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Standard and Non-standard Quadratures of Gaussian Type

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Dedicated to Borislav Bojanov on the occasion of his 60th birthday

In this paper we consider standard and non-standard (interval) quadratures of Gaussian type. We give a short account of these quadratures and then we derive a new algorithm for the numerical construction of the interval Gaussian quadratures with respect to the Jacobi measure on $(-1,1)$. A numerical example is included.

1. Introduction

Let $d\mu(t)$ be a finite positive Borel measure on the real line such that its support supp (μ) is an infinite set. Suppose that the moments $\mu_k = \int_{\mathbb{R}} t^k \, d\mu(t)$, $k = 0, 1, \ldots$, exist and are finite. For real-valued functions $f, g \in L^2(d\mu)$ we define an inner product

$$(f,g) = (f,g)_{d\mu} = \int_{\mathbb{R}} f(t)g(t) \, d\mu(t). \tag{1.1}$$

With $\mathcal{P}$ we denote the set of all algebraic polynomials and with $\mathcal{P}_n$ its subset formed by all polynomials of degree at most n $(\in \mathbb{N}_0)$.

Let $\{\pi_k(d\mu)\}_{k=0}^{\infty}$ be the corresponding system of monic polynomials

$$\pi_k(t) = \pi_k(d\mu; t) = t^k + \text{lower degree terms},$$

orthogonal with respect to the measure $d\mu(t)$. Because of the property of the inner product $(tf, g) = (f, tg)$, these polynomials satisfy the fundamental three-term recurrence relation

$$\begin{aligned} \pi_{k+1}(t) &= (t - \alpha_k)\pi_k(t) - \beta_k \pi_{k-1}(t), \qquad k = 1, 2, \ldots, \\ \pi_{-1}(t) &= 0, \qquad \pi_0(t) = 1, \end{aligned}$$

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where $\alpha_k = \alpha_k(d\mu)$, $\beta_k = \beta_k(d\mu)$, and by convention $\beta_0 := \int_{\mathbb{R}} d\mu(t)$. The all zeros of $\pi_n(t)$ are simple and lie in the smallest interval containing the support $E = \operatorname{supp}(d\mu)$.

The standard quadrature formulae of Gaussian type have a direct connection with orthogonal polynomials. However, such connections for non-standard (interval) quadratures are not known in general. In this paper we give first a short account on standard and interval quadratures in Sections 2 and 3, respectively, and then, in Section 4, we give a new algorithm for the numerical construction of the interval Gaussian quadratures with respect to the Jacobi measure on $(-1, 1)$.

2. Standard Quadratures of Gaussian Type

The n-point quadrature formula

$$\int_{\mathbb{R}} f(t)\, d\mu(t) = \sum_{\nu=1}^{n} A_\nu f(\tau_\nu) + R_n(f), \tag{2.1}$$

which provides an approximation to the integral $I(f) = \int_{\mathbb{R}} f(t) d\mu(t)$, is called *interpolatory* if its *degree of exactness* is at least $d = n - 1$, i.e., if $R_n(f) = 0$ for each $f \in \mathcal{P}_d$. The degree of exactness for a quadrature formula can be increased by a special selection of the nodes τ_ν in the *node* polynomial

$$\omega_n(t) = \prod_{\nu=1}^{n} (t - \tau_\nu), \tag{2.2}$$

and it is connected with the theory of orthogonality. Namely, according to the essential result of Jacobi [18] (see also Gautschi [13]), the interpolatory quadrature (2.1) has degree of exactness $d = n - 1 + m$ $(m > 0)$ *if and only if* the node polynomial (2.2) is *orthogonal* to all polynomials from $\mathcal{P}_{m-1}$, i.e.,

$$(\omega_n, p)_{d\mu} = \int_{\mathbb{R}} \omega_n(t) p(t)\, d\mu(t) = 0 \qquad \text{for each } p \in \mathcal{P}_{m-1}. \tag{2.3}$$

It is clear that $m \leq n$; otherwise, taking $p = \omega_n$ $(\in \mathcal{P}_n)$ we would get $(\omega_n, \omega_n) = \|\omega_n\|^2 = 0$, which is impossible. Thus, the optimal value is $m = n$ and (2.1) has the maximal degree of exactness $d = 2n - 1$. If we fix some nodes in (2.2), then the degree of exactness is reduced.

