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MOMENT-PRESERVING SPLINE APPROXIMATION AND QUADRATURES*

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Abstract. In this survey we discuss the problem of approximating a function f by a spline function of degree m and defect d, with n (variable) knots, matching as many of the initial moments of f as possible. The problem is connected with Gauss-Turán type of quadrature rules.

1. Introduction

Following earlier work of Laframboise and Stauffer [12] and Calder and Laframboise [1], Gautschi [7] considered the problem of approximating a spherically symmetric function $t \mapsto f(t),\ t = \|\boldsymbol{x}\|,\ 0 \le t < \infty$, in $\mathbb{R}^d$, $d \ge 1$, by a piecewise constant function

$$t \mapsto s_n(t) = \sum_{\nu=1}^{n} a_\nu H(\tau_\nu - t) \qquad (a_\nu \in \mathbb{R},\ 0 < \tau_1 < \cdots < \tau_n < +\infty),$$

where H is the Heaviside step function. Also, he considered an approximation by a linear combination of Dirac delta functions. The approximation was to preserve as many moments of f as possible. This work was extended to spline approximation of arbitrary degree by Gautschi and Milovanović [9]. Namely, they considered a spline function of degree $m \ge 0$ on $[0, +\infty)$, vanishing at $t = +\infty$, with $n \ge 1$ positive knots τ_ν $(\nu = 1, \ldots, n)$, which can be written in the form

$$s_{n,m}(t) = \sum_{\nu=1}^{n} a_\nu (\tau_\nu - t)_+^m \qquad (a_\nu \in \mathbb{R},\ 0 \le t < +\infty), \tag{1.1}$$

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where the plus sign on the right is the cutoff symbol, $u_+ = u$ if $u > 0$ and $u_+ = 0$ if $u \le 0$. Given a function $t \mapsto f(t)$ on $[0, +\infty)$, they determined $s_{n,m}$ such that

$$\int_0^{+\infty} s_{n,m}(t) t^j \, dV = \int_0^{+\infty} f(t) t^j \, dV \qquad (j = 0, 1, \ldots, 2n-1), \tag{1.2}$$

where dV is the volume element depending on the geometry of the problem. (For example, $dV = Ct^{d-1}\, dt$ if $d > 1$, where C is some constant, and $dV = dt$ if $d = 1$ were used in [9]. For some details see Gautschi [8].) In any case, the spline $s_{n,m}$ is such to faithfully reproduce the first $2n$ moments of f. Under suitable assumptions on f, it was shown that the problem has a unique solution if and only if certain Gauss-Christoffel quadratures exist corresponding to a moment functional or weight distribution depending on f. Existence, uniqueness and pointwise convergence of such approximation were analyzed. We mention two main results (Gautschi and Milovanović [9]) in the case when $dV = dt$.

Theorem 1.1. *Let $f \in C^{m+1}[0, +\infty]$ and*

$$\int_0^{+\infty} t^{2n+m+1} |f^{(m+1)}(t)| \, dt < +\infty \, . \tag{1.3}$$

Then a spline function $s_{n,m}$ of the form (1.1) *with positive knots τ_ν, that satisfies* (1.2), *exists and is unique if and only if the measure*

$$d\lambda(t) = \frac{(-1)^{m+1}}{m!} t^{m+1} f^{(m+1)}(t) \, dt \quad \text{on} \quad [0, +\infty) \tag{1.4}$$

admits an n-point Gauss-Christoffel quadrature formula

$$\int_0^{+\infty} g(t) \, d\lambda(t) = \sum_{\nu=1}^{n} \lambda_\nu^{(n)} g(\tau_\nu^{(n)}) + R_n(g; d\lambda), \tag{1.5}$$

with distinct positive nodes $\tau_\nu^{(n)}$, where $R_n(g; d\lambda) = 0$ for all $g \in \mathcal{P}_{2n-1}$. In that event, the knots τ_ν and weights a_ν in (1.1) *are given by*

$$\tau_\nu = \tau_\nu^{(n)}, \quad a_\nu = \tau_\nu^{-(m+1)} \lambda_\nu^{(n)} \qquad (\nu = 1, \ldots, n). \tag{1.6}$$

