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Gaussian quadrature rules using function derivatives

GRADIMIR V. MILOVANOVIĆ AND ALEKSANDAR S. CVETKOVIĆ
Faculty of Computer Sciences, Megatrend University Belgrade,

Bulevar umetnosti 29, 11070 Novi Beograd, Serbia.

Faculty of Sciences and Mathematics, University of Niš,

P.O. Box 224, 18000 Niš, Serbia.

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Abstract: For finite positive Borel measures supported on the real line we consider a new type of quadrature rule with maximal algebraic degree of exactness, which involves function derivatives. We prove the existence of such quadrature rules and describe their basic properties. Also, we give an application of these quadrature rules to the solution of a Cauchy problem without solving it directly. Numerical examples are included as well.

Keywords: quadrature rule; maximal algebraic degree of exactness; Borel measures; weight function; quadrature rules involving function derivatives; nodes; weight coefficients.

1. Introduction

Let $d\mu$ be a finite positive Borel measure on the real line such that its support $\mathrm{supp}(d\mu)$ is an unbounded set, and all its moments $\mu_k = \int_{\mathbb{R}} x^k \, d\mu(x)$, $k = 0, 1, \ldots$, exist and are finite. Let $\mathscr{P}_m$ be the set of all algebraic polynomials of degree at most m $(\in \mathbb{N}_0)$. The n-point quadrature formula of the highest degree of precision is the well-known Gaussian quadrature

$$\int_{\mathbb{R}} f(x)\, d\mu(x) = \sum_{k=1}^{n} w_k f(x_k) + R_n(f), \tag{1.1}$$

which is exact on the set $\mathscr{P}_{2n-1}$, i.e., $R_n(f) = 0$ for each $f \in \mathscr{P}_{2n-1}$. The nodes x_k, $k = 1, \ldots, n$, are the eigenvalues of a symmetric tridiagonal (so-called Jacobi) matrix $J_n(d\mu)$, whose elements are formed from the coefficients in the three-term recurrence relation for the monic polynomials $\{\pi_n(d\mu;\cdot)\}_{n\in\mathbb{N}_0}$, orthogonal with respect to the inner product

$$(f,g)_{d\mu} = \int_{\mathbb{R}} f(x) g(x)\, d\mu(x) \quad (f, g \in L^2(d\mu)). \tag{1.2}$$

The weight coefficients (Cotes-Christoffel numbers) w_k in (1.1) can be expressed in terms of the first components of the corresponding (normalized) eigenvectors $\mathbf{v}_k$ $(= [v_{k,1}\ v_{k,2}\ \ldots\ v_{k,n}]^T)$, $\mathbf{v}_k^T \mathbf{v}_k = 1$, $k = 1, \ldots, n$. In other words, x_k are zeros of the orthogonal polynomial $\pi_n(d\mu;\cdot)$ of degree n, and $w_k = \mu_0 v_{k,1}^2$, $k = 1, \ldots, n$ (see Golub & Welsch (1969)).

Gaussian quadrature formulae were generalized in several ways. In the middle of the last century, the idea of numerical integration involving multiple nodes was put forward, i.e.,

$$\int_{\mathbb{R}} f(x)\, d\mu(x) = \sum_{k=1}^{n} \sum_{i=0}^{2s_k} w_{k,i} f^{(i)}(x_k) + R(f),$$

together with a general concept of *power orthogonality* (see Chakalov (1948, 1954, 1957), Turán (1950), Popoviciu (1955), Ghizzetti & Ossicini (1970, 1975), etc.). A survey of such quadratures was given in Milovanović (2001). An efficient algorithm for constructing quadrature formulas with multiple Gaussian nodes in the presence of certain fixed nodes was recently presented by Milovanović, Spalević & Cvetković (2004) (see also Shi & Xu (2007)).

Further extensions dealing with quadratures with multiple nodes for ET (Extended Tschebycheff) systems were given by Karlin & Pinkus (1976), Barrow (1978), Bojanov, Braess & Dyn (1986), Bojanov (1997), etc. Recently, a method for constructing generalized Gaussian quadrature rules for Müntz polynomials on $(0,1)$ has been given by Milovanović & Cvetković (2005).

Another type of quadrature with multiple nodes are the so-called Birkhoff quadratures. Roughly speaking their quadrature sums do not include all derivatives. We mention here only a very special case of Birkhoff quadrature, the generalized $(0,m)$ quadrature problem

$$\int_{\mathbb{R}} f(x)\,d\mu(x) = \sum_{k=1}^{n}\Big[v_k f(x_k) + w_k f^{(m)}(x_k)\Big] + R(f) \tag{1.3}$$

of highest degree of precision, which was first stated in 1974 by Turán for $d\mu(x)=dx$ on $[-1,1]$, $m=2$, and with nodes taken as the zeros of the polynomial $\mathit{\Pi}_n(x) := (1-x^2)P'_{n-1}(x)$, where P_k is the Legendre polynomial of degree k (cf. Turán (1980)). For some particular results concerning (1.3) see Varma (1986), Dimitrov (1991), Milovanović & Varma (1997), Lénárd (2003).

Recently Masjed-Jamei (2007) proposed the idea of quadrature rules of the form

$$\int_{\alpha}^{\beta} w(x) f(x)\,dx = \sum_{k=1}^{n} w_k f^{(m)}(x_k) + R_{n,m}(f),$$

where w is a positive function and $f^{(\nu)}(\lambda)=0$ at some point $\lambda\in\mathbb{R}$ for $\nu=0,1,\ldots,m-1$.

Using this idea, for a restricted class of m-times continuously differentiable functions on $A\subset\mathbb{R}$, denoted by $C^m(\lambda;A)$, where $\lambda\in A$ and $\operatorname{supp}(d\mu)\subset A$, in this paper we consider Gaussian quadrature formulae of the form

$$\int_{\mathbb{R}} f(x)\,d\mu(x) = \sum_{k=1}^{n} w_k f^{(m)}(x_k) + R_{n,m}(f). \tag{1.4}$$

The restriction referred to in the previous sentence is that $f\in C^m(\lambda;A)$ has a zero of order m at $x=\lambda$, i.e., $f(x)=(x-\lambda)^m g(x)$. In this way we wish to construct the formula (1.4), which is exact for all algebraic polynomials of degree at most $2n+m-1$. The resulting quadrature formula will then be equivalent to the Gaussian formula

$$\int_{\mathbb{R}} g(x)(x-\lambda)^m\,d\mu(x) = \sum_{k=1}^{n} w_k \frac{d^m}{dx^m}\left(\frac{g(x)}{(x-\lambda)^m}\right)\bigg|_{x=x_k} \qquad (g\in\mathscr{P}_{2n-1}).$$

Although this formula resembles (1.3) for $v_k=0$, it is quite different from (1.4) and its construction and investigation require different tools.

To circumvent the conditions $f(\lambda)=f'(\lambda)=\cdots=f^{(m-1)}(\lambda)=0$, instead of (1.4), we can equivalently study the Gaussian rule

$$\int_{\mathbb{R}} \big(f(x)-P_{m-1}(x)\big)\,d\mu(x) = \sum_{k=1}^{n} w_k f^{(m)}(x_k) + R_{n,m}(f), \tag{1.5}$$

where $P_{m-1}(x) = P_{m-1}(f;x)$ is the Taylor polynomial of the function $f \in C^m(A)$ at a fixed point $x = \lambda \in A$ defined by

$$P_{m-1}(x;f) = \sum_{\nu=0}^{m-1} \frac{f^{(\nu)}(\lambda)}{\nu!}(x-\lambda)^\nu, \quad m \in \mathbb{N}.$$

An additional motivation for this type of quadrature comes also from its applications to initial-value problems for ordinary differential equations.

During the process of review of our manuscript, a paper of Welfert (2008) on the same subject was submitted and published. There are however some substantial differences between that and the present paper. First, there is a difference concerning the scope of the results. Our paper deals with positive Borel measures while the paper of Welfert (2008) is concerned only with positive weight functions supported on the interval (α,β). In Welfert (2008) there are only a few examples regarding the cases $m=1$ and $\lambda=\alpha$; also, the weight functions considered are only the simplest examples of the classical weight functions. In contrast, in this paper all classical weights are discussed, with an arbitrary position of $\lambda \in \mathbb{R}$, together with the most relevant examples of non-classical measures. Also, Welfert (2008) is concerned only with the definite case, and the non-definite case is not elaborated on at all. Finally, the numerical examples in Welfert (2008) involve only constructions in a very few points ($n=2$ or $n=3$).

