Extremal Problems for Polynomials: Old and New Results

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Abstract. In this paper we consider extremal problems of Markov-Bernstein type for polynomials involving the classical results of Markov and Bernstein and their extensions in the uniform norm. Also, we treat the corresponding problems in L^r norm, and problems in mixed norms. Finally, we investigate the Markov-Bernstein type inequalities for differential operators and connect them with the classical orthogonal polynomials giving some new characterizations of these polynomials.

1. Introduction

The first result on the extremal problems of Markov and Bernstein-type was connected with some investigations of the well-known Russian chemist Mendeleev [35]. In mathematical terms, Mendeleev's problem, after some reductions, was as follows: *If* $t \mapsto P(t)$ *is an arbitrary quadratic polynomial and* $|P(t)| \le 1$ *on* $[-1,1]$, *how large can* $|P'(t)|$ *be on* $[-1,1]$? Mendeleev found that $|P'(t)| \le 4$ on $[-1,1]$. This result is the best possible because for $P(t) = 1 - 2t^2$ we have $P(t) \le 1$ and $P'(\pm 1) = 4$. The corresponding problem for polynomials of degree n was considered by A. A. Markov [31]. An analogue of Markov's theorem for the unit disk in the complex plane instead of for the interval $[-1,1]$ was formulated by Bernstein [5].

Inequalities of Markov and Bernstein-type are fundamental for the proof of many inverse theorems in polynomial approximation theory (cf. [15], [25], [34]). There are many results on Markov's and Bernstein's theorems and their generalizations in various metrics and restricted classes of polynomials. Several monographs and papers have been published in this area (cf. [14], [37–38], [44], [46], [65]).

In this paper we consider such problems involving the classical results of Markov and Bernstein in the uniform norm (Section 2). In Section 3 we treat important special cases in L^2 norm, in Section 4 the corresponding results in L^r norm, and in Section 5 we deals with extremal problems in different norms. Finally, in Section 6

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we investigate the Markov-Bernstein type inequalities for differential operators and connect them with the classical orthogonal polynomials giving some new characterizations of these polynomials.

Taking some restricted polynomial classes, the corresponding Markov's inequalities can be improved. Here, such cases will not be considered.

2. Classical Results of Markov and Bernstein and Their Extensions

We begin this section by considering the following extremal problem: *Let $\mathcal{P}_n$ be the set of all algebraic polynomials P ($\not\equiv 0$) of degree at most n. For a given norm $\|.\|$, determine the best constant A_n such that*

$$\|P'\| \le A_n \|P\| \qquad (P \in \mathcal{P}_n), \tag{2.1}$$

i.e.,

$$A_n = \sup_{P \in \mathcal{P}_n} \frac{\|P'\|}{\|P\|}. \tag{2.2}$$

The first result in this area appeared in the year 1889. It was the well-known classical inequality of A. A. Markov [31]. For the maximum norm on $[-1, 1]$, i.e., $\|f\| = \|f\|_\infty = \max_{-1 \le t \le 1} |f(t)|$, Markov solved this extremal problem, giving a solution of Mendeleev's problem in a generalized form.

Theorem 2.1. *In the maximum norm, we have the following:*

$$\|P'\|_\infty \le n^2 \|P\|_\infty \qquad (P \in \mathcal{P}_n). \tag{2.3}$$

The equality holds only at ± 1 and only when $P(t) = cT_n(t)$, where T_n is the Chebyshev polynomial of the first kind of degree n and c is an arbitrary constant.

A natural question is how to get an upper bound for the k-th derivative of P. Iterating Markov's inequality (2.3) yields a crude result. The best possible inequality for the k-th derivative was found by V. A. Markov [32] in 1892. A version of this remarkable paper in German was published in 1916.

Theorem 2.2. *For each $k = 1, \ldots, n$, the inequality*

$$\|P^{(k)}\|_\infty \le \frac{1}{(2k-1)!!} \prod_{i=0}^{k-1} (n^2 - i^2) \|P\|_\infty \qquad (P \in \mathcal{P}_n) \tag{2.4}$$

holds. The extremal polynomial is T_n.

We note that the best constant in (2.4) is equal to $\|T_n^{(k)}\|_\infty = T_n^{(k)}(1)$. Thus the inequality (2.4) can be written in the following form $\|P^{(k)}\|_\infty \le T_n^{(k)}(1)\|P\|_\infty$.

Markov's proof of this result is based on a complicated variational method. A simple proof of Theorem 2.2 was given by Bernstein [5] and an elementary proof by Mohr [40]. Recently Shadrin [51] gave an elegant short proof of this inequality.