Theorem 1.2. *Given f as in Theorem 1.2, assume that the measure $d\lambda$ in (1.4) admits the n-point Gauss-Christoffel quadrature formula (1.5) with distinct positive nodes $\tau_\nu = \tau_\nu^{(n)}$ and the remainder term $R_n(g; d\lambda)$. Define*

$$\sigma_t(x) = x^{-(m+1)}(x-t)_+^m \,.$$

Then, for any $t > 0$, we have for the error of the spline approximation (1.1), (1.2),

$$f(t) - s_{n,m}(t) = R_n(\sigma_t; d\lambda). \tag{1.7}$$

Substituting (1.1) in (1.2) yields, since $\tau_\nu > 0$,

$$\sum_{\nu=1}^{n} a_\nu \int_0^{\tau_\nu} t^j (\tau_\nu - t)^m \, dt = \int_0^{+\infty} t^j f(t)\, dt \quad (j = 0, 1, \ldots, 2n-1),$$

i.e.,

$$\sum_{\nu=1}^{n} \big(a_\nu \tau_\nu^{m+1}\big) \tau_\nu^j = \mu_j \qquad (j = 0, 1, \ldots, 2n-1), \tag{1.8}$$

where

$$\mu_j = \frac{(j+m+1)!}{m!j!} \int_0^{+\infty} t^j f(t)\, dt \quad (j = 0, 1, \ldots). \tag{1.9}$$

For the proof of Theorem 1.1 we suppose that $j \le 2n-1$. Because of (1.3), the integral $\int_0^{+\infty} t^{j+m+2} f^{(m+1)}(t)\, dt$ exists and $\lim\limits_{t\to+\infty} t^{j+m+2} f^{(m+1)}(t) = 0$. Then, L'Hospital's rule implies

$$\lim_{t\to+\infty} t^{j+m+1} f^{(m)}(t) = 0.$$

Continuing in this manner, we find that

$$\lim_{t\to+\infty} t^{j+k+1} f^{(k)}(t) = 0 \qquad (k = m, m-1, \ldots, 0).$$

Under these conditions we can prove that (see [9])

$$\int_0^{+\infty} t^j f(t)\, dt = \frac{(-1)^{m+1}}{(j+1)(j+2)\cdots(j+m+1)} \int_0^{+\infty} t^{j+m+1} f^{(m+1)}(t)\, dt.$$

Therefore, the moments μ_j, defined by (1.9), exist and

$$\mu_j = \int_0^{+\infty} t^j \, d\lambda(t) \qquad (j = 0, 1, \ldots, 2n-1),$$

where $d\lambda(t)$ is given by (1.4). Hence, we conclude that Eqs. (1.2) are equivalent to Eqs. (1.8), which are precisely the conditions for τ_ν to be the nodes of the Gauss-Christoffel formula (1.5) and $a_\nu \tau_\nu^{m+1}$ their weights.

The nodes $\tau_\nu^{(n)}$, being the zeros of the orthogonal polynomial $\pi_n(\cdot; d\lambda)$ (if it exists), are uniquely determined, hence also the weights $\lambda_\nu^{(n)}$.

For example, if f is completely monotonic on $[0, +\infty)$ then $d\lambda(t)$ in (1.4) is a positive measure for every m. Also, the first $2n$ moments exist by virtue of the assumptions in Theorem 1.1. Then the Gauss-Christoffel quadrature formula exists uniquely, with n distinct and positive nodes $\tau_\nu^{(n)}$.

Using Taylor's formula "at $+\infty$", we find that

$$f(t) = \frac{(-1)^{m+1}}{m!} \int_t^{+\infty} (x-t)^m f^{(m+1)}(x) \, dx = \int_0^{+\infty} \sigma_t(x) \, d\lambda(x). \tag{1.10}$$

On the other hand, Theorem 1.1 gives

$$s_{n,m}(t) = \sum_{\nu=1}^{n} \lambda_\nu \tau_\nu^{-(m+1)} (\tau_\nu - t)_+^m = \sum_{\nu=1}^{n} \lambda_\nu \sigma_t(t_\nu). \tag{1.11}$$

Subtracting (1.11) from (1.10) yields (1.7).

Theorem 1.2 shows that $s_{n,m}$ converges pointwise to f as $n \to +\infty$ if the Gauss-Christoffel quadrature formula (1.5) converges for the particular function $x \mapsto g(x) = \sigma_t(x)$ $(x > 0)$.