Gauss quadrature formula. The formula with maximal degree of exactness $d = 2n - 1$ $(m = n)$ is called the *Gauss quadrature formula*. In that case, $(\omega_n, p) = 0$ for all $p \in \mathcal{P}_{n-1}$. Thus, ω_n must be orthogonal to all polynomials of lower degree with respect to the inner product (1.1), i.e., $\omega_n(t) = \pi_n(d\mu; t)$. The zeros of $\pi_n(d\mu; t)$ are nodes in the Gauss quadrature rule (2.1). His famous method of approximate integration, Gauss [11] discovered for the Legendre measure $d\mu(t) = dt$ on $[-1, 1]$ in 1814, and he obtained numerical values

of quadrature parameters, the nodes τ_ν and the weights A_ν, $\nu = 1, \ldots, n$, by solving nonlinear systems of equations for $n \leq 7$.

Computationally, today there are very stable methods for generating Gaussian rules. The standard method of Golub and Welsch [17] is based on Francis's QR algorithm for the eigenvalue problem for the following symmetric tridiagonal matrix

$$J_n(d\mu) = \begin{bmatrix} \alpha_0 & \sqrt{\beta_1} & & & \mathbf{0} \\ \sqrt{\beta_1} & \alpha_1 & \sqrt{\beta_2} & & \\ & \sqrt{\beta_2} & \alpha_2 & \ddots & \\ & & \ddots & \ddots & \sqrt{\beta_{n-1}} \\ \mathbf{0} & & & \sqrt{\beta_{n-1}} & \alpha_{n-1} \end{bmatrix}.$$

This matrix is known as the *Jacobi matrix* and its elements are formed from the coefficients in the three-term recurrence relation.

Theorem 1. *The nodes τ_ν of the n-point Gaussian quadrature rule* (2.1) *are the eigenvalues of the Jacobi matrix,*

$$J_n(d\mu)\boldsymbol{v}_\nu = \tau_\nu \boldsymbol{v}_\nu, \qquad \boldsymbol{v}_\nu^T \boldsymbol{v}_\nu = 1,$$

and the corresponding weights A_ν are given by

$$A_\nu = \beta_0 \boldsymbol{v}_{\nu,1}^2, \qquad \boldsymbol{v}_\nu = [\boldsymbol{v}_{\nu,1} \ \ldots \ \boldsymbol{v}_{\nu,n}]^T.$$

Gauss-Radau formula. Suppose that the support of the measure $d\mu(t)$ is bounded from below, i.e., $\inf \operatorname{supp}(d\mu) = a > -\infty$. If we want to have a quadrature formula in which one of the nodes is a, say $\tau_1 = a$, then (2.2) becomes $\omega_n(t) = (t-a)\omega_{n-1}(t)$, and (2.3) reduces to $(\omega_n, p)_{d\mu} = (\omega_{n-1}, p)_{d\sigma} = 0$ for each $p \in \mathcal{P}_{m-1}$, where $d\sigma(t) = (t-a)d\mu(t)$. The optimal value of m is now $m = n-1$. Thus, according to (2.3), ω_{n-1} must be orthogonal to all polynomials of degree at most $n-2$ with respect to the measure $d\sigma(t)$, i.e., $\omega_{n-1}(t) = \pi_{n-1}(d\sigma; t)$. The corresponding formula is called the n-point *Gauss-Radau formula*

$$\int_{\mathbb{R}} f(t)\, d\mu(t) = A_1^R f(a) + \sum_{\nu=2}^{n} A_\nu^R f(\tau_\nu^R) + R_n^R(f),$$

where $\pi_{n-1}(d\sigma; \tau_\nu^R) = 0$, $\nu = 2, \ldots, n$. Here, $d = 2n-2$, i.e., $R_n^R(f) = 0$ when $f \in \mathcal{P}_{2n-2}$.

Gauss-Lobatto formula. Suppose now that $\inf \operatorname{supp}(d\mu) = a > -\infty$ and $\sup \operatorname{supp}(d\mu) = b < +\infty$, and $n \geq 2$. Taking $\tau_1 = a$ and $\tau_n = b$ (two nodes are fixed), the polynomial (2.2) becomes $\omega_n(t) = -(t-a)(b-t)\omega_{n-2}(t)$, and (2.3) reduces to $(\omega_n, p)_{d\mu} = (\omega_{n-2}, p)_{d\lambda} = 0$ for each $p \in \mathcal{P}_{m-1}$, where

$d\lambda(t) = (t-a)(b-t)d\mu(t)$. The optimal value of m is now $m = n-2$, and ω_{n-2} must be $\omega_{n-2}(t) = \pi_{n-2}(d\lambda; t)$. The corresponding formula is called the n-point *Gauss-Lobatto formula*

$$\int_a^b f(t)\, d\mu(t) = A_1^L f(a) + \sum_{\nu=2}^{n-1} A_\nu^L f(\tau_\nu^L) + A_n^L f(b) + R_n^L(f),$$

where $\pi_{n-2}(d\lambda; \tau_\nu^L) = 0$, $\nu = 2, \dots, n-1$. Here, $d = 2n-3$, i.e., $R_n^L(f) = 0$ when $f \in \mathcal{P}_{2n-3}$.

