GERONIMUS CONCEPT OF ORTHOGONALITY FOR POLYNOMIALS ORTHOGONAL ON A CIRCULAR ARC

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Abstract. For complex polynomials orthogonal on the semicircle [2–4], [7], or on a circular arc [1], with respect to a complex-valued inner product, we consider an other type of orthogonality on a simple closed curve with respect to a *complex weight* $\chi(z)$, with a singularity in $z = 0$. In some cases, the weight can be found explicitly.

1. Introduction

One new type of orthogonality, so-called *orthogonality on the semicircle*, has been introduced by Gautschi and Milovanović [2], [3]. The inner product is given by

$$(f,g) = \int_{\Gamma} f(z)g(z)(iz)^{-1}\,dz, \tag{1.1}$$

where Γ is the semicircle $\Gamma = \{z \in \mathbb{C} \,:\, z = e^{i\theta},\, 0 \le \theta \le \pi\}$. Alternatively,

$$(f,g) = \int_0^{\pi} f(e^{i\theta})g(e^{i\theta})\,d\theta. \tag{1.2}$$

This inner product is not Hermitian, but the corresponding (monic) orthogonal polynomials $\{\pi_k\}$ exist uniquely and satisfy a three-term recurrence relation of the form

$$\begin{aligned} &\pi_{k+1}(z) = (z - i\alpha_k)\pi_k(z) - \beta_k\pi_{k-1}(z), \qquad k = 0,1,2,\dots\,, \\ &\pi_{-1}(z) = 0, \quad \pi_0(z) = 1. \end{aligned} \tag{1.3}$$

Notice that the inner product (1.1) possesses the property $(zf, g) = (f, zg)$.

In the paper [4] Gautschi, Landau and Milovanović have considered a general case of orthogonality with respect to a *complex weight function*. Namely, let $w : (-1,1) \mapsto \mathbf{R}_+$ be a weight function which can be extended to a function $w(z)$ holomorphic in the half disc $D_+ = \{z \in \mathbb{C} \,:\, |z| < 1,\, \mathrm{Im}\, z > 0\}$, and

$$(f,g) = \int_{\Gamma} f(z)g(z)w(z)(iz)^{-1}\,dz = \int_0^{\pi} f(e^{i\theta})g(e^{i\theta})w(e^{i\theta})\,d\theta. \tag{1.4}$$

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Under the assumption

$$\operatorname{Re}(1,1) = \operatorname{Re}\int_0^{\pi} w(e^{i\theta})\,d\theta \neq 0, \tag{1.5}$$

the monic, complex polynomials $\{\pi_k\}$ orthogonal with respect to the inner product (1.4) exist and satisfy the recurrence relation like (1.3).

Several interesting properties of such polynomials, especially for Gegenbauer weight, were shown in [4] and [7]. Besides of polynomials orthogonal on the semicircle, the coresponding functions of the second kind and associated polynomials were introduced and investigated in [7]. Also, some applications in numerical integration and numerical differentiation were given.

Recently M. G. de Bruin [1] has given a generalization of such orthogonal polynomials. Namely, he considered the polynomials $\{\pi_k^R\}$ orthogonal on a circular arc with respect to the complex inner product

$$(f,g) = \int_{\varphi}^{\pi-\varphi} f_1(\theta) g_1(\theta) w_1(\theta)\,d\theta, \tag{1.6}$$

where $\varphi \in (0, \pi/2)$, and for $f(z)$ the function $f_1(\theta)$ is defined by

$$f_1(\theta) = f\bigl(-iR + e^{i\theta}\sqrt{R^2+1}\bigr), \qquad R = \tan\varphi.$$

Alternatively, the inner product (1.6) can be expressed in the form

$$(f,g) = \int_{\Gamma_R} f(z) g(z) w(z) (iz - R)^{-1}\,dz, \tag{1.7}$$

where $\Gamma_R = \{z \in \mathbb{C} : z = -iR + e^{i\theta}\sqrt{R^2+1},\ \varphi \le \theta \le \pi - \varphi,\ \tan\varphi = R\}$.

For $R = 0$ the arc Γ_R reduces to the semicircle Γ.

In this paper, we study an another type of orthogonality of these polynomials, so-called Geronimus' version of orthogonality [5] on a contour with respect to a complex weight.

2. Geronimus' version of orthogonality on a contour

In the paper [6], J. W. Jayne considered the Geronimus' concept of orthogonality for recursively generated polynomials. Ya. L. Geronimus proved that a sequence of polynomials $\{\pi_k\}$, which is orthogonal on a finite interval on real line, is also orthogonal in the sense that there is a weight function $z \to \chi(z)$ having one or more singularities inside a simple curve C and such that

$$\langle \pi_k, \pi_m \rangle = \frac{1}{2\pi i}\oint_C \pi_k(z)\pi_m(z)\chi(z)\,dz = \begin{cases} 0, & k \neq m, \\ h_m, & k = m. \end{cases} \tag{2.1}$$

Following Geronimus [5] and Jayne [6], we will determine a such complex weight function $z \to \chi(z)$, for (monic) polynomials $\{\pi_k\}$ orthogonal on the semicircle Γ, and also for the corresponding polynomials $\{\pi_k^R\}$ orthogonal on the circular arc Γ_R $(R > 0)$.