Another type of these inequalities goes back to Bernstein [3] in 1912, who considered the following problem: *Let $z \mapsto P(z)$ be a polynomial of degree n and $|P(z)| \le 1$ in the unit disk $|z| \le 1$. Determine how large can $|P'(z)|$ be for $|z| \le 1$.* In other words, if we define $\|f\| = \max\limits_{|z|\le 1} |f(z)|$, this problem can be reduced to the inequality (2.1). Thus, Bernstein's theorem can be stated in the following form:

THEOREM 2.3. *Let $P \in \mathcal{P}_n$, then $\|P'\| \le n\|P\|$. The equality holds for $P(z) = cz^n$, $c = \text{const}$.*

This Bernstein's theorem can be stated in several different forms:

THEOREM 2.4. *Let $\theta \mapsto T(\theta)$ be a trigonometric polynomial of degree n and $|T(\theta)| \le M$, then*

$$|T'(\theta)| \le nM. \tag{2.5}$$

The equality holds for $T(\theta) = \gamma \sin n(\theta - \theta_0)$, where $|\gamma| = 1$.

THEOREM 2.5. *Let $P \in \mathcal{P}_n$ and $|P(t)| \le 1$ $(-1 \le t \le 1)$, then*

$$|P'(t)| \le \frac{n}{\sqrt{1-t^2}}, \qquad -1 < t < 1. \tag{2.6}$$

The equality is attained at the points $t = t_\nu = \cos\frac{(2\nu-1)\pi}{2n}$, $1 \le \nu \le n$, if and only if $P(t) = \gamma T_n(t)$, where $|\gamma| = 1$.

This result was proved by Bernstein [3] at the same time as Theorem 2.4, except that in (2.5) he had $2n$ in place of n. Inequality (2.6) in the present form first appeared in print in a paper of Fekete [1], who attributes the proof to Fejér [18]. Bernstein [4] attributes the proof to E. Landau.

Bernstein's proof of Theorem 2.4 was based on a variational method. Simpler proofs of this theorem have been obtained by M. Riesz [48], F. Riesz [47] and de la Vallée Poussin [57].

Schaeffer and Duffin [49] gave a proof of (2.4), i.e.,

$$\|P^{(k)}\|_\infty \le \frac{n^2(n^2-1^2)(n^2-2^2)\cdots(n^2-(k-1)^2)}{1\cdot 3\cdot 5\cdots(2k-1)}\,. \tag{2.7}$$

Also, Duffin and Schaeffer [13] proved that for this inequality to hold it is only necessary to assume that $|P(t)| \le 1$ at $n+1$ selected points in $[-1,1]$.

THEOREM 2.6. *Let $P \in \mathcal{P}_n$ such that $|P(\cos\nu\pi/n)| \le 1$ $(\nu = 0, 1, \ldots, n)$, then inequality (2.7) is satisfied for $k = 1, \ldots, n$. The equality occurs only if $P(t) = \gamma T_n(t)$, where $|\gamma| = 1$.*

An interesting question is whether or not there are $n+1$ other points in the interval $(-1,1)$ satisfying the same property. Duffin and Schaeffer [13] gave a negative answer to this question. In fact, they showed that if E is any closed subset of $(-1,1)$ which does not contain all the points $\tau_\nu = \cos(\nu\pi/n)$, then there is a polynomial $P \in \mathcal{P}_n$ which is bounded by 1 in E but (2.7) is not satisfied. The above refined inequality of Markov is known as *Markov-Duffin-Schaeffer inequality* (cf. [45]).

According to Theorems 2.1 and 2.5, we can state the following result:

THEOREM 2.7. *If $P \in \mathcal{P}_n$ then*

$$|P'(t)| \le \min\Big\{n^2, \frac{n}{\sqrt{1-t^2}}\Big\}\|P\|_\infty, \qquad -1 \le t \le 1.$$

Instead of the condition $|P(t)| \le 1$ on $[-1,1]$, Bernstein [5] used a more general condition

$$|P(t)| \le \sqrt{H(t)} \qquad (-1 \le t \le 1), \tag{2.8}$$

where H is an arbitrary positive polynomial on $[-1,1]$ of degree s. If $n \ge s/2$, the polynomial H can be uniquely represented in the form

$$H(t) = M_n(t)^2 + (1-t^2)N_{n-1}(t)^2, \tag{2.9}$$

where M_n and N_{n-1} are polynomials of degree n and $n-1$, respectively, such that all their zeros belong to $(-1,1)$ satisfying an interlacing property, and $M_n(1) > 0$, $N_{n-1}(1) > 0$.

THEOREM 2.8. *Let $P \in \mathcal{P}_n$. Under the condition* (2.8)*, where H is given by* (2.9)*, the inequality $|P'(t)| \le |(M_n(t) + i\sqrt{1-t^2}N_{n-1}(t))'|$ holds, for $-1 < t < 1$. The equality is attained for $P(t) = \gamma M_n(t)$, where $|\gamma| = 1$.*

Videnskiĭ [61] proved the corresponding inequality for the k-th derivative of P, i.e., $|P^{(k)}(t)| \le |(M_n(t)+i\sqrt{1-t^2}N_{n-1}(t))^{(k)}|$, where $k = 1, \dots, n$ and $-1 < t < 1$, with the same condition for the equality case. He also proved the following result ([62–63]):

THEOREM 2.9. *Let $P \in \mathcal{P}_n$ and $|P(t)| \le |\alpha t + i\sqrt{1-t^2}|$ $(\alpha \ge 0,\ -1 \le t \le 1)$. Then, for $k = 1, \dots, n$ and $-1 \le t \le 1$, we have that $|P^{(k)}(t)| \le M_n^{(k)}(1)$, where $M_n(t) = \frac{1}{2}(\alpha+1)T_n(t) + \frac{1}{2}(\alpha-1)T_{n-2}(t)$. The equality is attained only for $P(t) = \gamma M_n(t)$ at the endpoints $t = \pm 1$, where $|\gamma| = 1$.*

Several inequalities of this type were given by Videnskiĭ [59–63], and others.