2. Approximation by Defective Splines

A spline function of degree $m \geq 2$ and defect k on the interval $0 \leq t < +\infty$, vanishing at $t = +\infty$, with $n \geq 1$ positive knots τ_ν $(\nu = 1, \ldots, n)$, can be written in the form

$$s_{n,m}(t) = \sum_{\nu=1}^{n} \sum_{i=m-k+1}^{m} a_{i,\nu} (\tau_\nu - t)_+^i \tag{2.1}$$

where $a_{i,\nu}$ are real numbers.

As in Section 1 we consider moment-preserving approximation of a given function $t \mapsto f(t)$ on $[0, +\infty)$ by the defective spline $s_{n,m}$, defined by (2.1). Under suitable assumptions on f and $k = 2s+1$, Milovanović and Kovačević [15] showed that the problem has a unique solution if and only if certain generalized Turán quadratures exist corresponding to a measure depending on f.

The generalized Gauss-Turán quadrature

$$\int_{\mathbb{R}} g(t)\, d\lambda(t) = \sum_{\nu=1}^{n} \sum_{i=0}^{2s} A_{i,\nu}^{G} g^{(i)}(\tau_\nu^{(n)}) + R_n^G(g; d\lambda), \tag{2.2}$$

where $d\lambda(t)$ is a nonnegative measure on the real line $\mathbb{R}$, with compact or infinite support, for which all moments $\mu_\nu = \int_{\mathbb{R}} t^\nu \, d\lambda(t)$, $\nu = 0, 1, \ldots$, exist and are finite, and $\mu_0 > 0$. The formula (2.2) is exact for all polynomials of degree at most $2(s+1)n - 1$, i.e.,

$$R_n^G(g; d\lambda) = 0 \qquad \text{for} \quad g \in \mathcal{P}_{2(s+1)n-1}.$$

The knots $\tau_\nu^{(n)}$ $(\nu = 1, \ldots, n)$ in (2.2) are zeros of a (monic) polynomial $\pi_n(t)$, which minimizes the following integral

$$\int_{\mathbb{R}} \pi_n(t)^{2s+2}\, d\lambda(t),$$

where $\pi_n(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0$. In the other words, the polynomial π_n satisfies the following generalized orthogonality conditions

$$\int_{\mathbb{R}} \pi_n(t)^{2s+1} t^i \, d\lambda(t) = 0, \qquad i = 0, 1, \ldots, n-1. \tag{2.3}$$

This polynomial π_n is known as s-orthogonal (or s-self associated) polynomial with respect to the measure $d\lambda(t)$. For $s = 0$, we have the standard case of orthogonal polynomials, and (2.3) then becomes well-known Gauss-Christoffel formula.

The "orthogonality condition" (2.3) can be interpreted as (see Milovanović [14])

$$\int_{\mathbb{R}} \pi_\nu^{s,n}(t) t^i \, d\mu(t) = 0, \qquad i = 0, 1, \ldots, \nu - 1,$$

where $\{\pi_\nu^{s,n}\}$ is a sequence of standard monic polynomials orthogonal on $\mathbb{R}$ with respect to the new measure $d\mu(t) = d\mu^{s,n}(t) = (\pi_n^{s,n}(t))^{2s} d\lambda(t)$. The polynomials $\{\pi_\nu^{s,n}\}$, $\nu = 0, 1, \dots$, are implicitly defined because the measure $d\mu(t)$ depends on $\pi_n^{s,n}(t)$ $\big(= \pi_n(t)\big)$. We will write only π_ν instead of $\pi_\nu^{s,n}(\cdot)$. These polynomials satisfy a three-term recurrence relation

$$\begin{aligned} &\pi_{\nu+1}(t) = (t - \alpha_\nu)\pi_\nu(t) - \beta_\nu \pi_{\nu-1}(t), \quad \nu = 0, 1, \dots, \\ &\pi_{-1}(t) = 0, \quad \pi_0(t) = 1, \end{aligned}$$

where, because of orthogonality,

$$\begin{aligned} \alpha_\nu = \alpha_\nu(s,n) &= \frac{(t\pi_\nu, \pi_\nu)}{(\pi_\nu, \pi_\nu)} = \frac{\int_{\mathbb{R}} t\pi_\nu^2(t)\, d\mu(t)}{\int_{\mathbb{R}} \pi_\nu^2(t)\, d\mu(t)}, \\ \beta_\nu = \beta_\nu(s,n) &= \frac{(\pi_\nu, \pi_\nu)}{(\pi_{\nu-1}, \pi_{\nu-1})} = \frac{\int_{\mathbb{R}} t\pi_\nu^2(t)\, d\mu(t)}{\int_{\mathbb{R}} \pi_{\nu-1}^2(t)\, d\mu(t)}, \end{aligned}$$

and, for example, $\beta_0 = \int_{\mathbb{R}} d\mu(t)$.

