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ZERO DISTRIBUTION OF A CLASS OF POLYNOMIALS ASSOCIATED WITH THE GENERALIZED HERMITE POLYNOMIALS*

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Abstract. We prove that polynomials $P_N^{(m,q)}(t)$ associated with the generalized Hermite polynomials, where $m \in \mathbb{N}$ and $q \in \{0, 1, \ldots, m-1\}$, has only real postive zeros for every $N \in \mathbb{N}$.

The sequence of polynomials $\{h_{n,m}^{\lambda}(x)\}_{n=0}^{+\infty}$, where λ is a real parameter and m is an arbitrary positive integer, was studied in [4]. For $m = 2$, the polynomial $h_{n,m}^{\lambda}(x)$ reduces to $H_n(x, \lambda)/n!$, where $H_n(x, \lambda)$ is the Hermite polynomial with a parameter. For $\lambda = 1$, $h_{n,2}^{1}(x) = H_n(x)/n!$, where $H_n(x)$ is the classical Hermite polynomial. Taking $\lambda = 1$ and $n = mN + q$, where $N = [n/m]$ and $0 \le q \le m-1$, Đorđević [4] introduced the polynomials $P_N^{(m,q)}(t)$ by $h_{n,m}^{1}(x) = (2x)^q P_N^{(m,q)}((2x)^m)$, and proved that they satisfy an $(m+1)$-term linear recurrence relation of the form

$$\sum_{i=0}^{m} A_N(i,q) P_{N+1-i}^{(m,q)}(t) = B_N(q) t P_N^{(m,q)}(t), \tag{1}$$

where $B_N(q)$ and $A_N(i,q)$ $(i = 0, 1, \ldots, m)$ are constants depending only on N, m and q. Recently, one of us [7] determined the explicit expressions for the coefficients in (1) using some combinatorial identities.

An explicit representation of the polynomial $P_N^{(m,q)}(t)$ can be given in the form (see [4], [7]),

$$P_N^{(m,q)}(t) = \sum_{k=0}^{N} (-1)^{N-k} \frac{t^k}{(N-k)!(q+mk)!}\,, \tag{2}$$

where $m \in \mathbb{N}$ and $q \in \{0, 1, \ldots, m-1\}$.

In this note we prove a zero distribution of polynomials (2):

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Theorem 1. *The polynomial* $P_N^{(m,q)}(t)$ *defined by* (2)*, where* $m \in \mathbb{N}$ *and* $q \in \{0, 1, \dots, m-1\}$*, has only real and positive zeros for every* $N \in \mathbb{N}$*.*

In the proof of this theorem we use the following result (cf. Obreschkoff [11, p. 107]):

Theorem A. *Let* $a_0 + a_1 x + \cdots + a_n x^n$ *be a polynomial with only real zeros and let* $x \mapsto f(x)$ *be an entire function of the second kind without positive zeros. Then the polynomial*

$$a_0 f(0) + a_1 f(1) x + \cdots + a_n f(n) x^n$$

has only real zeros.

It is known that an entire function of the second kind (in the Laguerre-Pólya class) can be expressed in the form

$$f(x) = Ce^{-ax^2+bx} x^m \prod_{n=1}^{+\infty} \Big(1 - \frac{x}{\alpha_n}\Big) e^{x/\alpha_n}, \tag{3}$$

where $C, a, b \in \mathbb{R}$, $m \in \mathbb{N}_0$, $\alpha_n \in \mathbb{R}$ $(n = 1, 2, \dots)$ and $\sum_{n=1}^{+\infty} 1/\alpha_n^2 < +\infty$.

We first prove an auxiliary results regarding the ratio $\Gamma(x+1)/\Gamma(mx+q+1)$, where $\Gamma(x)$ is the gamma function.

Lemma 1. *Let* $m \in \mathbb{N}$ *and* $q \in \{0, 1, \dots, m-1\}$*. The equality*

$$\frac{\Gamma(x+1)}{\Gamma(mx+q+1)} = Ae^{\gamma(m-1)x} \prod_{n=1}^{+\infty} \Big(1 + \frac{(m-1)x+q}{n+x}\Big) e^{-((m-1)x+q)/n} \tag{3}$$

holds, where A *and* γ *are constants* ($\gamma = 0.57721566\ldots$ is known as Euler's constant).