These inequalities can be considered as inequalities of Markov type for curved majorants. In 1970 at a conference on *Constructive Function Theory* held in Varna, Bulgaria, the late Professor Paul Turán asked the following question: *Let $P \in \mathcal{P}_n$ and $|P(t)| \le \varphi(t)$ for $-1 \le t \le 1$, where the majorant φ is a nonnegative function. How large can $|P^{(k)}(t)|$ be at a given point $t = \tau$ in $[-1,1]$?* In (2.8) we have that $\sqrt{H(t)} = \varphi(t)$.

Defining $\|P\|_\varphi = \sup\limits_{-1<t<1} \big(|P(t)|/\varphi(t)\big)$ $(P \in \mathcal{P}_n)$, where the majorant $t \mapsto \varphi(t)$ is a nonnegative function on $[-1,1]$, and puting $\|P\| = \|P\|_\infty = \max\limits_{-1\le t\le 1} |P(t)|$, Turán's problem can be stated in the following form: *If $\|P\|_\varphi \le 1$, how large can $\|P^{(k)}\|$ be?*

In the case of the circular majorant $\big(\varphi(t) = \sqrt{1-t^2}\big)$ for $k = 1$ we have that $\|P'\| \le 2(n-1)$ (see [43]). Notice that this result is a special case of Theorem 2.9

for $\alpha = 0$. Pierre and Rahman [41] considered a more general case when $\varphi(t) = (1-t)^{\lambda/2}(1+t)^{\mu/2}$, where λ, μ are non-negative integers. They solved the case when $(\lambda+\mu)/2 \le k \le n$. The case when $1 \le k < (\lambda+\mu)/2$, for $(\lambda+\mu)/2 > 1$ was left unresolved. An asymptotic estimate when $n \to +\infty$, for $\lambda = \mu = 2$, was recently considered by Pierre, Rahman and Schmeisser [42].

3. Extremal Problems in L^2 Norm

In the L^2 metric we mention the following result of Schmidt [50] and Turán [56]:

Theorem 3.1. (a) *Let* $(a,b) = (-\infty, +\infty)$ *and* $\|f\|^2 = \int_{-\infty}^{\infty} e^{-t^2} f(t)^2\,dt$. *Then the best constant in* (2.2) *is* $A_n = \sqrt{2n}$. *An extremal polynomial is Hermite's polynomial* H_n.

(b) *Let* $(a,b) = (0, +\infty)$ *and* $\|f\|^2 = \int_0^{\infty} e^{-t} f(t)^2\,dt$. *Then* $A_n = \left(2\sin\frac{\pi}{4n+2}\right)^{-1}$. *The extremal polynomial is* $P(t) = \sum\limits_{\nu=1}^{n} \sin\frac{\nu\pi}{2n+1} L_\nu(t)$, *where* L_ν *is Laguerre polynomial.*

Theorem 3.1 b), in this form, was formulated by Turán [56]. Schmidt [50] gave only an asymptotic estimate.

Mirsky [39] considered the case of L^2 metric with a weight function and found an estimate for the best constant. Dörfler [12] considered the analogous problem for derivatives of higher order and determined the best possible constant $A_{n,k}$ in $\|P^{(k)}\| \le A_{n,k}\|P\|$ as the largest singular value of an $(n-k+1)\times(n+1)$-matrix.

Milovanović [37] showed that the exact constant in (2.1) can be found as the maximal eigenvalue of a matrix of Gram's type. He considered a more general case with a given nonnegative measure $d\lambda(t)$ on the real line $\mathbb{R}$, with compact or infinite support, for which all moments $\mu_\nu = \int_{\mathbb{R}} t^\nu\,d\lambda(t)$, $\nu = 0, 1, \ldots$, exist and are finite, and $\mu_0 > 0$. Then there exists a unique set of orthonormal polynomials $\pi_\nu(\cdot) = \pi_\nu(\cdot\,; d\lambda)$, $\nu = 0, 1, \ldots$, defined by

$$\pi_\nu(t) = a_\nu t^\nu + \text{lower degree terms}, \quad a_\nu > 0,$$

and

$$(\pi_\nu, \pi_\mu) = \int_{\mathbb{R}} \pi_\nu(t)\pi_\mu(t)\,d\lambda(t) = \delta_{\nu\mu}, \qquad \nu, \mu \ge 0.$$