This paper is organized as follows. In Section 2 we give some preliminary and auxiliary results. The main result, some properties of the quadrature rules (1.5), as well as a method for their construction, are stated in Section 3. Special cases are analyzed in Section 4. Finally, numerical examples are given in the last section.

2. Preliminary and auxiliary results

For an arbitrary $\lambda \in \mathbb{R}$ we introduce a subset of $\mathscr{P}_{2n+m-1}$ by

$$\mathscr{P}_{2n-1}^{\lambda,m} = \Big\{ p \;\Big|\; p \in \mathscr{P}_{2n+m-1},\ p^{(k)}(\lambda) = 0,\ k = 0,1,\ldots,m-1 \Big\}. \tag{2.1}$$

An element $p \in \mathscr{P}_{2n-1}^{\lambda,m}$ can be expressed in the form $p(x) = (x-\lambda)^m q(x)$, where $q \in \mathscr{P}_{2n-1}$, or as

$$p(x) = \frac{1}{(m-1)!}\int_\lambda^x (x-t)^{m-1} p^{(m)}(t)\,dt. \tag{2.2}$$

The last expression can be obtained using Taylor's formula for $p \in \mathscr{P}_{2n+m-1}$ at $x=\lambda$, with the remainder term in Cauchy's form,

$$p(x) = \sum_{\nu=0}^{m-1} \frac{p^{(\nu)}(\lambda)}{\nu!}(x-\lambda)^\nu + \frac{1}{(m-1)!}\int_\lambda^x (x-t)^{m-1} p^{(m)}(t)\,dt,$$

which reduces to (2.2) if $p \in \mathscr{P}_{2n-1}^{\lambda,m}$.

Also, for given $\lambda, x \in \mathbb{R}$ we introduce an auxiliary function $t \mapsto \psi(\lambda,x;t)$ by

$$\psi(\lambda,x;t) = \begin{cases} \chi_{(\lambda,x)}(t), & \lambda \leqslant x, \\ -\chi_{(x,\lambda)}(t), & \lambda > x, \end{cases} \tag{2.3}$$

where χ_A is the characteristic function of the set A.

LEMMA 2.1 For each $p \in \mathscr{P}_{2n-1}^{\lambda,m}$ ($\lambda \in \mathbb{R},\ n,m \in \mathbb{N}$) we have the following transformation

$$\int_{\mathbb{R}} p(x)\,d\mu(x) = \int_{\mathbb{R}} p^{(m)}(t)w(t)\,dt, \tag{2.4}$$

where

$$w(t) = \frac{1}{(m-1)!}\int_{\mathbb{R}} (x-t)^{m-1}\psi(\lambda,x;t)\,d\mu(x). \tag{2.5}$$

Proof. Let $\psi(\lambda,x;t)$ be defined as in (2.3). Integrating (2.2) with respect to the measure $d\mu(x)$ and using Fubini's theorem (cf. Lieb & Loss (1997, p. 25)) we can exchange the order of integration, so that we have

$$\begin{aligned}
\int_{\mathbb{R}} p(x)\,d\mu(x) &= \int_{\mathbb{R}} d\mu(x)\frac{1}{(m-1)!}\int_{\lambda}^{x}(x-t)^{m-1}p^{(m)}(t)\,dt\\
&= \int_{\mathbb{R}} d\mu(x)\frac{1}{(m-1)!}\int_{\mathbb{R}}(x-t)^{m-1}\psi(\lambda,x;t)p^{(m)}(t)\,dt\\
&= \frac{1}{(m-1)!}\int_{\mathbb{R}} p^{(m)}(t)\,dt\int_{\mathbb{R}}(x-t)^{m-1}\psi(\lambda,x;t)\,d\mu(x),
\end{aligned}$$

which is (2.4), with $w(t)$ given by (2.5). □

Let $a = \inf\ \mathrm{Co}(\mathrm{supp}(d\mu))$ and $b = \sup\ \mathrm{Co}(\mathrm{supp}(d\mu))$, where $\mathrm{Co}(\mathrm{supp}(d\mu))$ denotes the convex hull of the supporting set of the measure $d\mu$. In the following lemmas we give some properties of the function w, defined by (2.5).

In the sequel we consider only nontrivial cases, which are different from the simple case when $a = b = \lambda$.

LEMMA 2.2 (a) If m is an even integer, then for every $\lambda \in \mathbb{R}$ the function w is a nonnegative weight function supported on the set

$$\mathrm{supp}(w) = \mathrm{Co}(\mathrm{supp}(d\mu)) \cup (a,\lambda] \cup [\lambda,b).$$

(b) If m is an odd integer, then for the function w we have:

- for $\lambda \leqslant a$, w is positive on (λ,b) and is supported on the set $\mathrm{Co}(\mathrm{supp}(d\mu)) \cup [\lambda,b)$;
- for $\lambda \in \mathrm{Co}(\mathrm{supp}(d\mu))$, $w(x) < 0$ on (a,λ) and $w(x) > 0$ on (λ,b), and is supported on the set $\mathrm{Co}(\mathrm{supp}(d\mu))$;
- for $\lambda \geqslant b$, w is negative on (a,λ) and is supported on the set $\mathrm{Co}(\mathrm{supp}(d\mu)) \cup (a,\lambda]$.

Proof. The point is to investigate the integral (2.5). Let us suppose first that $\lambda \leqslant a$. Then it can be easily shown that, for $x \in \mathrm{Co}(\mathrm{supp}(d\mu))$, we have

$$\psi(\lambda,x;t) = \chi_{(\lambda,x)}(t) = \chi_{(t,+\infty)}(x)\chi_{(\lambda,+\infty)}(t),$$

so that (2.5) becomes

$$\begin{aligned}
w(t) &= \frac{1}{(m-1)!}\int_{\mathbb{R}}(x-t)^{m-1}\chi_{(t,+\infty)}(x)\chi_{(\lambda,+\infty)}(t)\,d\mu(x)\\
&= \frac{\chi_{(\lambda,+\infty)}(t)}{(m-1)!}\int_{\mathbb{R}}(x-t)^{m-1}\chi_{(t,+\infty)}(x)\chi_{\mathrm{Co}(\mathrm{supp}(d\mu))}(x)\,d\mu(x)\\
&= \frac{\chi_{(\lambda,+\infty)}(t)}{(m-1)!}\int_{\mathbb{R}}(x-t)^{m-1}\chi_{(t,+\infty)\cap\mathrm{Co}(\mathrm{supp}(d\mu))}(x)\,d\mu(x).
\end{aligned} \tag{2.6}$$

We now clearly see that w is a positive weight function on (λ, b) and it is supported on the set $[\lambda, +\infty) \cap \mathrm{Co}(\mathrm{supp}(d\mu))$, regardless of m.

Consider now the case $\lambda \geqslant b$. It is easy to conclude that in this case, for $x \in \mathrm{Co}(\mathrm{supp}(d\mu))$, we have

$$\psi(\lambda, x; t) = -\chi_{(x,\lambda)}(t) = -\chi_{(-\infty,\lambda)}(t)\chi_{(-\infty,t)}(x),$$

so that

$$\begin{aligned} w(t) &= \frac{1}{(m-1)!}\int_{\mathbb{R}} (x-t)^{m-1}\psi(\lambda, x; t)\, d\mu(x) \\ &= -\frac{\chi_{(-\infty,\lambda)}(t)}{(m-1)!}\int_{\mathbb{R}} (x-t)^{m-1}\chi_{(-\infty,t)\cap \mathrm{Co}(\mathrm{supp}(d\mu))}(x)\, d\mu(x). \end{aligned} \tag{2.7}$$

Using this formula we conclude that the supporting set for w is the set $(-\infty, \lambda] \cap \mathrm{Co}(\mathrm{supp}(\mu))$. Also, it is checked directly that the integrand is positive on the set of integration, so that this function is positive on (a, λ), provided m is even, while in the case of odd m the function is negative.