For finding nodes and weights in the Gauss-Radau and Gauss-Lobatto rules a similar procedure with some modified matrix eigenvalue problems can be used (see Golub [16] and Gautschi [12]).

Gaussian-type quadratures with multiple nodes. More than hundred years after the famous Gauss quadrature method, which was enriched by significant contributions of Jacobi and Christoffel, there appeared the idea of numerical integration involving multiple nodes, i.e.,

$$\int_{\mathbb{R}} f(t)d\mu(t) \approx \sum_{\nu=1}^{n} \sum_{k=0}^{2s_\nu} A_{\nu,k} f^{(k)}(\tau_\nu),$$

as well as a general concept of the *power orthogonality* (see Chakalov [8, 9, 10], Turán [34], Popoviciu [31], Ghizzetti and Ossicini [14, 15], etc.). For a positive measure $d\mu(t)$, a such quadrature rule is unique and its nodes τ_ν are contained in the supp (μ). It is also known that it integrates exactly polynomials of degree at most $2\sum_{\nu=1}^{n} s_\nu + 2n - 1$ (cf. [22]). An algorithm for constructing quadrature formulas with multiple Gaussian nodes in the presence of certain fixed nodes was recently presented by Milovanović, Spalević and Cvetković [25].

Further extensions dealing with quadratures with multiple node for ET (Extended Tschebycheff) systems are given by Karlin and Pinkus [19], Barrow [2], Bojanov, Braess, and Dyn [4], Bojanov [3], etc.

3. Non-standard (Interval) Gaussian Quadratures

If the information data $\{f(\tau_\nu)\}_{\nu=1}^n$ in the standard quadrature (2.1) is replaced by $\{(\mathcal{A}_{h_\nu} f)(\tau_\nu)\}_{\nu=1}^n$, where $\mathcal{A}_h$ is an extension of some linear operator $\mathcal{A}_h : \mathcal{P} \to \mathcal{P}$, $h \geq 0$, we get a non-standard quadrature formula

$$\int_{\mathbb{R}} f(t)d\mu(t) = \sum_{\nu=1}^{n} w_\nu (\mathcal{A}_{h_\nu} f)(\tau_\nu) + R_n(f). \tag{3.1}$$

Notice that we use the same notation for the linear operator defined on the space of all algebraic polynomials and for its extension to the certain class of

integrable functions X ($f \in X$). As a typical example for such operators is the *average operator*

$$(\mathcal{A}_h p)(x) = \frac{1}{2h} \int_{x-h}^{x+h} p(t)\, dt, \qquad h > 0,\ p \in \mathcal{P}. \tag{3.2}$$

In the case $h = 0$ this operator is interpreted as the *identity operator* $\mathcal{A}_0 = \mathcal{I}$, so that, for continuous f, its value at x is $f(x)$, i.e.,

$$(\mathcal{A}_0 p)(x) = \lim_{h \to 0} (\mathcal{A}_h p)(x) = (\mathcal{I}p)(x) = p(x). \tag{3.3}$$

In many cases, especially in physics and technics, the average operator (3.2) can be used instead of the identity operator, which appears in the standard quadrature (2.1).

In 1976 Omladič, Pahor, and Suhadolc [28] considered (3.1) with the average operator. Such kinds of quadratures are known as the *interval quadrature formulae*, and they have been also investigated by Pitnauer and Reimer [30], Kuz'mina [20], Sharipov [32], Babenko [1], Motornyi [26].

Let w be a given weight function on a finite interval $[a, b]$, i.e., a nonnegative Lebesgue integrable function, such that for each subinterval $I = (\alpha, \beta) \in [a, b]$, $\alpha < \beta$, we have $\int_I w(t)\, dt > 0$. In that case, a *weighted average operator* can be considered in the form

$$(\mathcal{A}_h^w p)(x) = \frac{(\mathcal{A}_h pw)(x)}{(\mathcal{A}_h w)(x)} = \frac{\displaystyle\int_{x-h}^{x+h} p(t)w(t)\, dt}{\displaystyle\int_{x-h}^{x+h} w(t)\, dt}, \qquad h > 0,\ p \in \mathcal{P}, \tag{3.4}$$

or simpler as

$$(\mathcal{B}_h^w p)(x) = (\mathcal{A}_h pw)(x) = \frac{1}{2h} \int_{x-h}^{x+h} p(t)w(t)\, dt, \qquad h > 0,\ p \in \mathcal{P}. \tag{3.5}$$

Let $h_1, \ldots, h_n$ be nonnegative numbers such that

$$a < \tau_1 - h_1 \le \tau_1 + h_1 < \tau_2 - h_2 \le \tau_2 + h_2 < \cdots < \tau_n - h_n \le \tau_n + h_n < b. \tag{3.6}$$

By using these inequalities it is obvious that we have $2(h_1 + \cdots + h_n) < b - a$.