Finding the coefficients α_ν, β_ν $(\nu = 0, 1, \dots, n-1)$ gives us access to the first $n+1$ orthogonal polynomials π_0, π_1, $\dots$, π_n. Of course, we are interested only in the last of them, i.e., $\pi_n \equiv \pi_n^{s,n}$. Thus, for $n = 0, 1, \dots$, the diagonal (boxed) elements in Table 2.1 are our s-orthogonal polynomials $\pi_n^{s,n}$.

Table 2.1

n	$d\mu^{s,n}(t)$	Orthogonal Polynomials			
0	$(\pi_0^{s,0}(t))^{2s}\, d\lambda(t)$	$\boxed{\pi_0^{s,0}}$			
1	$(\pi_1^{s,1}(t))^{2s}\, d\lambda(t)$	$\pi_0^{s,1}$	$\boxed{\pi_1^{s,1}}$		
2	$(\pi_2^{s,2}(t))^{2s}\, d\lambda(t)$	$\pi_0^{s,2}$	$\pi_1^{s,2}$	$\boxed{\pi_2^{s,2}}$	
3	$(\pi_3^{s,3}(t))^{2s}\, d\lambda(t)$	$\pi_0^{s,3}$	$\pi_1^{s,3}$	$\pi_2^{s,3}$	$\boxed{\pi_3^{s,3}}$
$\vdots$					

A stable algorithm for constructing such (s-orthogonal) polynomials was found by Milovanović [14].

Using the similar method as in Section 1, we can prove (see Milovanović and Kovačević [15]) the following result:

Theorem 2.1. *Let $f \in C^{m+1}[0,+\infty]$ and*

$$\int_0^{+\infty} t^{2(s+1)n+m+1} |f^{(m+1)}(t)|\, dt < +\infty\,.$$

Then a spline function $s_{n,m}$ of the form (2.1) *with $k = 2s+1$ and positive knots τ_ν, that satisfies* (1.2), *with $j = 0, 1, \ldots, 2(s+1)n-1$, exists and is unique if and only if the measure*

$$d\lambda(t) = \frac{(-1)^{m+1}}{m!} t^{m+1} f^{(m+1)}(t)\, dt \quad \text{on} \quad [0,+\infty)$$

admits a generalized Gauss-Turán quadrature formula

$$\int_0^{+\infty} g(t)\, d\lambda(t) = \sum_{\nu=1}^{n} \sum_{i=0}^{2s} A_{i,\nu}^{(n)} g^{(i)}(\tau_\nu^{(n)}) + R_n^G(g; d\lambda), \tag{2.4}$$

with distinct positive nodes $\tau_\nu^{(n)}$, where $R_n^G(g; d\lambda) = 0$ for all $g \in \mathcal{P}_{2(s+1)n-1}$. The knots τ_ν in (2.1) *are given by $\tau_\nu = \tau_\nu^{(n)}$, and coefficients $a_{i,\nu}$ by the following triangular system*

$$A_{i,\nu}^{(n)} = \sum_{j=i}^{2s} \frac{(m-j)!}{m!} \binom{j}{i} \left[D^{j-i} t^{m+1}\right]_{t=\tau_\nu} a_{m-j,\nu} \quad (i = 0, 1, \ldots, 2s),$$

where D is the standard differentiation operator.

Theorem 2.2. *Given f as in Theorem* 2.1, *assume that the measure $d\lambda(t)$ admits the n-point generalized Gauss-Turán quadrature formula* (2.4) *with distinct positive nodes $\tau_\nu = \tau_\nu^{(n)}$ and the remainder term $R_n^G(g; d\lambda)$. Then the error of the spline approximation is given by*

$$f(t) - s_{n,m}(t) = R_n^G(\sigma_t; d\lambda). \qquad (t > 0),$$

where $x \mapsto \sigma_t(x) = x^{-(m+1)}(x-t)_+^m$.

Again, if f is completely monotonic on $[0,+\infty)$ then $d\lambda(t)$ is a positive measure for every m. Also, the first $2(s+1)n$ moments exist by virtue of the assumptions in Theorem 2.1. Then the generalized Gauss-Turán quadrature formula exists uniquely, with n distinct and positive nodes $\tau_\nu^{(n)}$.

