ZERO DISTRIBUTION OF A CLASS OF POLYNOMIALS ASSOCIATED WITH THE GENERALIZED HERMITE POLYNOMIALS

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Abstract. We prove that polynomials $P_{N}^{(m,q)}(t)$ associated with the generalized Hermite polynomials, where $m \in \mathbb{N}$ and $q \in \{0,1,\ldots,m-1\}$, has only real positive zeros for every $N \in \mathbb{N}$.

The sequence of polynomials $\{h_{n,m}^{\lambda}(x)\}_{n=0}^{+\infty}$, where $\lambda$ is a real parameter and $m$ is an arbitrary positive integer, was studied in [4]. For $m = 2$, the polynomial $h_{n,m}^{\lambda}(x)$ reduces to $H_{n}(x,\lambda)/n!$, where $H_{n}(x,\lambda)$ is the Hermite polynomial with a parameter. For $\lambda = 1$, $h_{n,2}^{1}(x) = H_{n}(x)/n!$, where $H_{n}(x)$ is the classical Hermite polynomial. Taking $\lambda = 1$ and $n = mN + q$, where $N = [n/m]$ and $0 \leq q \leq m - 1$, Đorđević [4] introduced the polynomials $P_{N}^{(m,q)}(t)$ by $h_{n,m}^{1}(x) = (2x)^{q}P_{N}^{(m,q)}((2x)^{m})$, and proved that they satisfy an $(m+1)$-term linear recurrence relation of the form

$$\sum_{i=0}^{m} A_{N}(i,q)P_{N+1-i}^{(m,q)}(t) = B_{N}(q)tP_{N}^{(m,q)}(t),$$

where $B_{N}(q)$ and $A_{N}(i,q)$ ($i = 0,1,\ldots,m$) are constants depending only on $N$, $m$ and $q$. Recently, one of us [7] determined the explicit expressions for the coefficients in (1) using some combinatorial identities.

An explicit representation of the polynomial $P_{N}^{(m,q)}(t)$ can be given in the form (see [4], [7]),

$$P_{N}^{(m,q)}(t) = \sum_{k=0}^{N} (-1)^{N-k} \frac{t^{k}}{(N-k)!(q+mk)!},$$

where $m \in \mathbb{N}$ and $q \in \{0,1,\ldots,m-1\}$.

In this note we prove a zero distribution of polynomials (2):

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Theorem 1. The polynomial $P_{N}^{(m,q)}(t)$ defined by (2), where $m \in \mathbb{N}$ and $q \in \{0, 1, \ldots, m - 1\}$, has only real and positive zeros for every $N \in \mathbb{N}$.

In the proof of this theorem we use the following result (cf. Obreschkoff [11, p. 107]):

Theorem A. Let $a_0 + a_1 x + \cdots + a_n x^n$ be a polynomial with only real zeros and let $x \mapsto f(x)$ be an entire function of the second kind without positive zeros. Then the polynomial

$$a_0 f(0) + a_1 f(1) x + \cdots + a_n f(n) x^n$$

has only real zeros.

It is known that an entire function of the second kind (in the Laguerre-Pólya class) can be expressed in the form

$$f(x) = C e^{-ax^2 + bx^m} \prod_{n=1}^{+\infty} \left(1 - \frac{x}{\alpha_n}\right) e^{x/\alpha_n},$$

where $C, a, b \in \mathbb{R}$, $m \in \mathbb{N}_0$, $\alpha_n \in \mathbb{R}$ ($n = 1, 2, \ldots$) and $\sum_{n=1}^{+\infty} 1/\alpha_n^2 < +\infty$.

We first prove an auxiliary results regarding the ratio $\Gamma(x+1)/\Gamma(mx+q+1)$, where $\Gamma(x)$ is the gamma function.

Lemma 1. Let $m \in \mathbb{N}$ and $q \in \{0, 1, \ldots, m - 1\}$. The equality

$$\frac{\Gamma(x+1)}{\Gamma(mx+q+1)} = A e^{\gamma(m-1)x} \prod_{n=1}^{+\infty} \left(1 + \frac{(m-1)x+q}{n} \right) e^{-(m-1)x+q)/n}$$

holds, where $A$ and $\gamma$ are constants ($\gamma = 0.57721566 \ldots$ is known as Euler’s constant).

Proof. In 1856 Weierstrass proved the formula

$$\frac{1}{\Gamma(z+1)} = e^{\gamma z} \prod_{n=1}^{+\infty} \left[1 + \frac{z}{n}\right] e^{-z/n}.$$
According to this equality we have

\[
\frac{\Gamma(x + 1)}{\Gamma(mx + q + 1)} = \frac{e^{-\gamma x}e^{\gamma(mx+q)}}{\prod_{n=1}^{\infty} \left( 1 + \frac{x}{n} \right) e^{-x/n}} \prod_{n=1}^{\infty} \left( 1 + \frac{mx + q}{n} \right) e^{-(mx+q)/n}
\]

\[
= e^{\gamma((m-1)x+q)} \prod_{n=1}^{\infty} \left( 1 + \frac{(m-1)x + q}{n + x} \right) e^{-(m-1)x+q)/n}
\]

\[
= Ae^{\gamma(m-1)x} \prod_{n=1}^{\infty} \left( 1 + \frac{(m-1)x + q}{n + x} \right) e^{-(m-1)x+q)/n}.
\]

Since \( m \in \mathbb{N} \), the set of poles of \( \Gamma(x + 1) \) (the numerator in (3)), i.e., \( \{-1, -2, \ldots\} \), is contained in the set of poles of the denominator

\[
\left\{ (-1 - q - i)/m \mid i = 0, 1, \ldots \right\},
\]

so that the function (3) is an entire function without positive zeros. □

Now, we are ready to prove Theorem 1.

**Proof of Theorem 1.** Consider the polynomial

\[
(t - 1)^N = \sum_{k=0}^{N} \binom{N}{k} (-1)^{N-k} t^k = N! \sum_{k=0}^{N} \frac{(-1)^{N-k}}{(N-k)!} \frac{t^k}{k!},
\]

which zeros are evidently all real. Taking \( f(x) = \Gamma(x + 1) / \Gamma(mx + q + 1) \), we find that

\[
f(k) = \frac{k!}{(mk + q)!}.
\]

Then, according to Theorem A we conclude that all zeros of the polynomial

\[
N! \sum_{k=0}^{N} (-1)^{N-k} \frac{1}{(N-k)!k!} \frac{k!}{(mk + q)!} t^k
\]

are real. Notice that this polynomial is exactly our polynomial \( P_N^{(m,q)}(t) \). Changing \( t \) by \( -t \) in \( P_N^{(m,q)}(t) \) we conclude that these zeros are positive. □

At the end we mention that there are many results on transformations of polynomials by multiplier sequences (see [1–3], [8–13]), as well as the so-called zero-mapping transformations which map polynomials with zeros in
a certain interval into polynomials with zeros in another interval. A general technique for the construction of such transformations was developed by Iserles and Norsett [5] (see also [6]).

REFERENCES


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DISTRIBUCIJA NULA JEDNE KLASE POLINOMA KOJI SU PRIDRUŽENI GENERALISANIM HERMITEOVIM POLINOMIMA

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U radu se dokazuje da polinomi $P_{N}^{(m,q)}(t)$ pridruženi generalisanim Hermiteovim polinomima, sa $m \in \mathbb{N}$ i $q \in \{0,1,\ldots,m-1\}$, imaju samo realne pozitivne nule za svako $N \in \mathbb{N}$. 