Taking operators (3.5), Bojanov and Petrov [5] proved that the Gaussian interval quadrature rule of the maximal algebraic degree of exactness $2n - 1$ exists, i.e.,

$$\int_a^b f(t)w(t)\, dt = \sum_{\nu=1}^{n} \frac{w_\nu}{2h_\nu} \int_{\tau_\nu - h_\nu}^{\tau_\nu + h_\nu} f(t)w(t)\, dt + R_n(f), \tag{3.7}$$

with positive weight coefficients w_ν, $\nu = 1, \dots, n$, where $R_n(f) = 0$ for each $f \in \mathcal{P}_{2n-1}$. Also, in the same paper they proved that for $h_\nu = h$, $1 \le \nu \le n$, this Gaussian interval quadrature formula is unique. In that case, for each $f \in C^{2n}[a, b]$ there exists a point $\xi \in (a, b)$ such that

$$R_n(f) = \frac{f^{(2n)}(\xi)}{(2n)!} \int_a^b Q_{2n}(t) w(t)\, dt,$$

where $Q_{2n}(t)$ is a unique monic polynomial of degree $2n$ such that

$$\int_{\tau_\nu - h}^{\tau_\nu + h} Q_{2n}(t) w(t)\, dt = \int_{\tau_\nu - h}^{\tau_\nu + h} t Q_{2n}(t) w(t)\, dt = 0, \qquad k = 1, \dots, n.$$

Moreover, in [6] Bojanov and Petrov proved the uniqueness of (3.7) for the Legendre weight ($w(x) = 1$) for any set of lengths $h_\nu \ge 0$, $k = 1, \dots, n$, satisfying the condition (3.6).

The question of the existence for bounded a, b is proved in [5] in a much broader context.

Theorem 2. *Given an ordered set $\{m_1, \dots, m_n\}$ of odd integers, $m_k = 2s_k + 1$, $s_k \ge 0$, $k = 1, \dots, n$, with the property $n + 2\sum_{k=1}^{n} s_k = N + 1$, given a Chebyshev system of functions $\{u_0, u_1, \dots, u_N\}$ on $[a, b]$ and a set of lengths $h_1 \ge 0, \dots, h_n \ge 0$, with $2(h_1 + \cdots + h_n) < b - a$, then there exists a generalized Gaussian quadrature formula of the form*

$$\int_a^b f(x) w(x)\, dx \approx \sum_{k=1}^{n} \sum_{\nu=0}^{2s_k} \frac{\mu_{k,\nu}}{2h_k} \int_{I_k} f(x) u_\nu(x) w(x)\, dx,$$

where intervals I_k, $k = 1, \dots, n$, are non-overlapping and all are subintervals of $[a, b]$, with length of I_k equals $2h_k$, which integrates exactly every element from the linear span of $\{u_0, u_1, \dots, u_N\}$.

Recently in [23], using properties of the topological degree of non-linear mappings (see [29, 33]), it was proved that Gaussian interval quadrature formula is unique for the Jacobi weight function on $[-1, 1]$.

Theorem 3. *Let $w(x) = (1 - x)^\alpha (1 + x)^\beta$, $\alpha, \beta > -1$. For any given $n \in \mathbb{N}$, $h_k \ge 0$, $k = 1, \dots, n$, $h_1 + \cdots + h_n \le 1 - \varepsilon$, $\varepsilon > 0$, there exists a unique interval Gaussian quadrature rule*

$$\int_{-1}^{1} f(x) w(x)\, dx = \sum_{k=1}^{n} \frac{\sigma_k}{2h_k} \int_{x_k - h_k}^{x_k + h_k} f(x) w(x)\, dx + R_n(f), \tag{3.8}$$

with non-overlapping subintervals $(x_k - h_k, x_k + h_k) \subset (-1, 1)$, $k = 1, \dots, n$, which is exact in $\mathcal{P}_{2n-1}$.

An algorithm for numerical construction of (3.8) has also been investigated and some suitable solutions are proposed. A new appropriate algorithm will be presented in the next section of this paper.