Finally, we consider the case $\lambda \in (a, b)$. Then, for $x \in \mathrm{Co}(\mathrm{supp}(d\mu))$, we have

$$\psi(\lambda, x; t) = \begin{cases} \chi_{(t,+\infty)}(x)\chi_{\mathrm{Co}(\mathrm{supp}(d\mu))}(t), & \lambda < t, \\ -\chi_{(-\infty,t)}(x)\chi_{\mathrm{Co}(\mathrm{supp}(d\mu))}(t), & \lambda > t, \end{cases}$$

and therefore

$$\begin{aligned} w(t) &= \frac{1}{(m-1)!}\int_{\mathbb{R}} (x-t)^{m-1}\psi(\lambda, x; t)\, d\mu(x) \\ &= \frac{\chi_{\mathrm{Co}(\mathrm{supp}(d\mu))}(t)}{(m-1)!} \begin{cases} \int_{\mathbb{R}} (x-t)^{m-1}\chi_{(t,+\infty)}(x)\, d\mu(x), & \lambda < t, \\ -\int_{\mathbb{R}} (x-t)^{m-1}\chi_{(-\infty,t)}(x)\, d\mu(x), & \lambda > t. \end{cases} \end{aligned} \tag{2.8}$$

For $w(\lambda)$ we may choose an arbitrary value, for example, $w(\lambda) = 0$. Now, it is easy to check that in the case when m is even, the function w is positive on $(a, \lambda) \cup (\lambda, b)$ and is supported on the interval $\mathrm{Co}(\mathrm{supp}(d\mu))$. In the case when m is odd, the function w is negative on the interval (a, λ) and positive on the interval (λ, b), and is supported on $\mathrm{Co}(\mathrm{supp}(d\mu))$. □

In order to further elaborate on the properties of the quadrature rule (1.5) we need the concept of regularity of a complex measure $d\eta$.

Definition 2.1 The sequence of polynomials $\{p_n\}_{n\in\mathbb{N}_0}$ is said to be a sequence of (formal) orthogonal polynomials with respect to the complex measure $d\eta$ if and only if:

1) the degree of the polynomial p_n is n,

2) there exists a sequence of complex numbers $||p_n|| \neq 0$, $n \in \mathbb{N}_0$, such that for every $n, m \in \mathbb{N}_0$

$$\int_{\mathbb{R}} p_n p_m\, d\eta = ||p_n||^2 \delta_{n,m}.$$

A complex measure $d\eta$ is said to be regular if and only if there exists a sequence of (formal) orthogonal polynomials with respect to $d\eta$. If a regular measure $d\eta$ is absolutely continuous with respect to the Lebesgue measure, i.e., $d\eta(x) = v(x)\, dx$, then the function v is also regular.

It can be proved rather easily that a complex measure $d\eta$ is regular if and only if all of the associated Hankel determinants are different from zero, i.e.,

$$(\forall n \in \mathbb{N}) \quad H_n = \begin{vmatrix} \eta_0 & \eta_1 & \eta_2 & \cdots & \eta_{n-1} \\ \eta_1 & \eta_2 & \eta_3 & \cdots & \eta_n \\ \eta_2 & \eta_3 & \eta_4 & \cdots & \eta_{n+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \eta_{n-1} & \eta_n & \eta_{n+1} & \cdots & \eta_{2n-2} \end{vmatrix} \neq 0,$$

where η_k, $k \in \mathbb{N}_0$, is the corresponding sequence of moments with respect to the measure $d\eta$ (see Chihara (1978, p. 11)). In the special case of a positive measure $d\mu$ all of the Hankel determinants are greater than zero, and then the sequence of orthogonal polynomials does exist. Similarly, if the measure is negative, it is easy to show that $H_{2n-1} < 0$ and $H_{2n} > 0$ for each $n \in \mathbb{N}$. Thus, in this case, the corresponding sequence of orthogonal polynomials also exists.

According to Lemma 2.2, there is only one case in which we cannot claim the existence of orthogonal polynomials with respect to the measure $w(t)\,dt$, i.e., when $\lambda \in (a,b)$. However, in that case we can characterize the positions of the zeros of orthogonal polynomials with respect to the regular measure $w(t)\,dt$, where w is given by (2.5).

LEMMA 2.3 Let $\lambda \in (a,b)$ and assume that the measure $w(t)\,dt$ is regular, with the associated sequence of (formal) orthogonal polynomials $\{p_n\}_{n\in\mathbb{N}_0}$. Then for every $n \in \mathbb{N}$ at most one zero of the polynomial p_n is not contained in (a,b) and all zeros of the polynomial p_n are simple and real.

Proof. The crucial part in the proof is that the function w changes its sign only at the point λ. Assume that there are two zeros of p_n, e.g. x_1 and x_2, which are outside the interval (a,b) and that the zeros $x_k,\ldots,x_n$, $k > 2$, with odd multiplicities, are inside the interval (a,b).

We consider the integral

$$\int_a^b (x-\lambda)\left(\prod_{\ell=k}^{n}(x-x_\ell)\right) p_n(x)w(x)\,dx \tag{2.9}$$

and claim that it cannot be zero. Namely, it is easily seen that $(x-\lambda)w(x)$ does not change sign on (a,b); actually it is positive. Also $p_n(x)\prod_{\ell=k}^{n}(x-x_\ell)$ does not change its sign for $x \in (a,b)$, since all zeros with odd multiplicities of p_n, which are inside (a,b), are contained in the product, so that this product of two polynomials has only zeros with even multiplicities in (a,b). Thus, this consideration shows that the integral cannot be zero. However, since $(x-\lambda)\prod_{\ell=k}^{n}(x-x_\ell)$ is a polynomial of degree at most $n-1$, because of orthogonality, the integral (2.9) must be equal to zero, which is a contradiction.

Assume now that all zeros of p_n, with odd multiplicities, are inside (a,b) and are listed by $x_k,\ldots,x_n$, $k > 2$. Then consider the integral given in (2.9). Using the same arguments as above, we conclude that such an integral cannot be zero since the integrand does not change its sign on (a,b). This is again a contradiction to the orthogonality condition, since the polynomial in the product is at most of degree $n-1$. It does not produce a contradiction with the orthogonality condition if and only if $k = 2$, in which case the polynomial p_n has $n-1$ simple zeros, which implies that it has n simple zeros.

Since the polynomial p_n has n simple zeros and is a real polynomial with at least $n-1$ zeros contained in the interval (a,b), we must have that if a zero is not in (a,b) then it is real. $\square$

Definition 2.2 We say that a given complex measure $d\eta$ has the so-called GQR (Generalized Quadrature Rule) property if and only if for every $n \in \mathbb{N}_0$ there exist nodes $x_1,\ldots,x_n \in \mathbb{C}$ and weights $w_1,\ldots,w_n \in$

$\mathbb{C}$ such that for each $p \in \mathscr{P}_{2n-1}$ we have

$$\int_{\mathbb{R}} p\, d\eta = \sum_{k=1}^{n} w_k p(x_k).$$

Now, we are able to prove the following result:

LEMMA 2.4 For a given positive measure $d\mu$, let w be defined by (2.5) and let the corresponding measure $w(t)\,dt$ be regular. Then, for every $n \in \mathbb{N}$, it has the GQR property, with real nodes $x_1, \ldots, x_n$ and real weights $w_1, \ldots, w_n$.

Proof. Assume that the measure $d\eta(t) = w(t)\,dt$ is regular. Then there exists a sequence of polynomials $\{p_n\}_{n\in\mathbb{N}_0}$, orthogonal with respect to this measure $d\eta$. As we proved in Lemma 2.3, all zeros of p_n are simple and real. For a fixed $n \in \mathbb{N}$, we consider the interpolatory quadrature rule

$$\int_{\mathbb{R}} p\, d\eta = \sum_{k=1}^{n} w_k p(x_k), \quad p \in \mathscr{P}_{n-1},$$

where x_k, $k = 1, \ldots, n$, are zeros of the polynomial p_n. Such a quadrature rule always exists uniquely since the matrix of the linear system for computing the weights w_k, $k = 1, \ldots, n$, is the well-known Vandermonde matrix, which is always non-singular (see Horn & Johnson (1991, Section 6.1)).