Proof. In 1856 Weierstrass proved the formula

$$\frac{1}{\Gamma(z+1)} = e^{\gamma z} \prod_{n=1}^{+\infty} \Big[\Big(1 + \frac{z}{n}\Big) e^{-z/n}\Big].$$

According to this equality we have

$$
\begin{aligned}
\frac{\Gamma(x+1)}{\Gamma(mx+q+1)} &= \frac{e^{-\gamma x}e^{\gamma(mx+q)}}{\prod\limits_{n=1}^{+\infty}\Big(1+\dfrac{x}{n}\Big)e^{-x/n}}\prod_{n=1}^{+\infty}\Big(1+\frac{mx+q}{n}\Big)e^{-(mx+q)/n} \\
&= e^{\gamma((m-1)x+q)}\prod_{n=1}^{+\infty}\Big(1+\frac{(m-1)x+q}{n+x}\Big)e^{-((m-1)x+q)/n} \\
&= Ae^{\gamma(m-1)x}\prod_{n=1}^{+\infty}\Big(1+\frac{(m-1)x+q}{n+x}\Big)e^{-((m-1)x+q)/n}.
\end{aligned}
$$

Since $m \in \mathbb{N}$, the set of poles of $\Gamma(x+1)$ (the numerator in (3)), i.e., $\{-1,-2,\dots\}$, is contained in the set of poles of the denominator

$$\big\{(-1-q-i)/m \mid i=0,1,\dots\big\},$$

so that the function (3) is an entire function without positive zeros. □

Now, we are ready to prove Theorem 1.

Proof of Theorem 1. Consider the polynomial

$$(t-1)^N = \sum_{k=0}^{N}\binom{N}{k}(-1)^{N-k}t^k = N!\sum_{k=0}^{N}(-1)^{N-k}\frac{t^k}{(N-k)!k!}\,,$$

which zeros are evidently all real. Taking $f(x) = \Gamma(x+1)/\Gamma(mx+q+1)$, we find that

$$f(k) = \frac{k!}{(mk+q)!}\,.$$

Then, according to Theorem A we conclude that all zeros of the polynomial

$$N!\sum_{k=0}^{N}(-1)^{N-k}\frac{1}{(N-k)!k!}\cdot\frac{k!}{(mk+q)!}t^k$$

are real. Notice that this polynomial is exactly our polynomial $P_N^{(m,q)}(t)$. Changing t by $-t$ in $P_N^{(m,q)}(t)$ we conclude that these zeros are positive. □

At the end we mention that there are many results on transformations of polynomials by multiplier sequences (see [1–3], [8–13]), as well as the so-called zero-mapping transformations which map polynomials with zeros in a certain interval into polynomials with zeros in another interval. A general technique for the construction of such transformations was developed by Iserles and Nørsett [5] (see also [6]).

REFERENCES

1. T. Craven and G. Csordas: *An inequality for the distribution of zeros of polynomials and entire functions.* Pacific J. Math. **95** (1981), 263–280.
2. T. Craven and G. Csordas: *On the number of real roots of polynomials.* Pacific J. Math. **102** (1982), 15–28.
3. T. Craven and G. Csordas: *The Gauss-Lucas and Jensen polynomials.* Trans. Amer. Math. Soc. **278** (1983), 415–429.
4. G. B. Đorđević: *Generalized Hermite polynomials.* Publ. Inst. Math. (Beograd) (N.S.) **53 (67)** (1993), 69–72.
5. A. Iserles and S. P. Nørsett: *Zeros of transformed polynomials.* SIAM J. Math. Anal. **21** (1990), 483–509.
6. A. Iserles, S. P. Nørsett, and E. B. Saff: *On transformations and zeros of polynomials.* Rocky Mountain J. Math. **21** (1991), 331–357.
7. G. V. Milovanović: *Recurrence relation for a class of polynomials associated with the generalized Hermite polynomials.* Publ. Inst. Math. (Beograd) (N.S.) **54 (68)** (1993), 35–37.
8. G. V. Milovanović, D. S. Mitrinović, and Th. M. Rassias: *Topics in Polynomials: Extremal Problems, Inequalities, Zeros.* World Scientific, Singapore – New Jersey – London – Hong Kong, 1994.
9. N. Obreškov: *Über einige Multiplikatoren in der Theorie der algebraischen Gleichungen.* Jber. Deutschen Math. Ver. **35** (1926), 301–304.
10. N. Obreškov: *Quelques classes de fonctions entières limites de polynomes et de fonctions méromorphes limites de fonctions rationnelles.* Actualités scientifiques et industrielles, Paris, 1941.
11. N. Obreškov: *Zeros of Polynomials.* Bulgar. Acad. Sci., Sofia, 1963 (in Bulgarian).
12. G. Pólya: *Über Annäherung durch Polynome mit lauter reellen Wurzeln.* Rend. Circ. Matem. Palermo **36** (1913), 279–295.

13. G. Pólya and I. Schur: *Über zwei Arten der Faktorfolgen in der Theorie der algebraischen Gleichungen.* J. Reine Angew. Math. **144** (1914), 89–113.

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DISTRIBUCIJA NULA JEDNE KLASE POLINOMA KOJI SU PRIDRUŽENI GENERALISANIM HERMITEOVIM POLINOMIMA

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U radu se dokazuje da polinomi $P_N^{(m,q)}(t)$ pridruženi generalisanim Hermiteovim polinomima, sa $m \in \mathbb{N}$ i $q \in \{0, 1, \dots, m-1\}$, imaju samo realne pozitivne nule za svako $N \in \mathbb{N}$.