For the special case of the Chebyshev weight of the first kind and the special set of lengths an analytic solution can be given ([23]).

Theorem 4. *Let*

$$\delta_k = \frac{(2k-1)\pi}{2n} \quad \textit{and} \quad h_k = \sin\delta_k \sin\delta, \qquad k = 1, \dots, n,$$

where δ is chosen such that $0 < \delta < \pi/(2n)$. The parameters of the interval Gaussian quadrature with the Chebyshev weight,

$$\int_{-1}^{1} \frac{f(x)dx}{\sqrt{1-x^2}} = \sum_{k=1}^{n} \frac{\sigma_k}{2h_k} \int_{x_k-h_k}^{x_k+h_k} \frac{f(x)dx}{\sqrt{1-x^2}} + R_n(f), \tag{3.9}$$

can be expressed in the analytic form

$$x_k = -\cos\delta_k \cos\delta, \quad \sigma_k = \frac{\pi \sin\delta_k \sin\delta}{n\delta}, \qquad k = 1, \dots, n. \tag{3.10}$$

In the case when $\delta \to 0$, the nodes and weights given in (3.10) reduce to $x_k = -\cos\delta_k$ and $\sigma_k = \pi \sin\delta_k/n$, $k = 1, \dots, n$, and (3.9) becomes the well-known Gauss-Chebyshev quadrature formula.

Interval quadrature rules of Gauss-Radau and Gauss-Lobatto type with respect to the Jacobi weight functions are considered in [24].

Recently, Bojanov and Petrov [7] proved the existence and uniqueness of the weighted Gaussian interval quadrature formula for a given system of continuously differentiable functions, which constitute an ET system.

4. Algorithm for Numerical Construction of Interval Quadrature with Jacobi Weight

In this section we give a new algorithm for the numerical construction of the Gaussian interval quadrature rule for the Jacobi weight function. An alternative algorithm based on Gaussian quadrature rules for the calculation of the derivative of an analytic function in some bounded domain of the complex plane was presented in [23]. Although this method is more accurate, it requires complex arithmetic. Our new method is much simpler than the previous one presented in [23].

At first, we give some preliminary results using the following notation:

$$\begin{aligned} \phi(x) &= 1 - x^2, \\ \psi(x) &= -(\alpha+1)(1+x) + (\beta+1)(1-x), \\ w(x) &= (1-x)^\alpha (1+x)^\beta. \end{aligned}$$

It is easy to see that

$$(\phi p w)' = (p'\phi + p\psi)w, \qquad p \in \mathcal{P}.$$

We also use the following notation

$$\Omega(x) = \prod_{k=1}^{n}(x - x_k - h_k)(x - x_k + h_k),$$
$$\Omega_k(x) = \frac{\Omega(x)}{(x - x_k - h_k)(x - x_k + h_k)}, \qquad k = 1, \dots, n.$$

Theorem 5. *For any polynomial $p \in \mathcal{P}_{2n-2}$, we have*

$$\sum_{k=1}^{n} L_k(p, \Omega) = 0,$$

where

$$L_k(p, \Omega) = \begin{cases} \dfrac{p(x_k + h_k)}{2h_k\Omega_k(x_k + h_k)} - \dfrac{p(x_k - h_k)}{2h_k\Omega_k(x_k - h_k)}, & h_k \neq 0, \\ \dfrac{(p'\,\Omega_k - p\,\Omega_k')(x_k)}{(\Omega_k(x_k))^2}, & h_k = 0, \end{cases}$$

and $-1 < x_1 - h_1$, $x_n + h_n < 1$, $x_k - h_k \leq x_k + h_k$, $x_k + h_k < x_{k+1} - h_{k+1}$, $k = 1, \dots, n-1$.

Proof. It can be proved applying the Cauchy Residue Theorem to the function p/Ω over the circle $|z| = R > 1$ and letting $R \to +\infty$, taking into account that $(p/\Omega)(z) = O(z^{-2})$ as $z \to \infty$ and using the simple identity

$$\Omega'(x_k \pm h_k) = \pm 2h_k\Omega_k(x_k \pm h_k). \qquad \square$$

For a simpler presentation of the results we adopt also the following notation

$$\Delta_k(\Omega_k\phi w) = \frac{(\Omega_k\phi w)(x_k + h_k) - (\Omega_k\phi w)(x_k - h_k)}{2h_k}, \qquad k = 1, \dots, n.$$