In order to prove that this quadrature rule is exact on $\mathscr{P}_{2n-1}$, we take an arbitrary polynomial $p \in \mathscr{P}_{2n-1}$, which can be expressed uniquely as $p = qp_n + r$ $(q, r \in \mathscr{P}_{n-1})$, and integrate it with respect to the measure $d\eta(t) = w(t)\,dt$. Then, we get

$$\int_{\mathbb{R}} p\, d\eta = \int_{\mathbb{R}} qp_n\, d\eta + \int_{\mathbb{R}} r\, d\eta = \int_{\mathbb{R}} r\, d\eta = \sum_{k=1}^{n} w_k r(x_k) = \sum_{k=1}^{n} w_k p(x_k),$$

where we used the orthogonality $\int_{\mathbb{R}} qp_n\, d\eta = 0$ $(q \in \mathscr{P}_{n-1})$ and the fact that

$$p(x_k) = q(x_k)p_n(x_k) + r(x_k) = r(x_k), \quad k = 1, \ldots, n.$$

Thus, $w(t)\,dt$ has the GQR property. □

3. Main Result

Now we are able to state the main result.

THEOREM 3.1 Let $d\mu$ be a finite positive Borel measure on the real line such that all of its moments exist and are finite and let m be a positive integer. If the function w, given by (2.5), is regular, then the quadrature rule (1.5) exists uniquely with real nodes $x_1, \ldots, x_n$ and weights $w_1, \ldots, w_n$, which are parameters of the n-point Gaussian quadrature formula with respect to the measure $w(t)\,dt$, i.e.,

$$\int_{\mathbb{R}} g(t)w(t)\,dt = \sum_{k=1}^{n} w_k g(x_k) + R_n(g), \tag{3.1}$$

where $R_n(g) = 0$ for each $g \in \mathscr{P}_{2n-1}$.

Moreover, we have:

(i) If m is an even integer, all weights w_k are positive and all nodes x_k are contained in the interval $\mathrm{Co}(\mathrm{supp}(d\mu)) \cup (\lambda, b) \cup (a, \lambda)$, where $a = \inf\, \mathrm{Co}(\mathrm{supp}(d\mu))$ and $b = \sup\, \mathrm{Co}(\mathrm{supp}(d\mu))$;

(ii) If m is an odd integer, then, for $\lambda \leqslant a$ ($\lambda \geqslant b$), all weights w_k are positive (negative) and all nodes x_k are contained in the interval (λ, b) ((a, λ));

(iii) If m is an odd integer and $\lambda \in (a,b)$, there is at most one node outside the interval (a,b).

Proof. According to Lemma 2.1, we have

$$\int_{\mathbb{R}} p\,d\mu = \int_{\mathbb{R}} p^{(m)}(t)w(t)\,dt, \quad p \in \mathscr{P}_{2n-1}^{\lambda,m},$$

where the function w is given by (2.5). Assuming that the measure $w(t)\,dt$ is regular, according to Lemma 2.4, this measure has the GQR property. Hence, there exists the corresponding Gaussian quadrature rule (3.1), with nodes $x_1,\ldots,x_n$ and weights $w_1,\ldots,w_n$, such that

$$\int_{\mathbb{R}} p\,d\mu = \int_{\mathbb{R}} p^{(m)}(t)w(t)\,dt = \sum_{k=1}^{n} w_k p^{(m)}(x_k), \quad p \in \mathscr{P}_{2n-1}^{\lambda,m},$$

since $p^{(m)} \in \mathscr{P}_{2n-1}$.

According to Lemma 2.2 the statements (i) and (ii) immediately follow. Finally, the statement (iii) follows from Lemma 2.3. □

Now, we are able to derive a quadrature formula of the previous type for any polynomial $p \in \mathscr{P}_{2n+m-1}$.

THEOREM 3.2 Under the conditions of Theorem 3.1, for any $p \in \mathscr{P}_{2n+m-1}$, we have

$$\int_{\mathbb{R}} p\,d\mu = \sum_{k=0}^{m-1} \frac{p^{(k)}(\lambda)}{k!} \int_{\mathbb{R}} (x-\lambda)^k\,d\mu(x) + \sum_{\nu=1}^{n} w_\nu p^{(m)}(x_\nu), \tag{3.2}$$

where x_ν and w_ν, $\nu = 1,\ldots,n$, are the nodes and the weights of the quadrature rule (2.5).

Proof. It is easy to see that, for any $p \in \mathscr{P}_{2n+m-1}$, the polynomial

$$q(x) = p(x) - \sum_{k=0}^{m-1} \frac{p^{(k)}(\lambda)}{k!}(x-\lambda)^k$$

belongs to the space $\mathscr{P}_{2n-1}^{\lambda,m}$. Then our quadrature rule (1.5) is exact for this polynomial q, so that

$$\int_{\mathbb{R}} q\,d\mu = \sum_{\nu=1}^{n} w_\nu q^{(m)}(x_\nu).$$

If we substitute the expression for q we get what is stated. □

The following result is related to the convergence of the quadrature rules (1.5).

COROLLARY 3.1 Let $\operatorname{supp}(d\mu)$ be a bounded subset of the real line. If m $(\in \mathbb{N})$ is even, λ arbitrary and $f \in C^m(\operatorname{supp}(w))$, then

$$\int_{\mathbb{R}} f\,d\mu = \sum_{k=0}^{m-1} \frac{f^{(k)}(\lambda)}{k!} \int_{\mathbb{R}} (x-t)^k\,d\mu(x) + \lim_{n\to+\infty} \sum_{\nu=1}^{n} w_\nu f^{(m)}(x_\nu). \tag{3.3}$$

If m $(\in \mathbb{N})$ is odd, $\lambda \in \mathbb{R}\backslash\operatorname{Co}(\operatorname{supp}(\mu))$ and $f \in C^m(\operatorname{supp}(w))$, then (3.3) also holds.

Proof. Using Taylor's formula with the remainder term in Cauchy's form, we have

$$f(x)=\sum_{k=0}^{m-1}\frac{f^{(k)}(\lambda)}{k!}(x-\lambda)^k+\frac{1}{(m-1)!}\int_\lambda^x (x-t)^{m-1}f^{(m)}(t)\,dt.$$

Integrating with respect to the measure $d\mu$, we get

$$\int_{\mathbb{R}} f\,d\mu=\sum_{k=0}^{m-1}\frac{f^{(k)}(\lambda)}{k!}\int_{\mathbb{R}}(x-\lambda)^k\,d\mu(x)+\int_{\mathrm{supp}(w)} f^{(m)}(t)w(t)\,dt,$$

where w is given by (2.5). Applying the Gaussian quadrature rule for the weight w in the last integral we get

$$\int_{\mathrm{supp}(w)} f^{(m)}(t)w(t)\,dt\approx\sum_{\nu=1}^{n} w_\nu f^{(m)}(x_\nu).$$

According to the fact that $f^{(m)}\in C(\mathrm{supp}(w))$, the sequence of Gaussian quadrature rules for the weight w converges to the integral on the left, which finishes the proof. □

One possible application of the previous result is the following.

COROLLARY 3.2 Suppose that we are given the Cauchy problem

$$f^{(k)}(\lambda)=a_k,\quad k=0,1,\ldots,m-1,\quad f^{(m)}(x)=g(x),\quad x\in[\lambda,b],$$

with $g\in C[\lambda,b]$, $b\in\mathbb{R}$. Then, for $\mathrm{supp}(d\mu)\subset[\lambda,b]$, we have

$$\int_{\mathbb{R}} f\,d\mu=\sum_{k=0}^{m-1}\frac{a_k}{k!}\int_{\mathbb{R}}(x-\lambda)^k\,d\mu(x)+\lim_{n\to+\infty}\sum_{\nu=1}^{n} w_\nu g(x_\nu),$$

where x_ν and w_ν, $\nu=1,\ldots,n$, are the nodes and the weights of the Gaussian quadrature rule for the weight function (2.5).