In the special case when $s = 1$, the coefficients of the spline function $s_{n,m}$ are

$$\begin{aligned}
a_{m-2,\nu} &= m(m-1) A_{2,\nu}^{(n)} \tau_\nu^{-(m+1)}, \\
a_{m-1,\nu} &= m\big(A_{1,\nu}^{(n)} \tau_\nu - 2(m+1) A_{2,\nu}^{(n)}\big) \tau_\nu^{-(m+2)}, \\
a_{m,\nu} &= \big((m+1)(m+2) A_{2,\nu}^{(n)} - (m+1) A_{1,\nu}^{(n)} \tau_\nu + A_{0,\nu}^{(n)} \tau_\nu^2\big) \tau_\nu^{-(m+3)}.
\end{aligned}$$

3. Spline Approximation on Finite Intervals

Frontini, Gautschi and Milovanović [2] and Frontini and Milovanović [3] considered analogous problems on an arbitrary finite interval, which can be standardized to $[a, b] = [0, 1]$. If the approximations exist, they can be represented in terms of generalized Gauss-Lobatto and Gauss-Radau quadrature formulas relative to appropriate measures depending on f, when the defect $k = 1$. Using defective splines with odd defect $k = 2s+1$, approximation problems reduce to certain generalized Turán-Lobatto and Turán-Radau quadrature formulas.

A spline function of degree $m \geq 2$ and defect $k = 2s + 1$, with n distinct knots τ_ν $(\nu = 1, \ldots, n)$ in $(0, 1)$, can be written in the form

$$s_{n,m}(t) = p_m(t) + \sum_{\nu=1}^{n} \sum_{i=m-2s}^{m} a_{i,\nu}(\tau_\nu - t)_+^i \qquad (0 \leq t \leq 1), \tag{3.1}$$

where $a_{i,\nu}$ are real numbers and $t \mapsto p_m(t)$ is a polynomial of degree $\leq m$. Evidently, for $t \geq 1$ we have $s_{n,m}(t) \equiv p_m(t)$.

There are two interesting approximation problems:

Problem I. Determine $s_{n,m}$ in (3.1) such that

$$\int_0^1 t^j s_{n,m}(t)\,dt = \int_0^1 t^j f(t)\,dt, \qquad j = 0, 1, \ldots, 2(s+1)n + m. \tag{3.2}$$

Problem I*. Determine $s_{n,m}$ in (3.1) such that

$$s_{n,m}^{(k)}(1) = p_m^{(k)}(1) = f^{(k)}(1), \qquad k = 0, 1, \ldots, m, \tag{3.3}$$

and such that (3.2) holds for $j = 0, 1, ..., 2(s + 1)n - 1$.

The both problems can be reduced to the s-orthogonality and generalized Gauss-Turán quadratures by restricting the class of functions f as before.

Putting

$$\phi_k = \frac{(-1)^k}{m!} f^{(k)}(1), \quad b_k = \frac{(-1)^k}{m!} p_m^{(k)}(1) \quad (k = 0, 1, ..., m) \tag{3.4}$$

and applying $m+1$ integration by parts to the integrals in the moment equation (3.2), we obtain after much calculations

$$\sum_{k=0}^{m} b_k \left[D^{m-k} t^{m+1+j}\right]_{t=1} + \sum_{\nu=1}^{n}\sum_{k=0}^{2s} \frac{(m-k)!}{m!} a_{m-k,\nu} \left[D^k (t^{m+1+j})\right]_{t=\tau_\nu}$$

$$= \sum_{k=0}^{m} \phi_k \left[D^{m-k} t^{m+1+j}\right]_{t=1} + \int_0^1 t^{m+1+j}\, d\lambda(t), \tag{3.5}$$

$$j = 0, 1, \ldots, 2(s+1)n + m,$$

where the measure $d\lambda(t)$ is defined again by

$$d\lambda(t) = \frac{(-1)^{m+1}}{m!} f^{(m+1)}(t)\, dt \quad \text{on} \quad [0,1]. \tag{3.6}$$

The main result for *Problem I* can be stated in the form:

Theorem 3.1. *Let $f \in C^{m+1}[0,1]$. There exists a unique spline function* (3.1) *on $[0,1]$, with $d = 2s+1$, satisfying* (3.2) *if and only if the measure $d\lambda(t)$ in* (3.6) *admits a generalized Gauss-Lobatto-Turán quadrature*

$$\int_0^1 g(t)\, d\lambda(t) = \sum_{k=0}^{m} \left[\alpha_k g^{(k)}(0) + \beta_k g^{(k)}(1)\right]$$
$$+ \sum_{\nu=1}^{n}\sum_{i=0}^{2s} A_{i,\nu}^{L} g^{(i)}(\tau_\nu^{(n)}) + R_{n,m}^{L}(g; d\lambda), \tag{3.7}$$

where

$$R_{n,m}^{L}(g; d\lambda) = 0 \quad \text{for all} \quad g \in \mathcal{P}_{2(s+1)n+2m+1}, \tag{3.8}$$

with distinct real zeros $\tau_\nu^{(n)}$ $(\nu = 1, 2, \ldots, n)$ all contained in the open interval $(0,1)$. The spline function in (3.1) *is given by*

$$\tau_\nu = \tau_\nu^{(n)}, \quad a_{m-k,\nu} = \frac{m!}{(m-k)!} A_{k,\nu}^{L} \quad (\nu = 1, \ldots, n;\ k = 0, 1, \ldots, 2s), \tag{3.9}$$

where $\tau_\nu^{(n)}$ are the interior nodes of the generalized Gauss-Lobatto-Turán quadrature formula and $A_{k,\nu}^{L}$ are the corresponding weights, while the polynomial $p_m(t)$ is given by

$$p_m^{(k)}(1) = f^{(k)}(1) + (-1)^k m! \beta_{m-k} \quad (k = 0, 1, \ldots, m), \tag{3.10}$$

where β_{m-k} is the coefficient of $g^{(m-k)}(1)$ in (3.7).

The solution of *Problem I** can be given in a similar way:

Theorem 3.2. *Let $f \in C^{m+1}[0,1]$. There exists a unique spline function on* $[0,1]$,

$$s^*_{n,m}(t) = p^*_m(t) + \sum_{\nu=1}^{n} \sum_{i=m-2s}^{m} a^*_{i,\nu} (\tau^*_\nu - t)^i_+, \tag{3.11}$$
$$0 < \tau^*_\nu < 1, \ \tau^*_\nu \neq \tau^*_\mu \ \ \textit{for } \nu \neq \mu,$$

satisfying (3.3) *and* (3.2), *for* $j = 0, 1, \ldots, 2(s+1)n - 1$, *if and only if the measure* $d\lambda(t)$ *in* (3.6) *admits a generalized Gauss-Radau-Turán quadrature*

$$\int_0^1 g(t)\, d\lambda(t) = \sum_{k=0}^{m} \alpha^*_k g^{(k)}(0) + \sum_{\nu=1}^{n} \sum_{i=0}^{2s} A^R_{i,\nu} g^{(i)}(\tau^{(n)*}_\nu) + R^R_{n,m}(g; d\lambda), \tag{3.12}$$

where

$$R^R_{n,m}(g; d\lambda) = 0 \quad \textit{for all} \quad g \in \mathcal{P}_{2(s+1)n+m},$$

with distinct real zeros $\tau^{(n)*}_\nu$, $\nu = 1, 2, \ldots, n$, *all contained in the open interval* $(0,1)$. *The knots* τ^*_ν *in* (3.11) *are then precisely these zeros,*

$$\tau^*_\nu = \tau^{(n)*}_\nu \qquad (\nu = 1, \ldots, n) \tag{3.13}$$

and

$$a^*_{m-k,\nu} = \frac{m!}{(m-k)!} A^R_{k,\nu} \quad (\nu = 1, 2, \ldots, n;\ k = 0, 1, \ldots, 2s), \tag{3.14}$$

while the polynomial $t \mapsto p^*_m(t)$ *is given by*

$$p^*_m(t) = \sum_{k=0}^{m} \frac{f^{(k)}(1)}{k!} (t-1)^k. \tag{3.15}$$

The following statement gives the error of spline approximations:

Theorem 3.3. *Define*

$$\rho_t(x) = (x-t)^m_+, \qquad 0 \leq x \leq 1.$$

Under conditions of Theorem 3.1 *and Theorem* 3.2, *we have*

$$f(t) - s_{n,m}(t) = R^L_{n,m}(\rho_t;\, d\lambda) \qquad (0 < t < 1)$$

and

$$f(t) - s^*_{n,m}(t) = R^R_{n,m}(\rho_t;\, d\lambda) \qquad (0 < t < 1),$$

respectively, where $R^L_{n,m}(g;\, d\lambda)$ *and* $R^R_{n,m}(g;\, d\lambda)$ *are the remainder terms in the corresponding Gauss-Turán formulas of Lobatto and Radau type.*

For proofs of Theorems 3.1–3.3, we refer to [3]. The case $s = 0$ of these results has been obtained by Frontini, Gautschi and Milovanović [2]. A more general case with variable defects was considered by Gori and Santi [10]. In that case, approximation problems reduce to Gauss-Turán-Stancu type of quadratures and σ-orthogonal polynomials (cf. Gautschi [4], Gori, Lo Cascio and Milovanović [11]).