We denote by C any positively oriented simple closed contour surrounding some circle $|z| = r > 1$. We assume that

$$\chi(z) = \sum_{k=1}^{\infty} \omega_k z^{-k}, \qquad \omega_1 = 1, \tag{2.2}$$

for $|z| > r$.

At first, we express z^n as a linear combination of the monic polynomials π_m $(m = 0, 1, \dots, n)$, which are orthogonal on the semicircle Γ, with respect to the inner product (1.4), i.e. (1.5). Namely, we have

$$z^n = \sum_{m=0}^{n} \gamma_{n,m} \pi_m(z), \tag{2.3}$$

where

$$(z^n, \pi_m) = \gamma_{n,m}(\pi_m, \pi_m), \qquad m = 0, 1, \dots, n. \tag{2.4}$$

Using the inner product (2.1) and the representation (2.2), we obtain

$$\begin{aligned} \langle z^n, 1\rangle &= \frac{1}{2\pi i} \oint_C z^n \chi(z)\, dz \\ &= \frac{1}{2\pi i} \oint_C \sum_{k=1}^{\infty} \omega_k z^{n-k}\, dz = \omega_{n+1}. \end{aligned}$$

On the other hand, because of (2.3) and the orthogonality condition (2.1), we find

$$\langle z^n, 1\rangle = \langle \sum_{m=0}^{n} \gamma_{n,m} \pi_m(z), 1\rangle = \sum_{m=0}^{n} \gamma_{n,m} \langle \pi_m, 1\rangle,$$

i.e.,

$$\langle z^n, 1\rangle = \gamma_{n,0} \langle \pi_0, \pi_0\rangle = \gamma_{n,0} h_0.$$

Thus, we have

$$w_{n+1} = \gamma_{n,0} h_0 = \gamma_{n,0},$$

because $h_0 = \omega_1 = 1$.

Finally, using (2.4) and the moments $\mu_n = (z^n, 1)$, we obtain

$$\omega_{n+1} = \frac{\mu_n}{\mu_0}, \quad n \geq 0,$$

and

$$\chi(z) = \frac{1}{\mu_0} \sum_{k=1}^{\infty} \mu_{k-1} z^{-k} \qquad (|z| > r). \tag{2.5}$$

So, we need the convergence of this series for $|z| > r > 1$.

Let w be a weight function, nonnegative on $(-1, 1)$, holomorphic in

$$D_+ = \{z \in \mathbb{C} \,:\, |z| < 1,\, \mathrm{Im}\, z > 0\},$$

integrable over ∂D_+, and such that (1.5) is satisfied.

The moments μ_k can be expressed in the form

$$\mu_0 = \int_\Gamma w(z)(iz)^{-1}\, dz = \frac{1}{i}\left(i\pi w(0) - \text{v.p.} \int_{-1}^{1} \frac{w(x)}{x}\, dx \right) \tag{2.6}$$

and

$$\mu_k = \int_\Gamma z^k w(z)(iz)^{-1}\, dz = i \int_{-1}^{1} x^{k-1} w(x)\, dx, \quad k \geq 1. \tag{2.7}$$

These moments are included in the series (2.5).

Additionally, we suppose that the weight function w has such moments μ_k, which provide the convergence of the series (2.5) for all z outside some circle $|z| = r > 1$ lying interior to C.

Theorem 2.1. *Let w be a weight function satisfying the above conditions. Then the monic polynomials $\{\pi_k\}$, which are orthogonal on the semicircle Γ with respect to the inner product* (1.4)*, are also orthogonal in the sense of* (2.1)*, where*

$$\chi(z) = \frac{1}{z}\left(1 + \frac{i}{\mu_0} \int_{-1}^{1} \frac{w(x)}{z - x}\, dx \right) \qquad (|z| > r > 1)$$

and

$$\mu_0 = \pi w(0) + i\, \text{v.p.} \int_{-1}^{1} \frac{w(x)}{x}\, dx.$$

Proof. Let $|z| > r > 1$ and let the moments are given by (2.6) and (2.7). Then, (2.5) becomes