Note that in the case $h_k = 0$, by the continuity argument, we have

$$\Delta_k(\Omega_k\phi w) = \partial_{x_k}(\Omega_k\phi w)(x_k).$$

Theorem 6. *Suppose that the distribution of nodes $-1 < x_1 - h_1$, $x_k + h_k < x_{k+1} - h_{k+1}$, $k = 1, \dots, n$, $x_n + h_n < 1$, satisfies the following conditions*

$$\Delta_k(\Omega_k\phi w) = 0, \qquad k = 1, \dots, n. \tag{4.1}$$

Then there exists a positive constant C such that, for any $q \in \mathcal{P}_{2n-1}$,

$$\int_{-1}^{1} q(x)w(x)\,dx = C\sum_{k=1}^{n} \frac{1}{(\Omega_k\phi w)(x_k + h_k)} \frac{1}{2h_k} \int_{x_k - h_k}^{x_k + h_k} q(x)w(x)\,dx. \tag{4.2}$$

Proof. Let $p \in \mathcal{P}_{2n-2}$. Note that in the case $h_k \neq 0$, equation $\Delta_k(\Omega_k \phi w) = 0$ means

$$(\Omega_k \phi w)(x_k + h_k) = (\Omega_k \phi w)(x_k - h_k).$$

If we assume all $h_k \neq 0$, $k = 1, \ldots, n$, and under the condition (4.1), the equality from the previous theorem can be transformed to

$$\begin{aligned} 0 &= \sum_{k=1}^{n} \frac{1}{2h_k} \left[\frac{(p\phi w)(x_k + h_k)}{(\Omega_k \phi w)(x_k + h_k)} - \frac{(p\phi w)(x_k - h_k)}{(\Omega_k \phi w)(x_k - h_k)} \right] \\ &= \sum_{k=1}^{n} \frac{1}{2h_k} \frac{(p\phi w)(x_k + h_k) - (p\phi w)(x_k - h_k)}{(\Omega_k \phi w)(x_k + h_k)} \\ &= \sum_{k=1}^{n} \frac{1}{(\Omega_k \phi w)(x_k + h_k)} \frac{1}{2h_k} \int_{x_k - h_k}^{x_k + h_k} (p\phi w)' dx \\ &= \sum_{k=1}^{n} \frac{1}{(\Omega_k \phi w)(x_k + h_k)} \frac{1}{2h_k} \int_{x_k - h_k}^{x_k + h_k} (p'\phi + p\psi) w \, dx. \end{aligned}$$

If $h_k = 0$, the respective term which appears in the previous sum can be transformed, using the condition (4.1), as follows

$$\begin{aligned} \frac{(p'\,\Omega_k - p\,\Omega_k')(x_k)}{(\Omega_k(x_k))^2} &= \frac{(p\phi w)'(x_k)(\Omega_k \phi w)(x_k) - (p\phi w)(x_k)(\Omega_k \phi w)'(x_k)}{(\Omega_k \phi w)^2(x_k)} \\ &= \frac{(p\phi w)'(x_k)}{(\Omega_k \phi w)(x_k)} = \frac{1}{(\Omega_k \phi w)(x_k)} \frac{1}{2h_k} \int_{x_k - h_k}^{x_k + h_k} (p\phi w)' \, dx, \end{aligned}$$

where we used the convention (3.3). As we can see the final expression is the same regardless if $h_k = 0$ or $h_k \neq 0$.

On the other hand,

$$\int_{-1}^{1} (p'\phi + p\psi) w \, dx = \int_{-1}^{1} (p\phi w)' dx = (p\phi w)(1) - (p\phi w)(-1) = 0.$$

Thus, we conclude that, for each $q \in \mathcal{L} \equiv \{p'\phi + p\psi \mid p \in \mathcal{P}_{2n-2}\}$ and any constant C,

$$\int_{-1}^{1} q(x) w(x) \, dx = C \sum_{k=1}^{n} \frac{1}{(\Omega' \phi w)(x_k + h_k)} \int_{x_k - h_k}^{x_k + h_k} q(x) w(x) \, dx = 0. \quad (4.3)$$

Note that $\mathcal{L}$ is a linear subspace of $\mathcal{P}_{2n-1}$, with dimension $2n - 1$. Taking monomials $p = x^{\nu-1}$, $\nu = 1, \ldots, 2n - 1$, one basis in $\mathcal{L}$ can be chosen as $\ell_\nu = (\nu - 1)x^{\nu-2}\phi + x^{\nu-1}\psi$, $\nu = 1, \ldots, 2n - 1$, i.e.,

$$\ell_\nu = -(\nu + \alpha + \beta + 1)x^\nu + (\beta - \alpha)x^{\nu-1} + (\nu - 1)x^{\nu-2}, \quad \nu = 1, \ldots, 2n - 1,$$

since $\nu + \alpha + \beta + 1 > 0$, $\deg(\ell_\nu) = \nu$, and ℓ_ν, $\nu = 1, \ldots, 2n - 1$, are linearly independent.