Further extensions of the moment-preserving spline approximation on $[0, 1]$ are given by Micchelli [13]. He relates this approximation to the theory of the monosplines.

4. Construction of Spline Approximation on $[0, 1]$

Firstly, we mention two auxiliary results, which give a connection between the generalized Gauss-Turán quadrature and the corresponding formulas of Lobatto and Radau type (see [3]):

Lemma 4.1. *If the measure* $d\lambda(t)$ *in* (3.6) *admits the generalized Gauss-Lobatto-Turán quadrature* (3.7)*, with distinct real zeros* $\tau_\nu = \tau_\nu^{(n)}$ $(\nu = 1, \ldots, n)$ *all contained in the open interval* $(0, 1)$*, there exists then a generalized Gauss-Turán formula*

$$\int_0^1 g(t)\, d\sigma(t) = \sum_{\nu=1}^{n} \sum_{i=0}^{2s} A^G_{i,\nu} g^{(i)}(\tau_\nu^{(n)}) + R^G_n(g), \tag{4.1}$$

where $d\sigma(t) = [t(1-t)]^{m+1} d\lambda(t)$*, the nodes* $\tau_\nu^{(n)}$ *are the zeros of* s*-orthogonal polynomial* $\pi_n(\cdot;\, d\sigma))$*, while the weights* $A^G_{i,\nu}$ *are expressible in terms of those in* (3.7) *by*

$$A^G_{i,\nu} = \sum_{k=i}^{2s} \binom{k}{i} \left[D^{k-i} \left(t(1-t)\right)^{m+1} \right]_{t=\tau_\nu} A^L_{k,\nu} \qquad (i = 0, 1, \ldots, 2s). \tag{4.2}$$

Lemma 4.2. *If the measure $d\lambda(t)$ in* (3.6) *admits the generalized Gauss-Radau-Turán quadrature* (3.12)*, with distinct real zeros $\tau_\nu = \tau_\nu^{(n)*}$ ($\nu = 1,\dots,n$) all contained in the open interval $(0,1)$, there exists then a generalized Gauss-Turán formula* (4.1)*, where $d\sigma(t) = d\sigma^*(t) = t^{m+1}d\lambda(t)$, the nodes $\tau_\nu^{(n)*}$ are the zeros of s-orthogonal polynomial $\pi_n(\cdot; d\sigma^*)$, while the weights $A_{i,\nu}^G$ are expressible in terms of those in* (3.12) *by*

$$A_{i,\nu}^G = \sum_{k=i}^{2s} \binom{k}{i} \left[D^{k-i}t^{m+1}\right]_{t=\tau_\nu} A_{k,\nu}^R \qquad (i = 0, 1, \dots, 2s). \tag{4.3}$$

A construction procedure of our spline approximations can be stated in the form (see [3]):

1° For a given $t \mapsto f(t)$ and (n, m, s), we find the measure $d\lambda(t)$ and the corresponding Jacobi matrix $J_N(d\lambda)$, where $N = (s+1)n + 2m + 2$ in the Lobatto case, and $N = (s+1)n + m + 1$ in the Radau case. The latter can be computed by the discretized Stieltjes procedure (see [5, § 2.2]).

2° By repeated application of the algorithms in [6, § 4.1] corresponding to multiplication of a measure by $t(1-t)$ and t, from the above Jacobi matrices, we generate the Jacobi matrices $J_{(s+1)n}(d\sigma)$ and $J_{(s+1)n}(d\sigma^*)$, respectively. Here, $d\sigma(t) = (t(1-t))^{m+1}d\lambda(t)$ and $d\sigma^*(t) = t^{m+1}d\lambda(t)$.

3° Using the algorithm for the construction of s-orthogonal polynomials, given in [14], we obtain the Jacobi matrix $J_n(d\mu)$, where $d\mu(t) = (\pi_n(t))^{2s}d\sigma(t)$, or $d\mu(t) = (\pi_n(t))^{2s}d\sigma^*(t)$.