$$
\begin{aligned}
\chi(z) &= \frac{1}{z}\left(1+\frac{1}{\mu_0}\sum_{k=1}^{\infty}\mu_k z^{-k}\right)\\
&= \frac{1}{z}\left(1+\frac{i}{\mu_0}\sum_{k=1}^{\infty}z^{-k}\int_{-1}^{1}x^{k-1}w(x)\,dx\right)\\
&= \frac{1}{z}\left(1+\frac{i}{\mu_0 z}\sum_{k=1}^{\infty}z^{-(k-1)}\int_{-1}^{1}x^{k-1}w(x)\,dx\right)\\
&= \frac{1}{z}\left(1+\frac{i}{\mu_0 z}\int_{-1}^{1}w(x)\left(\sum_{k=1}^{\infty}\left(\frac{x}{z}\right)^{k-1}\right)dx\right)\\
&= \frac{1}{z}\left(1+\frac{i}{\mu_0 z}\int_{-1}^{1}w(x)\frac{1}{1-\frac{x}{z}}\,dx\right),
\end{aligned}
$$

i.e.,

$$\chi(z) = \frac{1}{z}\left(1+\frac{i}{\mu_0}\int_{-1}^{1}\frac{w(x)}{z-x}\,dx\right). \quad \square$$

In Gegenbauer case we obtain the following result:

Corollary 2.2. *Let $w(z) = (1-z^2)^{\lambda-1/2}$, $(\lambda > -1/2)$. The monic polynomials $\{\pi_k\}$, which are orthogonal on the unit semicircle with respect to the inner product* (1.4)*, are also orthogonal in the sense of* (2.1)*, where*

$$\chi(z) = \frac{1}{z} + \frac{i}{\sqrt{\pi}\, z^2}\cdot\frac{\Gamma(\lambda+\frac{1}{2})}{\Gamma(\lambda+1)}\,F\big(1,\frac{1}{2},\lambda+1;\frac{1}{z^2}\big), \tag{2.8}$$

where F is the Gauss hypergeometric series and Γ is the gamma function.

Proof. Using the above theorem for Gegenbauer weight, we obtain

$$\chi(z) = \frac{1}{z} + \frac{i}{\pi z}\int_{-1}^{1}\frac{(1-x^2)^{\lambda-1/2}}{z-x}\,dx,$$

i.e.,

$$\chi(z) = \frac{1}{z} + \frac{i}{\pi z^2}\int_{0}^{1}t^{-1/2}(1-t)^{\lambda-1/2}(1-tz^{-2})^{-1}\,dx,$$

which is equivalent to (2.8).

Remark. In Legendre case ($\lambda = 1/2$) we obtain

$$\chi(z) = \frac{1}{z} + \frac{i}{\pi z}\log\frac{z+1}{z-1},$$

where the interval from -1 to 1 on the real axis is connsidered as a branch cut.

Now, we consider the polynomials $\{\pi_k^R\}$ $(R>0)$ which are orthogonal on the circular arc.

Let w be a weight function, nonnegative on $(-1,1)$, holomorphic in

$$M_+ = \{z \in \mathbb{C} \,:\, |z+iR| < \sqrt{R^2+1},\, \mathrm{Im}\, z > 0\},$$

and integrable over ∂M_+.

In this case, the moments μ_k can be expressed in the form

$$\mu_k = \int_{\Gamma_R} z^k w(z)(iz-R)^{-1}\,dz = -\int_{-1}^{1} x^k (ix-R)^{-1} w(x)\,dx, \quad (k \ge 0),$$

i.e.,

$$\mu_k = \int_{-1}^{1} \frac{R+ix}{R^2+x^2}\, x^k w(x)\,dx \qquad (k\ge 0). \tag{2.9}$$

Again, we suppose that the weight function w has such moments μ_k, which provide the convergence of the series (2.5) for all z outside some circle $|z|=r>1$ lying interior to C.

Theorem 2.3. *Under the above conditions on the weight function w, the monic polynomials $\{\pi_k^R\}$, which are orthogonal on the circular arc Γ_R with respect to the inner product* (1.6), *i.e.,* (1.7), *are also orthogonal in the sense of* (2.1), *where*

$$\chi(z) = \frac{1}{\mu_0}\int_{-1}^{1} \frac{(R+ix)w(x)}{(R^2+x^2)(z-x)}\,dx \qquad (|z|>r>1) \tag{2.10}$$

and

$$\mu_0 = \int_{-1}^{1} \frac{R+ix}{R^2+x^2}\, w(x)\,dx$$

Proof. Let $|z|>r>1$. Using the moments, given by (2.9), we get

$$\begin{aligned}\sum_{k=1}^{\infty} \mu_{k-1} z^{-k} &= \sum_{k=1}^{\infty}\left(\int_{-1}^{1} \frac{R+ix}{R^2+x^2}\, x^{k-1} w(x)\,dx\right) z^{-k} \\ &= \int_{-1}^{1} \frac{R+ix}{R^2+x^2}\cdot \frac{w(x)}{z}\sum_{k=1}^{\infty}\left(\frac{x}{z}\right)^{k-1} dx \\ &= \int_{-1}^{1} \frac{(R+ix)w(x)}{(R^2+x^2)(z-x)}\,dx,\end{aligned}$$

i.e., (2.10). $\square$

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