For each polynomial $P \in \mathcal{P}_n$, with complex coefficients, we define

$$\|P\| = \left(\int_{\mathbb{R}} |P(t)|^2\,d\lambda(t)\right)^{1/2}$$

and consider the extremal problem

$$A_{n,k} = A_{n,k}(d\lambda) = \sup_{P\in\mathcal{P}_n} \frac{\|P^{(k)}\|}{\|P\|} \qquad (1 \le k \le n). \tag{3.1}$$

THEOREM 3.2. *The best constant $A_{n,k}$ defined in* (3.1) *is given by*

$$A_{n,k} = (\lambda_{\max}(B_{n,k}))^{1/2},$$

where $\lambda_{\max}(B_{n,k})$ is the maximal eigenvalue of the matrix $B_{n,k} = \big[b_{i,j}^{(k)}\big]_{k\le i,j\le n}$, whose elements are given by $b_{i,j}^{(k)} = \int_{\mathbb{R}} \pi_i^{(k)}(t)\pi_j^{(k)}(t)\,d\lambda(t)$ $(k \le i, j \le n)$. An extremal polynomial is $P^*(t) = \sum_{\nu=k}^{n} c_\nu \pi_\nu(t)$, where $[c_k, c_{k+1}, \dots, c_n]^T$ is an eigenvector of the matrix $B_{n,k}$ corresponding to the eigenvalue $\lambda_{\max}(B_{n,k})$.

In the Hermite case $d\lambda(t) = e^{-t^2}dt$, $-\infty < t < +\infty$ we have $\pi_\nu(t) = \hat{H}_\nu(t) = (\sqrt{\pi}\,2^\nu \nu!)^{-1/2} H_\nu(t)$, where H_ν is a Hermite polynomial of degree ν. Then, we find $A_{n,k} = 2^{k/2}\sqrt{n!/(n-k)!}$. This result also can be found in the unpublished Ph. D. thesis of Shampine [52] and [53]. For $k = 1$, this result reduces to the assertion (a) in Theorem 3.1.

In the generalized Laguerre case $d\lambda(t) = t^s e^{-t}dt$, $0 < t < +\infty$, when $k = 1$, we find that $C_{n,1} = -J_n$, where

$$J_n = \begin{bmatrix} \alpha_0 & \sqrt{\beta_1} & & & \mathrm{O} \\ \sqrt{\beta_1} & \alpha_1 & \sqrt{\beta_2} & & \\ & \sqrt{\beta_2} & \alpha_2 & \ddots & \\ & & \ddots & \ddots & \sqrt{\beta_{n-1}} \\ \mathrm{O} & & & \sqrt{\beta_{n-1}} & \alpha_{n-1} \end{bmatrix}$$

and $\alpha_0 = -(1+s)$, $\alpha_\nu = -\big(2 + s/(\nu+1)\big)$, $\beta_\nu = 1 + s/\nu$, $\nu = 1, \dots, n-1$. We see that J_n is the Jacobi matrix for monic orthogonal polynomials $\{Q_\nu\}$, which satisfy the following three-term recurrence relation

$$\begin{aligned} &Q_{k+1}(t) = (t - \alpha_k)Q_k(t) - \beta_k Q_{k-1}(t), \qquad k = 0, 1, 2, \dots \\ &Q_{-1}(t) = 0, \quad Q_0(t) = 1. \end{aligned}$$

The eigenvalues of $C_{n,1}$ are $\lambda_\nu = -t_\nu$, where $Q_n(t_\nu) = 0$ for $\nu = 1, \dots, n$.

The standard Laguerre case $(s = 0)$ can be exactly solved. In that case, we obtain the Turán result (Theorem 3.1 (b)). In the case when $k = 2$ and $s = 0$, we obtain a five-diagonal symmetric matrix $C_{n,2}$ of the order $n - 1$ (see Milovanović [37]). Using the minimal eigenvalue of such matrix, we obtain the best constant $A_{n,2} = (\lambda_{\min}(C_{n,2}))^{-1/2}$. For $n = 2$ and $n = 3$ we have exact values: $A_{2,2} = 1$ and $A_{3,2} = \big(3 + 2\sqrt{2}\big)^{1/2}$ respectively.

A case with a special even weight function, involving the Gegenbauer weight, was considered by Milovanović [37], but exact constant is not yet known.

4. Generalizations in L^r Norm

It is interesting to extend the inequalities of A. A. Markov and S. N. Bernstein to L^r spaces, where $\|P\| = \|P\|_r = \left((b-a)^{-1}\int_a^b |P(t)|^r\,dt\right)^{1/r}$ $(r \geq 1)$. The case $r = 2$ was considered in the previous section.

Let $\mathcal{T}_n$ denote the set of all trigonometric polynomials of degree at most n. Zygmund [65] proved the following theorem:

Theorem 4.1. *Let $r \geq 1$, $(a,b) = (0,2\pi)$, and $T \in \mathcal{T}_n$. Then $\|T'\|_r \leq n\|T\|_r$.*

Taking $r \to +\infty$, this Zygmund's inequality reduces to the Bernstein inequality.