Proof. Using Taylor's formula with the remainder term in Cauchy's form, we have

$$f(x)=\sum_{k=0}^{m-1}\frac{f^{(k)}(\lambda)}{k!}(x-\lambda)^k+\frac{1}{(m-1)!}\int_\lambda^x f^{(m)}(t)(x-t)^{m-1}\,dt,$$

i.e.,

$$\begin{aligned}\int_{\mathbb{R}} f\,d\mu &= \sum_{k=0}^{m-1}\frac{a_k}{k!}\int_{\mathbb{R}}(x-\lambda)^k\,d\mu(x)+\frac{1}{(m-1)!}\int_{\mathbb{R}}d\mu(x)\int_\lambda^x (x-t)^{m-1}g(t)\,dt\\ &= \sum_{k=0}^{m-1}\frac{a_k}{k!}\int_{\mathbb{R}}(x-\lambda)^k\,d\mu(x)+\int_\lambda^b g(t)w(t)\,dt.\end{aligned}$$

Thanks to the continuity of the function g we have

$$\int_\lambda^b g(t)w(t)dt=\lim_{n\to+\infty}\sum_{\nu=1}^{n} w_\nu g(x_\nu),$$

where x_ν and w_ν, $\nu=1,\ldots,n$, are nodes and weights of the Gaussian quadrature rule for the weight function w. □

4. Special cases

In the special case of certain Jacobi measures we can give an explicit expression for the function w. In this section we use the hypergeometric function ${}_2F_1\left(\begin{matrix} a,b \\ c \end{matrix}\ ;z\right)$ defined by the series

$$\sum_{n=0}^{+\infty} \frac{(a)_n (b)_n}{(c)_n} \cdot \frac{z^n}{n!} \tag{4.1}$$

for $|z| < 1$, and by continuation elsewhere. Here $(a)_n$ denotes the shifted factorial defined by $(a)_n = a(a+1)\ldots(a+n-1)$ for $n > 0$, $(a)_0 = 1$, where a is any real or complex number.

THEOREM 4.1 Let $d\mu(x) = (1-x)^\alpha (1+x)^\beta \chi_{[-1,1]}(x)\, dx$ be the Jacobi measure with the parameters $\alpha, \beta > -1$. Then for the function w, given by (2.5), we have:

(a) If $\lambda \leqslant -1$ and $\beta \in \mathbb{N}_0$, then

$$w(t) = \frac{\chi_{[\lambda,1]}(t)}{(m-1)!} \begin{cases} \beta!(1-t)^{m-1} 2^{\alpha+\beta+1} \displaystyle\sum_{\nu=0}^{m-1} \binom{m-1}{\nu} \left(\frac{-2}{1-t}\right)^\nu \frac{(\nu+\alpha)!}{(\nu+\alpha+\beta+1)!}, & \lambda < t < -1, \\ \alpha!(1-t)^{m+\alpha}(1+t)^\beta \displaystyle\sum_{\nu=0}^{\beta} \binom{\beta}{\nu} \left(\frac{1-t}{1+t}\right)^\nu \frac{(\nu+m-1)!}{(\nu+\alpha+m)!}, & t > -1; \end{cases}$$

(b) If $\lambda \in (-1,1)$ and $\alpha, \beta \in \mathbb{N}_0$, then

$$w(t) = \frac{\chi_{[-1,1]}(t)}{(m-1)!} \begin{cases} (-1)^m \beta!(1-t)^\alpha (1+t)^{m+\beta} \displaystyle\sum_{\nu=0}^{\alpha} \binom{\alpha}{\nu} \left(\frac{1+t}{1-t}\right)^\nu \frac{(\nu+m-1)!}{(\nu+\beta+m)!}, & t < \lambda, \\ \alpha!(1-t)^{m+\alpha}(1+t)^\beta \displaystyle\sum_{\nu=0}^{\beta} \binom{\beta}{\nu} \left(\frac{1-t}{1+t}\right)^\nu \frac{(\nu+m-1)!}{(\nu+\alpha+m)!}, & t > \lambda; \end{cases}$$

(c) If $\lambda \geqslant 1$ and $\alpha \in \mathbb{N}_0$, then

$$w(t) = \frac{(-1)^m \chi_{[-1,\lambda]}(t)}{(m-1)!} \begin{cases} \beta!(1-t)^\alpha (1+t)^{m+\beta} \displaystyle\sum_{\nu=0}^{\alpha} \binom{\alpha}{\nu} \left(\frac{1+t}{1-t}\right)^\nu \frac{(\nu+m-1)!}{(\nu+\beta+m)!}, & t < 1, \\ \alpha!(1+t)^{m-1} 2^{\alpha+\beta+1} \displaystyle\sum_{\nu=0}^{m-1} \binom{m-1}{\nu} \left(\frac{-2}{1+t}\right)^\nu \frac{(\nu+\beta)!}{(\nu+\alpha+\beta+1)!}, & 1 < t < \lambda. \end{cases}$$

Proof. First we prove the statement (b), i.e., the case when $\lambda \in (-1,1)$. Let $\lambda < t$. Then, using (2.8) and applying the substitution $u = (x-t)/(1-t)$, we have

$$\begin{aligned} w(t) &= \frac{\chi_{[-1,1]}(t)}{(m-1)!} \int_t^1 (x-t)^{m-1}(1-x)^\alpha (1+x)^\beta\, dx \\ &= \frac{\chi_{[-1,1]}(t)}{(m-1)!} (1-t)^{m+\alpha}(1+t)^\beta \int_0^1 u^{m-1}(1-u)^\alpha \left[1 - \left(-\frac{1-t}{1+t}\right) u\right]^\beta du. \end{aligned}$$

Using Euler's integral representation of the hypergeometric function ${}_2F_1$, defined by (see Andrews, Askey & Roy (1999, Theorem 2.2.1, p. 65))

$${}_2F_1\left(\begin{matrix} a,b \\ c \end{matrix}\ ;z\right) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 u^{b-1}(1-u)^{c-b-1}(1-zu)^{-a}\, du \qquad (\Re c > \Re b > 0),$$

for all z in the complex z-plane except for the cut along the real axis from 1 to ∞, we get

$$w(t) = \chi_{[-1,1]}(t)(1-t)^{m+\alpha}(1+t)^{\beta}\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+m+1)}\ {}_2F_1\left(\begin{matrix}-\beta, m\\ \alpha+m+1\end{matrix}; -\frac{1-t}{1+t}\right). \tag{4.2}$$

Now, thanks to the fact $\beta \in \mathbb{N}_0$, the hypergeometric series on the right-hand side terminates (see Andrews, Askey & Roy (1999, p. 67)), which in an expanded form (4.1) produces the desired result.

Consider now the case $t < \lambda$. According to (2.8) and (4.2), by the substitution $u = (t-x)/(1+t)$, we have

$$\begin{aligned} w(t) &= -\frac{\chi_{[-1,1]}(t)}{(m-1)!}\int_{-1}^{t}(x-t)^{m-1}(1-x)^{\alpha}(1+x)^{\beta}\,dx \\ &= (-1)^m\chi_{[-1,1]}(t)(1-t)^{\alpha}(1+t)^{m+\beta}\frac{\Gamma(\beta+1)}{\Gamma(\beta+m+1)}\ {}_2F_1\left(\begin{matrix}-\alpha, m\\ \beta+m+1\end{matrix}; -\frac{1+t}{1-t}\right), \end{aligned} \tag{4.3}$$

where again using the fact that $\alpha \in \mathbb{N}_0$, the hypergeometric series on the right-hand side terminates ($(-\alpha)_n = 0$ for $n \geqslant \alpha+1$) and produces the desired result.

(a) Consider the case $t \in (\lambda, -1)$, where $\lambda \leqslant -1$. According to (2.6) and using (4.2), we have

$$\begin{aligned} w(t) &= \frac{\chi_{[\lambda,1]}(t)}{(m-1)!}\int_{-1}^{1}(x-t)^{m-1}(1-x)^{\alpha}(1+x)^{\beta}\,dx \\ &= \frac{2^{\alpha+\beta+1}(1-t)^{m-1}\chi_{[\lambda,1]}(t)}{(m-1)!}\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}\ {}_2F_1\left(\begin{matrix}-m+1, \alpha+1\\ \alpha+\beta+2\end{matrix}; \frac{2}{1-t}\right), \end{aligned}$$

where we used the substitution $u = (1-x)/2$ in order to express our result in terms of a hypergeometric series. As we can see, the hypergeometric series terminates because $m \in \mathbb{N}$ ($(-m+1)_n = 0$ for $n \geqslant m$).