The constant C in (4.3) can be chosen such that (4.2) holds also for $q(x) = 1$, because

$$(\Omega_k \phi w)(x_k + h_k) > 0, \qquad k = 1, \dots, n,$$

and

$$\mu_0 = \int_{-1}^{1} w(x)\,dx, \;\; g_k = (\mathcal{B}_{h_k}^{w} 1)(x_k) = \frac{1}{2h_k} \int_{x_k - h_k}^{x_k + h_k} w(x)\,dx > 0, \;\; k = 1, \dots, n,$$

where the operator $\mathcal{B}_h^w$ is defined by (3.5). Therefore,

$$C = \frac{\mu_0}{\displaystyle\sum_{k=1}^{n} \frac{g_k}{(\Omega_k \phi w)(x_k + h_k)}} > 0. \tag{4.4}$$

Taking $\ell_0 = 1$, we can check easily that the set of polynomials $B = \{\ell_0, \ell_1, \dots, \ell_{2n-1}\}$ is linearly independent, so that $\operatorname{span}(B) = \mathcal{P}_{2n-1}$. Thus, (4.2), with the constant C as in (4.4), holds for every $q \in \mathcal{P}_{2n-1}$. □

Theorem 7. *The distribution* $-1 < x_1 - h_1,\ x_k + h_k < x_{k+1} - h_{k+1},\ k = 1, \dots, n,\ x_n + h_n < 1$, *is a solution of the system of equations* (4.1) *if and only if it defines the Gaussian interval quadrature rule for the Jacobi weight* w.

Proof. Suppose that (3.8) is the Gaussian interval quadrature rule for the Jacobi weight w, i.e.,

$$\int_{-1}^{1} p(x) w(x)\,dx = \sum_{k=1}^{n} \frac{\sigma_k}{2h_k} \int_{x_k - h_k}^{x_k + h_k} p(x) w(x)\,dx,$$

for each $p \in \mathcal{P}_{2n-1}$. Choose $P \in \mathcal{P}_{2n-1}$ such that

$$Pw = (\Omega_m' \phi + \Omega_m \psi) w = (\Omega_m \phi w)', \qquad m = 1, \dots, n.$$

Then

$$\begin{aligned} \int_{-1}^{1} (\Omega_m' \phi + \Omega_m \psi)(x) w(x)\,dx &= \int_{-1}^{1} (\Omega_m \phi w)'(x)\,dx \\ &= (\Omega_m \phi w)(1) - (\Omega_m \phi w)(-1) \\ &= 0, \end{aligned}$$

as well as

$$\begin{aligned} 0 = \sum_{k=1}^{n} \frac{\sigma_k}{2h_k} \int_{x_k - h_k}^{x_k + h_k} (\Omega_m \phi w)'(x)\,dx &= \sum_{k=1}^{n} \frac{\sigma_k}{2h_k} \Delta_k (\Omega_m \phi w) \\ &= \frac{\sigma_m}{2h_m} \Delta_m (\Omega_m \phi w), \qquad m = 1, \dots, n. \end{aligned}$$

Since $\sigma_m \neq 0$, we have $\Delta_m(\Omega_m \phi w) = 0$, $m = 1, \ldots, n$. □

Finally, it is easy to recognize the equations

$$\Delta_m(\Omega_m \phi w) = 0, \qquad m = 1, \ldots, n, \tag{4.5}$$

as the one already being used in [23] for the numerical construction of the nodes in the Gaussian interval quadrature rule. The real benefit are expressions for the weight coefficients. Combining (4.4) and (4.2), we can express the weight coefficients as follows

$$\sigma_\nu = \frac{\mu_0}{(\Omega_\nu \phi w)(x_\nu + h_\nu)} \left(\sum_{k=1}^{n} \frac{g_k}{(\Omega_k \phi w)(x_k + h_k)} \right)^{-1}, \qquad \nu = 1, \ldots, n, \tag{4.6}$$

where

$$g_k = \frac{1}{2h_k} \int_{x_k - h_k}^{x_k + h_k} w(x)\, dx, \qquad k = 1, \ldots, n.$$

According to the uniqueness property already presented in [23], (4.6) presents the weight coefficients in the Gauss interval quadrature formula for the Jacobi weight function.