Using the norm in the space $L^r(0,2\pi)$ of a function f, defined by

$$\|f\|_r = \left(\frac{1}{2\pi}\int_0^{2\pi} |f(\theta)|^r\,d\theta\right)^{1/r} \qquad (0 < r < +\infty),$$

as well as limiting cases: for $r \to +\infty$ the uniform norm $\|f\|_\infty$, and for $r \to 0$ the quasi-norm of $L^0(0,2\pi)$, defined by $\|f\|_0 = \exp\left((2\pi)^{-1}\int_0^{2\pi}\log|f(\theta)|\,d\theta\right)$, Golitschek and Lorentz [21] recently proved:

Theorem 4.2. *If $T \in \mathcal{T}_n$, then*

$$\left\| \frac{1}{n}T' \right\|_r \leq \|T\|_r \qquad (0 \leq r \leq +\infty). \tag{4.1}$$

For $0 < r < 1$, the inequality (4.1) is due to Máté and Nevai [33], but with an extra factor $(4e)^{1/r}$ on the right hand side. Later, Arestov [2] proved this inequality in the form (4.1), using subharmonic functions and Jensen's formula. Golitschek and Lorentz gave a new simpler proof of the inequality (4.1).

An important generalization of A. A. Markov's inequality for algebraic polynomials in an integral norm was given by Hille, Szegő, and Tamarkin [24], who proved the following theorem:

Theorem 4.3. *Let $r > 1$ and let $P \in \mathcal{P}_n$. Then*

$$\left(\int_{-1}^1 |P'(t)|^r\,dt\right)^{1/r} \leq An^2\left(\int_{-1}^1 |P(t)|^r\,dt\right)^{1/r}, \tag{4.2}$$

where the constant $A = A(n,r)$ is given by

$$A(n,r) = 2(r-1)^{1/r-1}\left(r+\frac{1}{n}\right)\left(1+\frac{r}{nr-r+1}\right)^{n-1+1/r},$$

for $r > 1$, and $A(n,1) = 2\left(1+1/n\right)^{n+1}$.

The constant $A(n,r)$ in Theorem 4.3 is not the best possible. We can see that $A(n,r) \le 6\exp(1+1/e)$, for every n and $r \ge 1$. Also,

$$A(n,r) \;\to\; \begin{cases} 2(1+1/(n-1))^{n-1} < 2e & (n \;\text{ fixed},\; r \to +\infty), \\ 2e & (r=1,\; n \to +\infty), \\ 2er(r-1)^{(1/r)-1} & (r>1 \;\text{ fixed},\; n \to +\infty). \end{cases}$$

Some improvements of the constant $A(n,r)$ have recently been obtained by Goetgheluck [20]. He found that $A(n,1) = \sqrt{8/\pi}\bigl(1+3/(4n)\bigr)^2$, as well as a very complicated expression for $r > 1$.

5. Extremal Problems in Different Norms

There are several interesting results, including different norms. In the following we will first discuss a few of those inequalities which involve the norms

$$\|f\|_\infty = \max_{-1\le t\le 1} |f(t)| \quad \text{ and } \quad \|f\|_2 = \left(\int_{-1}^{1} w(t)|f(t)|^2\,dt\right)^{1/2},$$

where $t \mapsto w(t)$ is a given weight function.

In the case when $w(t)=1$, Labelle [28] proved the following inequality

$$\|P^{(k)}\|_\infty \le \frac{(2k-1)!!(n+k+1)}{\sqrt{2(2k+1)}}\binom{n+k}{n-k}\|P\|_2, \tag{5.1}$$

with equality in (5.1) only for constant multiples of the polynomial

$$\sum_{i=0}^{n-k}(2i+2k+1)\binom{2k+i}{i}P_{k+i}(t),$$

where $P_\mu(t)$ denotes the Legendre polynomial of degree μ.

Lupaş [30] investigated a more general case using the Jacobi weight $w(t) = w(t;\alpha,\beta) = (1-t)^\alpha(1+t)^\beta \quad (\alpha,\beta > -1)$. He obtained the best constant in the following inequality

$$\|P^{(k)}\|_\infty \le A_n(k,\alpha,\beta)\|P\|_2 \qquad (P \in \mathcal{P}_n). \tag{5.2}$$

Theorem 5.1. *Let $P \in \mathcal{P}_n$ and $q = \max(\alpha,\beta) \ge -1/2$. Then the best constant in inequality* (5.2) *is given by*

$$A_n(k,\alpha,\beta) = \left(\frac{k!}{2^{2k+\alpha+\beta+1}}\sum_{\nu=k}^{n} C_{\nu,k}^{(\alpha,\beta)}\binom{\nu+\alpha+\beta+k}{k}\binom{\nu+q}{\nu-k}\right)^{1/2},$$

where

$$C_{\nu,k}^{(\alpha,\beta)} = \frac{\nu!(2\nu+\alpha+\beta+1)\Gamma(\nu+\alpha+\beta+k+1)}{\Gamma(\nu+\alpha+1)\Gamma(\nu+\beta+1)}\binom{\nu+q}{\nu-k}.$$

Equality is attained for $P(t) = C\sum_{\nu=k}^{n} C_{\nu,k}^{(\alpha,\beta)} P_\nu^{(\alpha,\beta)}(t)$, *where* C *is a constant and* $P_\nu^{(\alpha,\beta)}(t)$ *is the Jacobi orthogonal polynomial of degree* ν.