For $t \in (-1,1)$ we can use the expression (4.2) obtained earlier.

(c) Finally, consider the case $t \in (1,\lambda)$, where $\lambda \geqslant 1$. According to (2.7) and using (4.2), we have

$$\begin{aligned} w(t) &= -\frac{\chi_{[-1,\lambda]}(t)}{(m-1)!}\int_{-1}^{1}(x-t)^{m-1}(1-x)^{\alpha}(1+x)^{\beta}\,dx \\ &= \frac{2^{\alpha+\beta+1}(-1)^m(1+t)^{m-1}\chi_{[-1,\lambda]}(t)}{(m-1)!}\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}\ {}_2F_1\left(\begin{matrix}-m+1, \beta+1\\ \alpha+\beta+2\end{matrix}; \frac{2}{1+t}\right), \end{aligned}$$

where we used the substitution $u = (1+x)/2$ to express our result by a hypergeometric series. As we can see, the hypergeometric series terminates because $m \in \mathbb{N}$.

For the case $t \in (-1,1)$ we can use the expression (4.3) already obtained. □

The result becomes particularly simple for the Legendre measure ($\alpha = \beta = 0$). In that case, directly from (2.5), we get the following result.

COROLLARY 4.1 Let $d\mu(x) = \chi_{[-1,1]}(x)\,dx$ be the Legendre measure. Then for the function w, given by (2.5), we have:

(a) If $\lambda \leqslant -1$, then

$$w(t) = \frac{\chi_{[\lambda,1]}(t)}{m!}\Big[(1-t)^m - (\max\{-1,t\}-t)^m\Big];$$

(b) If $\lambda \in (-1,1)$, then

$$w(t) = \frac{\chi_{[-1,1]}(t)}{m!} \begin{cases} (-1-t)^m, & t < \lambda, \\ (1-t)^m, & t > \lambda; \end{cases}$$

(c) If $\lambda \geqslant 1$, then

$$w(t) = \frac{\chi_{[-1,\lambda]}(t)}{m!} \Big[(-1-t)^m - (\min\{1,t\} - t)^m \Big].$$

Proof. We consider only the case $\lambda \leqslant -1$. Using (2.6), we have

$$\begin{aligned} w(t) &= \frac{\chi_{[\lambda,1]}(t)}{(m-1)!} \begin{cases} \displaystyle\int_{\mathbb{R}} (x-t)^{m-1} \chi_{(t,+\infty)}(x) \chi_{[-1,1]}(x)\, dx, & \lambda < t, \\ \displaystyle -\int_{\mathbb{R}} (x-t)^{m-1} \chi_{(-\infty,t)}(x) \chi_{[-1,1]}(x)\, dx, & \lambda > t. \end{cases} \\ &= \frac{\chi_{[\lambda,1]}(t)}{(m-1)!} \int_{\max\{-1,t\}}^{1} (x-t)^{m-1}\, dx = \frac{\chi_{[\lambda,1]}(t)}{m!} \left[(1-t)^m - (\max\{-1,t\} - t)^m \right]. \end{aligned}$$

Similar proof holds for all other cases. □

In Fig. 1 and Fig. 2 we display the graph of the function $t \mapsto w(t)$ for different values of λ and m.

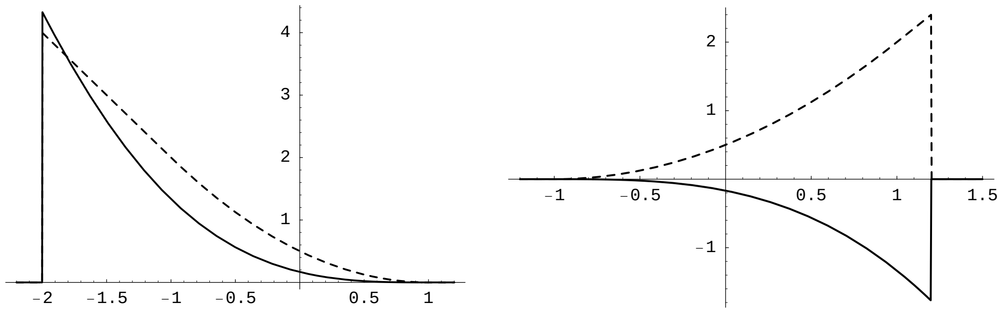

FIG. 1. The function $t \mapsto w(t)$ in the case of the Legendre measure for $m = 2$ (dashed line) and $m = 3$ (solid line), when $\lambda = -2$ (left) and $\lambda = 1.2$ (right)

For the generalized Laguerre measure $x^\alpha \mathrm{e}^{-x} \chi_{[0,+\infty)}(x)\, dx$, $\alpha \in \mathbb{N}_0$, we state the following result.

THEOREM 4.2 Let $d\mu$ be the generalized Laguerre measure $d\mu(x) = x^\alpha \mathrm{e}^{-x} \chi_{[0,+\infty)}(x)\, dx$, with $\lambda \leqslant 0$, $\alpha \in \mathbb{N}_0$, and $m \in \mathbb{N}$ arbitrary. Then

$$w(t) = \chi_{[\lambda,+\infty)}(t) \begin{cases} \displaystyle \mathrm{e}^{-t} \sum_{k=0}^{m-1} \frac{(m-k-1+\alpha)!}{(m-1-k)!k!} (-t)^k, & t \in (\lambda, 0), \\ \displaystyle \mathrm{e}^{-t} \sum_{k=0}^{\alpha} \frac{(m+k-1)!}{(m-1)!} \binom{\alpha}{k} t^{\alpha-k}, & t > 0. \end{cases} \tag{4.4}$$

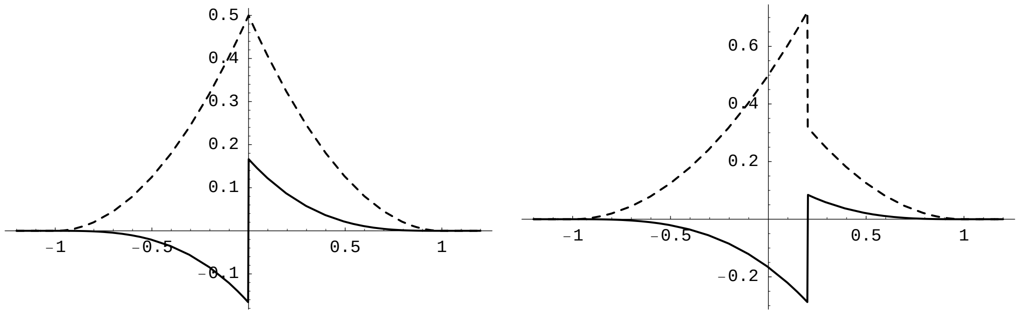

FIG. 2. The function $t \mapsto w(t)$ in the case of the Legendre measure for $m = 2$ (dashed line) and $m = 3$ (solid line), when $\lambda = 0$ (left) and $\lambda = 0.2$ (right)

Proof. Using (2.8) we have

$$\begin{aligned} w(t) &= \frac{\chi_{[\lambda,+\infty)}(t)}{(m-1)!} \int_0^{+\infty} (x-t)^{m-1} \chi_{(t,+\infty)}(x) \chi_{(0,+\infty)}(x) x^\alpha \mathrm{e}^{-x}\, dx \\ &= \frac{\chi_{[\lambda,+\infty)}(t)}{(m-1)!} \int_{\max\{0,t\}}^{+\infty} (x-t)^{m-1} (x-t+t)^\alpha \mathrm{e}^{-x} dx \\ &= \chi_{[\lambda,+\infty)}(t) \begin{cases} \displaystyle\sum_{k=0}^{m-1} \frac{1}{(m-k-1)!k!} (-t)^k \int_0^{+\infty} x^{m-k-1+\alpha} \mathrm{e}^{-x}\, dx, & t \in (\lambda, 0), \\ \displaystyle\frac{1}{(m-1)!} \sum_{k=0}^{\alpha} \binom{\alpha}{k} t^{\alpha-k} \int_t^{+\infty} (x-t)^{k+m-1} \mathrm{e}^{-x}\, dx, & t > 0, \end{cases} \end{aligned}$$

which reduces to (4.4). □

The result becomes particularly simple for the ordinary Laguerre measure $(\alpha = 0)$.

