2}(x) \tilde{A}^C_{m+1/2}(y) + \tilde{A}^S_{m+3/2}(x) \tilde{A}^S_{m+1/2}(y) \right) + 2(\cos x - \cos y) \left( \tilde{A}^C_{m+3/2}(x) \tilde{A}^C_{m+1/2}(y) + \tilde{A}^S_{m+3/2}(x) \tilde{A}^S_{m+1/2}(y) \right)
\]
\[
= \Delta^1_m \left( \tilde{A}^C_{m+3/2}(x) \tilde{A}^C_{m+1/2}(y) + \tilde{A}^S_{m+3/2}(x) \tilde{A}^S_{m+1/2}(y) \right) - \Delta^1_m \tilde{A}^C_{m+3/2}(y) \tilde{A}^C_{m+1/2}(x) - \Delta^1_m \tilde{A}^S_{m+3/2}(y) \tilde{A}^S_{m+1/2}(x)
\]
\[
+ \Delta^2_m \tilde{A}^S_{m+3/2}(x) \tilde{A}^S_{m+1/2}(y) + \Delta^2_m \tilde{A}^C_{m+3/2}(x) \tilde{A}^C_{m+1/2}(y)
\]
\[
+ \Delta^3_m \tilde{A}^C_{m+3/2}(x) \tilde{A}^C_{m+1/2}(y) + \Delta^3_m \tilde{A}^S_{m+3/2}(x) \tilde{A}^S_{m+1/2}(y)
\]
\[
\begin{align*}
\Delta_m^1 & \left( \tilde{A}_{m+3/2}^C(x)\tilde{A}_{m+1/2}^C(y) - \tilde{A}_{m+3/2}^C(y)\tilde{A}_{m+1/2}^C(x) \right) h_m \\
\Delta_m^c & \left( \tilde{A}_{m+3/2}^S(x)\tilde{A}_{m+1/2}^C(y) - \tilde{A}_{m+3/2}^S(y)\tilde{A}_{m+1/2}^C(x) \right) h_m \\
\Delta_m^a & \left( \tilde{A}_{m+3/2}^C(x)\tilde{A}_{m+1/2}^S(y) - \tilde{A}_{m+3/2}^C(y)\tilde{A}_{m+1/2}^S(x) \right) h_m \\
\Delta_m^2 & \left( \tilde{A}_{m+3/2}^S(x)\tilde{A}_{m+1/2}^S(y) - \tilde{A}_{m+3/2}^S(y)\tilde{A}_{m+1/2}^S(x) \right) h_m \\
= & \frac{\Delta_{m+1}^1 (\tilde{A}_{m+5/2}^C(x)\tilde{A}_{m+3/2}^C(y) - \tilde{A}_{m+3/2}^C(y)\tilde{A}_{m+5/2}^C(x)) h_{m+1}}{h_{m+1}} \\
+ & \frac{\Delta_{m+1}^c (\tilde{A}_{m+5/2}^S(x)\tilde{A}_{m+3/2}^C(y) - \tilde{A}_{m+3/2}^S(y)\tilde{A}_{m+5/2}^C(x)) h_{m+1}}{h_{m+1}} \\
+ & \frac{\Delta_{m+1}^a (\tilde{A}_{m+5/2}^C(x)\tilde{A}_{m+3/2}^S(y) - \tilde{A}_{m+3/2}^C(y)\tilde{A}_{m+5/2}^S(x)) h_{m+1}}{h_{m+1}} \\
+ & \frac{\Delta_{m+1}^2 (\tilde{A}_{m+5/2}^S(x)\tilde{A}_{m+3/2}^S(y) - \tilde{A}_{m+3/2}^S(y)\tilde{A}_{m+5/2}^S(x)) h_{m+1}}{h_{m+1}}.
\end{align*}
\]

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