A general extremal problem was studied by Daugavet and Rafal'son [10] and Konjagin [27], taking

$$\begin{aligned} \|f\|_{r,\mu} &= \left(\int_{-1}^{1} |f(t)(1-t^2)^\mu|^r\,dt\right)^{1/r}, \quad && 0 \le r < +\infty,\\ &= \operatorname*{ess\,sup}_{-1\le t\le 1} |f(t)|(1-t^2)^\mu, && r = +\infty, \end{aligned}$$

where μ is a real number such that $r\mu > -1$. (This condition, for $r = +\infty$, should be understood as $\mu \ge 0$.)

Konjagin [27] considered the following general extremal problem

$$A_{n,k}(r,\mu;\,p,\nu) = \sup_{P\in\mathcal{P}_n} \frac{\|P^{(k)}\|_{p,\nu}}{\|P\|_{r,\mu}}\,. \tag{5.3}$$

For example, the best constant in (2.4) is $A_{n,k}(+\infty,0;\,+\infty,0)$. Also, the Bernstein's inequality (2.6) can be represented in the form $\|P'\|_{\infty,1/2} \le n\|P\|_{\infty,0}$ $(P\in\mathcal{P}_n)$. The case when $p = r \ge 1$, $\mu = \nu = 0$, and $k = 1$ was considered by Hille, Szegő, and Tamarkin [24] (see Theorem 4.3).

Bojanov [7] considered the problem (5.3) in the case when $r = +\infty$, $\mu = \nu = 0$, and $1 \le p < +\infty$.

THEOREM 5.2. *Let* $P \in \mathcal{P}_n$ *and* $r \in [1,+\infty)$. *Then* $\|P'\|_r \le \|T_n'\|_r\|P\|_\infty$. *Equality is attained only for* $P(t) = \pm T_n(t)$.

In two boundary cases we have $\|T_n'\|_\infty = n^2$ and $\|T_n'\|_1 = 2n$. Recently, Ciesielski [9] has given theoretical estimates for $\|T_n'\|_r$ in the whole range of the parameters n and r and formulated a conjecture on the best estimates based on numerical calculations. Theorem 5.2 is a particular case of the following more general assertion:

THEOREM 5.3. *Let* $\mathcal{G}_n = \{P\in\mathcal{P}_n \mid \|P\|_\infty \le 1\}$. *For every increasing convex function* $x\mapsto\varphi(x)$, *for* $x>0$, *the quantity* $\sup\limits_{P\in\mathcal{G}_n}\int_{-1}^{1}\varphi(|P'(t)|)\,dt$ *is attained only for* $P(t) = \pm T_n(t)$.

This result was proved by Bojanov [8]. Since $x\mapsto\varphi(x) = x^p$ $(1<p<+\infty)$ is an increasing convex function on $[0,+\infty)$, Theorem 5.2 follows from Theorem 5.3. For $\varphi(x) = \sqrt{1+M^2x^2}$, Theorem 5.3 provides the solution of a conjecture on the longest polynomial stated by Erdős [17] (see [6]).

6. Markov-Bernstein Type Inequalities for Differential Operators

Using the interpolation in polynomial classes, in 1957 Stein [55] proved the following result:

Theorem 6.1. *Let $P \in \mathcal{P}_n$, $n \geq 1$, and $1 \leq r \leq +\infty$. For the Legendre differential operator $\mathcal{D} = \frac{d}{dt}(1-t^2)\frac{d}{dt}$, there exists a positive constant depending only on r such that $\|\mathcal{D}P\|_r \leq C_r n^2 \|P\|_r$ $(1 \leq r \leq +\infty)$, where $\|f\|_r = \left(\int_{-1}^{1} |f(t)|^r \, dt\right)^{1/r}$ and $\|f\|_\infty = \max\limits_{-1\leq t\leq 1} |f(t)|$.*

For $r = +\infty$ and $r = 2$, we have $C_\infty = 2$ and $C_2 = 1 + 1/n$, respectively.

Taking L^r norm with the Jacobi measure, Stein [55] also proved the corresponding results for the operator

$$\mathcal{D}_{\alpha,\beta} = (1-t)^{-\alpha}(1+t)^{-\beta}\frac{d}{dt}\Big[(1-t)^{\alpha+1}(1+t)^{\beta+1}\frac{d}{dt}\Big].$$

Since $\mathcal{D}_{\alpha,\beta}P_k^{(\alpha,\beta)}(t) = -k(k+\alpha+\beta+1)P_k^{(\alpha,\beta)}(t)$ $(\alpha, \beta > -1)$, where $P_k^{(\alpha,\beta)}(t)$ is the Jacobi polynomial of degree k, we see that the Jacobi polynomials are eigenfunctions of the operator $\mathcal{D}_{\alpha,\beta}$. Recently, Džafarov [16] has considered similar problems for operators which correspond to the classical orthogonal polynomials.