COROLLARY 4.2 Let $d\mu(x) = \mathrm{e}^{-x} \chi_{[0,+\infty)}(x)\, dx$, $\lambda = 0$ and let m be arbitrary. Then,

$$w(t) = \mathrm{e}^{-t} \chi_{[0,+\infty)}(t).$$

Hence, it follows that for the Laguerre measure and $\lambda = 0$, we have simply that the quadrature rule is unchanged when m increases and it is really the classical Gaussian quadrature rule for the Laguerre measure.

There is also an interesting result for the measure $\mathrm{e}^{-|x|} \chi_{\mathbb{R}}(x)\, dx$.

COROLLARY 4.3 Let $d\mu(x) = \mathrm{e}^{-|x|} \chi_{\mathbb{R}}(x)\, dx$ and $\lambda = 0$. Then, the function w in (2.5) is given by

$$w(x) = \begin{cases} (-1)^m \mathrm{e}^x, & x < 0, \\ \mathrm{e}^{-x}, & x > 0. \end{cases}$$

Proof. Using (2.8) with $\lambda = 0$, we conclude that $w(x) = (-1)^m w(-x)$, $x \in \mathbb{R}$. For $x > 0$, the weight w coincides with the weight alredy calculated for the Laguerre measure given in Corollary 4.2. □

There is also an interesting result for the measure with $\mathrm{supp}(d\mu) = \{-1, 1\}$ and $\mu(-1) = \mu(1) = 1$.

THEOREM 4.3 Let $\int_{\mathbb{R}} p\, d\mu = p(-1) + p(1)$. Then we have:

(a) For $\lambda < -1$,

$$w(t) = \frac{\chi_{[\lambda,1]}(t)}{(m-1)!} \begin{cases} (1-t)^{m-1} + (-1-t)^{m-1}, & \lambda < t < -1, \\ (1-t)^{m-1}, & -1 < t < 1; \end{cases}$$

(b) For $\lambda > 1$,

$$w(t) = -\frac{\chi[-1,\lambda](t)}{(m-1)!} \begin{cases} (-1-t)^{m-1}, & -1 < t < 1, \\ (-1-t)^{m-1} + (1-t)^{m-1}, & 1 < t < \lambda; \end{cases}$$

(c) For $\lambda \in [-1,1]$,

$$w(t) = \frac{\chi_{[-1,1]}(t)}{(m-1)!} \begin{cases} -(-1-t)^{m-1}, & -1 < t < \lambda, \\ (1-t)^{m-1}, & \lambda < t < 1. \end{cases}$$

Proof. The stated results follow by direct computation, using the results of Lemma 2.2. □

Inspecting this theorem,we see that for $\lambda = -1$ and $m = 1$ we get the weight function $\chi_{[-1,1]}$, which is the Legendre measure. In Figures 3, 4, and 5 we display the graphs of $w(t)$ from Theorem 4.3 for some selected values of λ and m.

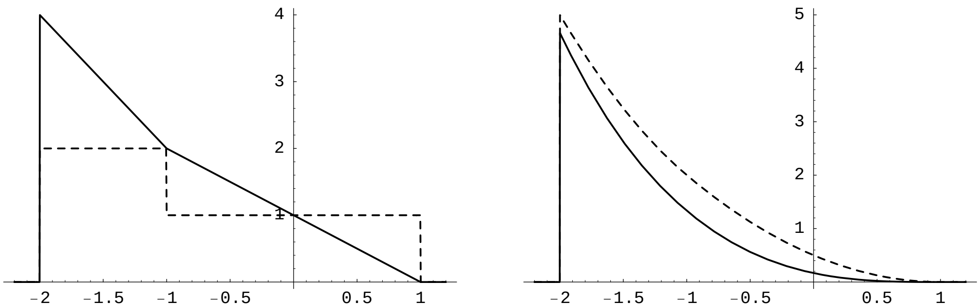

FIG. 3. The function $t \mapsto w(t)$ in Theorem 4.3 for $\lambda = -2$: Left figure $m = 1$ (dashed line) and $m = 2$ (solid line); Right figure $m = 3$ (dashed line) and $m = 4$ (solid line)

Finally, we give an explicit result for the measure supported on the set $\mathbb{N}_0$, with masses e^{-k} at the points $x = k$, $k \in \mathbb{N}_0$.

THEOREM 4.4 Let $d\mu$ be supported on the set $\mathbb{N}_0$, with mass e^{-k} at the point k $(\in \mathbb{N}_0)$. For $\lambda = 0$ and $m = 1$, the weight function w, defined in (2.5), is given by

$$w(t) = \frac{e^{-[t]}}{e-1} \chi_{[0,+\infty)}(t),$$

where $[t]$ is the integer part of t.

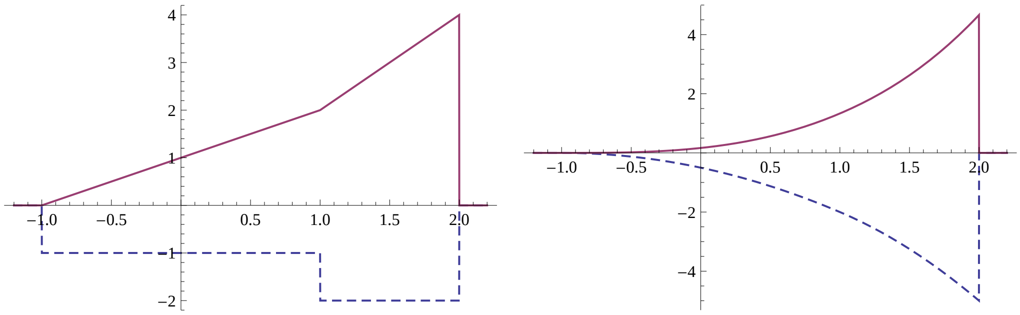

FIG. 4. The function $t \mapsto w(t)$ in Theorem 4.3 for $\lambda = 2$: Left figure $m = 1$ (dashed line) and $m = 2$ (solid line); Right figure $m = 3$ (dashed line) and $m = 4$ (solid line)

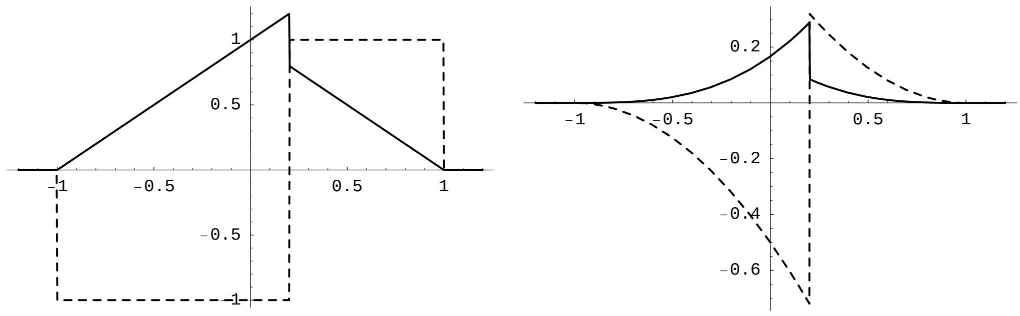

FIG. 5. The function $t \mapsto w(t)$ in Theorem 4.3 for $\lambda = 0.2$: Left figure $m = 1$ (dashed line) and $m = 2$ (solid line); Right figure $m = 3$ (dashed line) and $m = 4$ (solid line)

Proof. Using (2.6), we have

$$
\begin{aligned}
w(t) &= \chi_{[0,+\infty)}(t)\int_{\mathbb{R}} \chi_{(t,+\infty)}(x)\,d\mu(x) = \chi_{[0,+\infty)}(t)\sum_{k>t}^{+\infty} \mathrm{e}^{-k} \\
&= \chi_{[0,+\infty)}(t)\mathrm{e}^{-[t]-1}\sum_{k=0}^{+\infty} \mathrm{e}^{-k} = \chi_{[0,+\infty)}(t)\frac{\mathrm{e}^{-[t]-1}}{1-\mathrm{e}^{-1}} = \chi_{[0,+\infty)}(t)\frac{\mathrm{e}^{-[t]}}{\mathrm{e}-1}.
\end{aligned}
$$

□

5. Numerical examples

In this section we present three numerical examples.