4° From $J_n(d\mu)$ we determine the Gaussian nodes $\tau_\nu^{(n)}$ (resp. $\tau_\nu^{(n)*}$ in the Radau case) and the corresponding weights $A_{i,\nu}^G$ ($\nu = 1,\dots,n$; $i = 0, 1, \dots, 2s$).

5° From the triangular systems of linear equations (4.2) and (4.3), we find the coefficients $A_{k,\nu}^L$ and $A_{k,\nu}^R$, respectively.

6° Using (3.9) and (3.10), or (3.13), (3.14) and (3.15), we determine the spline approximation $s_{n,m}(t)$, or $s_{n,m}^*(t)$, respectively.

5. Numerical Example

In this section we consider a simple example – exponential distribution.

Let $f(t) = e^{-t}$ on $[0, +\infty)$. According to Theorem 2.1 we have here the generalized Laguerre measure

$$d\lambda(t) = \frac{1}{m!}\, t^{m+1} e^{-t}\, dt, \qquad 0 \le t < +\infty.$$

We analyzed the cases when $n = 2(1)5$, $m = 2(1)5$, and $s = 1$. All computations were done on the MICROVAX 3400 using VAX FORTRAN Ver. 5.3 in D-arithmetics (machine precision $\approx 2.76 \times 10^{-17}$).

The coefficients of the spline (2.1), i.e.,

$$s_{n,m}(t) = \sum_{\nu=1}^{n} \left[a_{m-2,\nu}(\tau_\nu - t)_+^{m-2} + a_{m-1,\nu}(\tau_\nu - t)_+^{m-1} + a_{m,\nu}(\tau_\nu - t)_+^{m}\right],$$

are given in Tables 5.1 and 5.2 (to 10 decimals only, to save space) for $n = 2$ and $n = 3$, respectively. Numbers in parenthesis idicate decimal exponents.

Table 5.1
The coefficients of spline function $s_{n,m}(t)$ for $n = 2$, $m = 5$, $s = 1$

ν	τ_ν	$a_{m-2,\nu}$	$a_{m-1,\nu}$	$a_{m,\nu}$
1	5.187737459(0)	5.298036250(−4)	−2.719217472(−3)	6.344798189(−3)
2	1.418519396(1)	2.992965707(−7)	−2.517818993(−6)	5.551582363(−6)

Table 5.2
The coefficients of spline function $s_{n,m}(t)$ for $n = 3$, $m = 5$, $s = 1$

ν	τ_ν	$a_{m-2,\nu}$	$a_{m-1,\nu}$	$a_{m,\nu}$
1	3.978424366(0)	1.048630112(−3)	−3.676925296(−3)	8.840144142(−3)
2	1.028050094(1)	5.332404617(−6)	−3.374135232(−5)	7.443616416(−5)
3	2.086562513(1)	5.370512250(−10)	−4.708093471(−9)	1.017579042(−8)

Table 5.3 shows the accuracy of the spline approximation $s_{n,m}$, i.e.,

$$e_{n,m} = \max_{0 \le t \le \tau_n} |s_{n,m}(t) - e^{-t}|,$$

for $n = 2(1)5$, $m = 2(1)5$, and $s = 1$. Clearly, for $t \geq \tau_n$, the absolute error is equal to $f(t) = e^{-t}$.

Table 5.3
Accuracy of the spline approximation $s_{n,m}(t)$

n	$m = 2$	$m = 3$	$m = 4$	$m = 5$
2	1.2(−1)	2.1(−2)	1.2(−2)	7.2(−3)
3	8.4(−2)	1.1(−2)	3.3(−3)	1.7(−3)
4	5.9(−2)	7.9(−3)	1.3(−3)	5.3(−4)
5	4.1(−2)	5.6(−3)	7.7(−4)	2.0(−4)

We can see that the approximation error is more easily reduced by increasing m rather than n.

A similar example of spline approximation on $[0, 1]$ was given in [3].

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SPLAJN APROKSIMACIJE KOJE OČUVAVAJU MOMENTE I QUADRATURNE FORMULE

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U radu se diskutuje problem aproksimacije funkcije f pomoću splajn funkcije stepena m i defekta d sa n (promenljivih) čvorova, očuvavajući pritom maksimalan broj početnih momenata. Problem se povezuje sa Gauss-Turánovim kvadraturnim formulama.