It is important to note that in the case $h_k = 0$, according to the continuity argument,

$$g_k\big|_{h_k=0} = \lim_{h_k \to 0} \frac{1}{2h_k} \int_{x_k - h_k}^{x_k + h_k} w(x)\, dx = w(x_k),$$

where we know that the nodes x_k have to be inside $(-1, 1)$ if the interval quadrature formula is going to be Gaussian, i.e., with maximal algebraic degree of exactness (see [23]).

Numerical construction should be performed in the same way it is proposed in [23], as far as the construction of nodes is considered. As it is shown in [23], the solution of the system (4.5) depends continuously on the lengths h_k, $k = 1, \ldots, n$. This means if we start with a known solution $x_k^{(0)}$, $k = 1, \ldots, n$, for some known set of lengths $h_k^{(0)}$, $k = 1, \ldots, n$, we can modify slightly the system of lengths and to expect the convergence of the Newton-Kantorovich method applied to the system of equations (4.5). Of course, there are at least two known solutions. First, when all lengths are zero, then it is the classical Gauss-Jacobi quadrature rule, and the second one when the sum of all lengths equals 2, i.e, when the subintervals cover the whole interval $[-1, 1]$. In the latter case, the solutions are obviously known and are given by

$$x_k^{(0)} = -1 + h_k^{(0)} + 2 \sum_{\nu=1}^{k-1} h_\nu^{(0)}, \qquad k = 1, \ldots, n. \tag{4.7}$$

Of course, we cannot apply the previous starting values if either $h_1^{(0)}$ or $h_n^{(0)}$ is zero, i.e., the method is safe only in the case when $h_1^{(0)} \neq 0$ and $h_n^{(0)} \neq 0$.

We can summarize the previous facts in the following algorithm:

1° *Using QR-algorithm, construct the classical Gauss-Jacobi quadrature rule or use equations* (4.7) *to get the starting values* $x_k^{(0)}$, $k = 1, \dots, n$, *denote the starting lengths* $h_k^{(0)}$, $k = 1, \dots, n$.

2° *Modify the lengths for some small amounts and solve* (4.5) *for the modified lengths. If during computations some solution goes out of* $[-1, 1]$ *or if there is overlapping between the intervals, the process should start again with a smaller modification of lengths.*

3° *If a desired lengths are reached, stop; if it is not go back to the step* 2°.

Of course, this construction is a painful process. However, there is some advantage if we start with one or another starting values.

When the nodes are constructed, we can calculate the weight safely using equations (4.6). Note that all the terms which appear in the formula for the weight are positive, since

$$(\Omega_k \phi w)(x_k + h_k) = (\phi w)(x_k + h_k) \prod_{\nu \neq k} (x_\nu + h_\nu - x_k - h_k)(x_\nu - h_\nu - x_k - h_k) > 0,$$

for each $k = 1, \dots, n$, according to the interlacing property of the solution for the nodes $-1 < x_1 - h_1$, $x_k + h_k < x_{k+1} - h_{k+1}$, $k = 1, \dots, n-1$, $x_n + h_n < 1$. This means that the formula is numerically perfect provided we can calculate safely quantities $(\Omega_k \phi w)(x_k + h_k)$, $k = 1, \dots, n$. In the case $x_{k+1} - h_{k+1} - x_k - h_k$ is not too small, we can say that the calculation is stable, but in general provided it is of order 10^{-r}, $r > 0$, we should be ready to lose r digits precision in calculations. This effect is not cumulative, as we know product of two numbers has the precision of the one with a smaller precision.

Nodes	Weights
−0.9513504367349746	0.9041875921607397(−1)
−0.8231056754448114	0.1642358982486300
−0.6282572988861806	0.2219171495928649
−0.3895449034466384	0.2467619011625343
−0.9834656451885066(−1)	0.3340427418096471
0.2315417963862776	0.3294578645377376
0.5116759367555432	0.2287997969490443
0.7270053532993432	0.1935137389125616
0.8893294771360129	0.1281503510913336
0.9801709253569534	0.5493199877807465(−1)

Table 1: Nodes and weights in the Gaussian interval quadrature formula for a set of lengths 2^{-7}, 2^{-7}, 2^{-7}, 2^{-7}, 2^{-3}, 2^{-3}, 2^{-6}, 2^{-6}, 2^{-6}, 2^{-6}, for the weight $w(x) = (1-x)^{-1/3}(1+x)^{2/3}$.

Finally, we give an example. Table 1 presents results for the case of the weight $w(x) = (1-x)^{-1/3}(1+x)^{2/3}$. The sets of nodes and weights are constructed for the system of lengths 2^{-7}, 2^{-7}, 2^{-7}, 2^{-7}, 2^{-3}, 2^{-3}, 2^{-6}, 2^{-6}, 2^{-6}, 2^{-6}.

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