The classical orthogonal polynomials $\{Q_n\}$ on (a, b) can be specificated as the Jacobi polynomials $P_n^{(\alpha,\beta)}(t)$ $(\alpha, \beta > -1)$ on $(-1, 1)$, the generalized Laguerre polynomials $L_n^s(t)$ $(s > -1)$ on $(0, +\infty)$, and finally as the Hermite polynomials $H_n(t)$ on $(-\infty, +\infty)$. Their weight functions $t \mapsto w(t)$ on an interval of orthogonality (a, b) satisfy the differential equation $(A(t)w(t))' = B(t)w(t)$, where the functions $t \mapsto A(t)$ and $t \mapsto B(t)$ are defined as in Table 6.1.

Table 6.1
The Classification of the Classical Orthogonal Polynomials

(a,b)	$w(t)$	$A(t)$	$B(t)$	λ_n
$(-1,1)$	$(1-t)^\alpha(1+t)^\beta$	$1-t^2$	$\beta-\alpha-(\alpha+\beta+2)t$	$n(n+\alpha+\beta+1)$
$(0,\infty)$	$t^s e^{-t}$	t	$s+1-t$	n
$(-\infty,\infty)$	e^{-t^2}	1	$-2t$	$2n$

The classical orthogonal polynomial $t \mapsto Q_n(t)$ is a particular solution of the following differential equation of the second order $L[y] = A(t)y'' + B(t)y' + \lambda_n y = 0$, where λ_n is given in Table 6.1.

Let $(f, g) = \int_a^b f(t)g(t)w(t)\,dt$ and $\|f\|^2 = (f, f)$. Similarly to the well-known Landau inequality ([29]) for continuously–differentiable functions and other generalizations (cf. [11], [26], [36], [54]), Agarwal and Milovanović [1] stated the following characterization of the classical orthogonal polynomials:

THEOREM 6.2. *For all $P_n \in \mathcal{P}_n$ the inequality*

$$(2\lambda_n + B'(0))\|\sqrt{A}P_n'\|^2 \le \lambda_n^2\|P_n\|^2 + \|AP_n''\|^2$$

holds, with equality if only if $P_n(t) = cQ_n(t)$, where Q_n is the classical orthogonal polynomial of degree n orthogonal to all polynomials of degree $\le n-1$ with respect to the weight function $t \mapsto w(t)$ on (a,b), and c is an arbitrary real constant. λ_n, $A(t)$ and $B(t)$ are given in Table 6.1.

The Hermite case was considered by Varma [58].

Recently Guessab and Milovanović [22] have considered a weighted L^2-analogues of the Bernstein's inequality (2.6), which can be stated in the following form:

$$\|\sqrt{1-t^2}P'(t)\|_\infty \le n\|P\|_\infty. \tag{6.1}$$

Let w be the weight of the classical orthogonal polynomials ($w \in CW$) and $t \mapsto A(t)$ be given as in Table 6.1. Using the norm $\|f\|_w^2 = (f,f)$, we consider the following problem connected with the Bernstein's inequality (6.1): *Determine the best constant $C_{n,m}(w)$ $(1 \le m \le n)$ such that the inequality*

$$\|A^{m/2}P^{(m)}\|_w \le C_{n,m}(w)\|P\|_w \tag{6.2}$$

holds for all $P \in \mathcal{P}_n$.

At first, we note if $w \in CW$, then the corresponding classical orthogonal polynomial $t \mapsto Q_n(t)$ is a particular solution of the differential equation of the second order

$$\frac{d}{dt}\left(A(t)w(t)\frac{dy}{dt}\right) + \lambda_n w(t)y = 0, \tag{6.3}$$

where $\lambda_n = -n\left(\frac{1}{2}(n-1)A''(0) + B'(0)\right)$. The k-th derivative of Q_n is also the classical orthogonal polynomial, with respect to the weight $t \mapsto w_k(t) = A(t)^k w(t)$, and satisfies the following differential equation

$$\frac{d}{dt}\left(A(t)w_k(t)\frac{dy}{dt}\right) + \lambda_{n,k}w_k(t)y = 0, \tag{6.4}$$

where $\lambda_{n,k} = -(n-k)\left(\frac{1}{2}(n+k-1)A''(0) + B'(0)\right)$. We note that $\lambda_{n,0} = \lambda_n$.

THEOREM 6.3. *For all $P \in \mathcal{P}_n$ the inequality* (6.2) *holds, with the best constant $C_{n,m}(w) = \sqrt{\lambda_{n,0}\lambda_{n,1}\cdots\lambda_{n,m-1}}$. The equality is attained in* (6.2) *if and only if P is a constant multiple of the classical polynomial Q_n orthogonal with respect to the weight function $w \in CW$.*

Now, we give the special cases:

COROLLARY 6.4. *Let $w(t) = (1-t)^\alpha(1+t)^\beta$ $(\alpha, \beta > -1)$. Then, for every $P \in \mathcal{P}_n$, the inequality*

$$\|(1-t^2)^{m/2}P^{(m)}\|_w \le \sqrt{\frac{n!\Gamma(n+\alpha+\beta+m+1)}{(n-m)!\Gamma(n+\alpha+\beta+1)}}\ \|P\|_w,$$

holds, with equality if and only if $P(t) = cP_n^{(\alpha,\beta)}(t)$.