EXAMPLE 5.1 We consider the existence of a quadrature rule with respect to the Chebyshev weight of the second kind:

$$
\int_{-1}^{1} p(x)\sqrt{1-x^2}\,dx = \sum_{\nu=1}^{n} w_\nu p^{(m)}(x_\nu), \quad p \in \mathscr{P}_{2n-1}^{\lambda,m}.
$$

If $\lambda \leqslant -1$, according to Theorem 3.1, the quadrature rule exists with positive weights w_ν, $\nu = 1,\ldots,n$, and with nodes contained in the interval $(\lambda,1)$. Choose, for example, $\lambda=-2$ and $m=3$. Then we have

$$\begin{aligned} w(t) &= \frac{\chi_{[\lambda,1]}(t)}{2!}\int_{\max\{-1,t\}}^{1}(x-t)^2\sqrt{1-x^2}\,dx \\ &= \chi_{[-2,1]}(t)\begin{cases} \dfrac{\pi}{16}(1+4t^2), & t\in(-2,-1),\\ \dfrac{\pi}{32}(1+4t^2)-\dfrac{1}{48}\left(t\sqrt{1-t^2}(13+2t^2)+3(1+4t^2)\arcsin t\right), & t\in[-1,1). \end{cases} \end{aligned}$$

The integral in the second branch of $w(t)$ when $t\in[-1,1)$ can be calculated using the substitution $x=\sin\varphi$, $\varphi\in(\arcsin t,\pi/2)$. The graph of this function is shown in Fig. 6 (left).

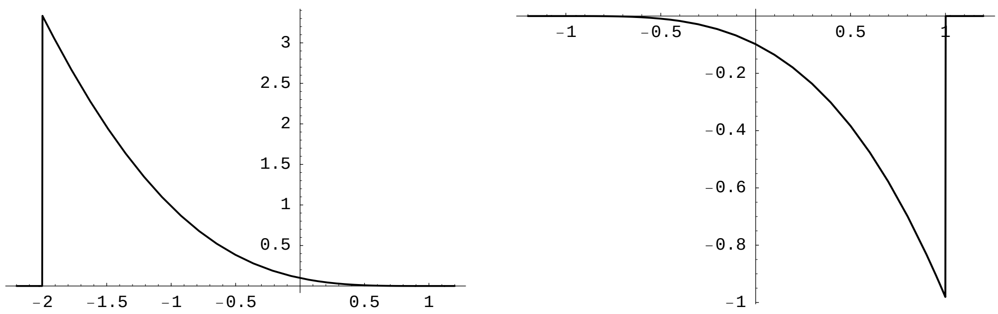

FIG. 6. The function $t\mapsto w(t)$ in the case of the Chebyshev measure of the second kind for $m=3$ and $\lambda=-2$ (left) and $\lambda=1$ (right)

Using the corresponding software for the numerical construction of Gaussian quadrature rules (see Cvetković & Milovanović (2004)), we are able to construct our quadrature rule and we present the results for $n=20$ in Table 1.

EXAMPLE 5.2 As before we consider the same measure with $\lambda=1$ and $m=3$. Then,

$$\begin{aligned} w(t) &= -\frac{\chi_{[-1,\lambda]}(t)}{2!}\int_{-1}^{\min\{1,t\}}(x-t)^2\sqrt{1-x^2}\,dx \\ &= -\frac{\chi_{[-1,1]}(t)}{96}\left[2t\sqrt{1-t^2}(13+2t^2)+3(1+4t^2)(\pi+2\arcsin(t)\right]. \end{aligned}$$

As we see directly by inspection, our weight function is negative on $(-1,1)$ (see Fig. 6 (right)). Table 2 contains the nodes and the weights of the quadrature rule (1.5) for this case.

EXAMPLE 5.3 Here we illustrate an application of Theorem 3.2. Consider the following Cauchy problem

$$f(-2)=f'(-2)=f''(-2)=0,\quad f'''(x)=\sin x. \tag{5.1}$$

Using numerical software (see Cvetković & Milovanović (2004)) we constructed quadrature rules for $n=5,10,15,20$ and obtained the following relative errors $1.8\times10^{-9}, 9.3\times10^{-23}, 3.9\times10^{-38}$ and

TABLE 1 *The nodes* x_ν *and weights* w_ν, $\nu = 1,\ldots,20$, *for the quadrature rule with* $m = 3$, $\lambda = -2$ *and* $d\mu(x) = \sqrt{1-x^2}\chi_{[-1,1]}(x)\,dx$

ν	x_ν	w_ν
1	−1.991086876748409	0.7563519052743393(−1)
2	−1.953242532290692	0.1683444846677203
3	−1.885990071707095	0.2438162358689971
4	−1.790720678320623	0.2950049352781838
5	−1.669419086762357	0.3184748642670166
6	−1.524602357280282	0.3146439234816484
7	−1.359253462117994	0.2875092121414554
8	−1.176738954492131	0.2438108114282358
9	−9.807133644862987(−1)	0.1917837747997080
10	−7.750572860202184(−1)	0.1396047614042675
11	−5.638568258155322(−1)	0.9361418736637676(−1)
12	−3.513019233023667(−1)	0.5738834354357883(−1)
13	−1.415990258816221(−1)	0.3180284668567313(−1)
14	6.111053029915573(−2)	0.1567744922341449(−1)
15	2.528311300770748(−1)	0.6716416518160149(−2)
16	4.297913946601915(−1)	0.2415211847657919(−2)
17	5.885203715054273(−1)	0.6901792356905729(−3)
18	7.259243143312154(−1)	0.1427314124417377(−3)
19	8.393823349996572(−1)	0.1778634334202285(−4)
20	9.270009273491675(−1)	8.3805091723525613(−7)

TABLE 2 *The nodes* x_ν *and weights* w_ν, $\nu = 1,\ldots,10$, *for the quadrature rule with* $m = 3$, $\lambda = 1$ *and* $d\mu(x) = \sqrt{1-x^2}\chi_{[-1,1]}(x)\,dx$

ν	x_ν	w_ν
1	−0.8418338174326530	−0.2647820886600818(−4)
2	−0.6599359783150116	−0.4899251895814333(−3)
3	−0.4374147243621220	−0.3266397489379376(−2)
4	−0.1878808658250491	−0.3023702670520688(−2)
5	0.7267469124647604(−1)	−0.1245464474562610(−1)
6	0.3273456878699940	−0.6280031935619400(−1)
7	0.5594682673260498	−0.9405600827603161(−1)
8	0.7537240590508397	−0.1095874208396626
9	0.8971750417623099	−0.9488044913732112(−1)
10	0.9802029946952789	−0.4815164570346023(−1)

6.4×10^{-55}, respectively. For comparison, we give the result for $n = 10$, i.e,

$$\int_{-1}^{1} f(x)\sqrt{1-x^2}\,dx \approx \sum_{\nu=1}^{10} w_\nu \sin x_\nu = -2.2095745911970091126655.$$

We can solve directly the Cauchy problem (5.1) and get

$$f(x) = \cos 2 - 2\sin 2 + (2\cos 2 - \sin 2)x + \frac{\cos 2}{2}x^2 + \cos x.$$

For checking the obtained result, we use

$$\int_{-1}^{1} f(x)\sqrt{1-x^2}dx = \pi\left(J_1(1) + \frac{9\cos 2}{16} - \sin 2\right) = -2.20957459119700911266571007 5\ldots,$$

where J_1 is the Bessel function of the first kind and order one. As we can see, the first 22 digits in the quadrature sum obtained (for $n = 10$) are exact.

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