COROLLARY 6.5. *Let $w(t) = t^s e^{-t}$ $(s > -1)$ on $(0, +\infty)$. Then for every $P \in \mathcal{P}_n$ we have $\|t^{m/2}P^{(m)}\|_w \le \sqrt{n!/(n-m)!}\ \|P\|_w$, with equality if and only if $P(t) = cL_n^s(t)$.*

The Hermite case with the weight $w(t) = e^{-t^2}$ on $(-\infty, +\infty)$ is the simplest. Then the best constant is $C_{n,m}(w) = 2^{m/2}\sqrt{n!/(n-m)!}$. This result was obtained in Section 3.

Finally, we consider extremal problems of Markov's type

$$C_{n,m}(d\sigma) = \sup_{P\in\mathcal{P}_n} \frac{\|\mathcal{D}_m P\|_{d\sigma}}{\|A^{m/2}P\|_{d\sigma}} \tag{6.5}$$

for the differential operator $\mathcal{D}_m$ defined by

$$\mathcal{D}_m P = \frac{d^m}{dt^m}\,[A^m P] \qquad (P \in \mathcal{P}_n,\ m \ge 1), \tag{6.6}$$

where $\|P\|_{d\sigma} = \left(\int_{\mathbb{R}} |P(t)|^2\, d\sigma(t)\right)^{1/2}$.

Guessab and Milovanović [23] found the best constant $C_{n,m}(d\sigma)$ in three following cases:

1° *The Legendre measure* $d\sigma(t) = dt$ on $[-1, 1]$;

2° *The Laguerre measure* $d\sigma(t) = e^{-t}\,dt$ on $[0, +\infty)$.

3° *The Hermite measure* $d\sigma(t) = e^{-t^2}\,dt$ on $(-\infty, +\infty)$.

Let $P \in \mathcal{P}_n$, $d\sigma(t) = w(t)\,dt$ on (a, b), and $\mathcal{D}_m$ be given by (6.6). An application of integration by parts and Cauchy-Schwarz inequality gives

$$\|\mathcal{D}_m P\|_{d\sigma}^2 \le \|A^{m/2}P\|_{d\sigma}\left(\int_a^b \frac{A^m}{w}\left([w\mathcal{D}_m P]^{(m)}\right)^2 dt\right)^{1/2},$$

with equality if and only if $\mathcal{F}_m P = (-1)^m w^{-1}[w\mathcal{D}_m P]^{(m)} = \lambda P$ $(P \in \mathcal{P}_n)$, where λ is an arbitrary constant. Taking a norm with respect to the measure $d\sigma_m(t) = A^m\, d\sigma(t) = A^m w\, dt$, we have

$$\frac{\|\mathcal{D}_m P\|_{d\sigma}}{\|A^{m/2}P\|_{d\sigma}} \le \left(\frac{\|\mathcal{F}_m P\|_{d\sigma_m}}{\|P\|_{d\sigma_m}}\right)^{1/2}, \tag{6.7}$$

with equality if and only if $\mathcal{F}_m P = \lambda P$. We are interested only in polynomial solutions of this equation. If they exist, then from the eigenvalue problem and the inequality (6.7), we can determine the best constant in the extremal problem (6.5). Namely,

$$C_{n,m}(d\sigma) = \sqrt{\max_{0\le\nu\le n} |\lambda_{\nu,m}|}\,,$$

where $\lambda_{\nu,m}$ are eigenvalues of the operator $\mathcal{F}_m$. Then, the extremal polynomial is the eigenfunction corresponding to the maximal eigenvalue.

THEOREM 6.6. *Let $d\sigma(t) = dt$ on $(-1,1)$. Then $C_{n,m}(d\sigma) = \sqrt{(n+2m)!/n!}$. The supremum in (6.5) is attained only if $P(t) = \gamma C_n^{m+1/2}(t)$, where C_n^μ is the Gegenbauer polynomial of degree n, and $\gamma\,(\neq 0)$ is an arbitrary real constant.*

THEOREM 6.7. *Let $d\sigma(t) = e^{-t}\,dt$ on $(0,+\infty)$. Then $C_{n,m}(d\sigma) = \sqrt{(n+m)!/n!}$. The supremum in (6.5) is attained only if $P(t) = \gamma L_n^m(t)$, where L_n^m is the generalized Laguerre polynomial of degree n, and $\gamma\,(\neq 0)$ is an arbitrary real constant.*

Finally, in the Hermite case when $d\sigma(t) = e^{-t^2}\,dt$ on the real line $\mathbb{R}$, the extremal problem (6.5) reduces to the corresponding problem (6.2).

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