# Summation of Series and Gaussian Quadratures, II

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*Dedicated to Luigi Gatteschi on the occasion of his 70th birthday*

Abstract. Continuing previous work, we discuss applications of our summation/integration procedure to some classes of complex slowly convergent series. Especially, we consider the series of the form $\sum_{k=1}^{+\infty}(\pm 1)^k k^{\nu-1}R(k)$, where $0<\nu\le 1$ and $R(s)$ is a rational function. Such cases were recently studied by Gautschi, using the Laplace transform method. Also, we gave an appropriate method for calculating values of the Riemann zeta function $\zeta(z)=\sum_{k=1}^{+\infty}k^{-z}$, which can be transformed to a weighted integral on $(0,+\infty)$ of the function $t\mapsto \exp\bigl(-(z/2)\log(1+\beta_m^2t^2)\bigr)\cos(z\arctan(\beta_m t))$, $\beta_m=2/((2m+1)\pi)$, $m\in\mathbb{N}_0$, involving the hyperbolic weight $w(t)=1/\cosh^2 t$. Numerical results are included to illustrate the method.

## 1. Introduction and Preliminaries

We consider the summation of slowly convergent series of the type

$$T_m=T_m(\nu,a,p)=\sum_{k=m}^{+\infty}\frac{k^{\nu-1}}{(k+a)^p} \tag{1.1}$$

and

$$S_m=S_m(\nu,a,p)=\sum_{k=m}^{+\infty}(-1)^k\frac{k^{\nu-1}}{(k+a)^p}\,, \tag{1.2}$$

where $m\in\mathbb{Z}$, $0<\nu\le 1$, and $a$ and $p$ are such to provide convergence of (1.1) and (1.2). In particular, we consider the Riemann zeta function

$$z\mapsto\zeta(z)=\sum_{k=1}^{+\infty}\frac{1}{k^z}=T_1(1,0,z).$$

Some methods of summation of series can be found, for example, in the books of Henrici [4], Lindelöf [5], and Mitrinović and Kečkić [8].

Using the Laplace transform method (see [3, §4]), Gautschi [2] considered the series $\sum_{k=1}^{+\infty}(\pm 1)^k k^{\nu-1}R(k)$, where $0<\nu\le 1$ and $R(\,\cdot\,)$ is a rational function $R(s)=$

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$P(s)/Q(s)$, with $P$, $Q$ real polynomials of degrees $\deg P \le \deg Q$. As he showed, the problem can be simplified by first obtaining the partial fraction decomposition of $R$ and then it is enough to consider only the case

$$R(s) = \frac{1}{(s+a)^m}\,, \qquad \operatorname{Re} a \ge 0,\ m \ge 1.$$

By interpreting the terms in such series as Laplace transforms at integer values, Gautschi expressed the sum of the series as a weighted integral over $\mathbb{R}_+$ of certain special functions related to the incomplete gamma function. Namely, if $0 < \nu < 1$,

$$\varepsilon(t) = \frac{t}{e^t - 1} \quad \text{(Einstein's weight)}, \qquad \varphi(t) = \frac{1}{e^t + 1} \quad \text{(Fermi's weight)},$$

and if the function $g_n(t)$ is defined for $t > 0$ by

$$g_n(t; a, \nu) = \frac{e^{-at} t^{\nu-1}}{n!\Gamma(1-\nu)} \int_0^t e^{a\tau} (t-\tau)^n \tau^{-\nu}\, d\tau, \tag{1.3}$$

where $n \in \mathbb{N}_0$, $\operatorname{Re} a \ge 0$, $\operatorname{Im} a \ge 0$, $a \ne 0$, Gautschi obtained the following representations:

$$\sum_{k=1}^{+\infty} \frac{k^{\nu-1}}{(k+a)^m} = \int_0^{+\infty} t^{-\nu} \varepsilon(t) g_{m-1}(t; a, \nu)\, dt, \qquad m \ge 1,$$

and

$$\sum_{k=1}^{+\infty} \frac{(-1)^{k-1} k^{\nu-1}}{(k+a)^m} = \int_0^{+\infty} t^{-\nu} \varphi(t) t g_{m-1}(t; a, \nu)\, dt, \qquad m \ge 0,$$

which suggest to apply Gaussian quadrature to the integrals on the right, using the weight functions $t^{-\nu}\varepsilon(t)$ and $t^{-\nu}\varphi(t)$, respectively. The first 80 recursion coefficients for the corresponding orthogonal polynomials, for $\nu = 1/2$, were given to 25 significant digits in [2, Tables 1 and 2 of the Appendix]. The case $\nu = 1$ of purely rational series was also considered. The first 40 recursion coefficients for the polynomials orthogonal with respect to $\varepsilon(t)$ and $\varphi(t)$ can be found to 25 significant digits in [3, Appendices A1 and A2].

An alternative summation/integration procedure for the series of the type $S_\pm(f; m) = \sum_{k=m}^{+\infty} (\pm 1)^k f(k)$, where $z \mapsto f(z)$ is a holomorphic function in the region

$$G_m = \{ z \in \mathbb{C} \mid \operatorname{Re} z \ge \alpha,\ m-1 < \alpha < m \}, \quad m \in \mathbb{N},$$

was derived in [7]. This method requires the indefinite integral $F$ of $f$ chosen so as to satisfy the following decay conditions:

(C1) *$F$ is a holomorphic function in the region $G_m$;*

(C2) $\lim_{|t| \to +\infty} e^{-c|t|} F(x + it/\pi) = 0$, *uniformly for $x \ge \alpha$;*

(C3) $\lim_{x \to +\infty} \int_{-\infty}^{+\infty} e^{-c|t|} \left| F(x + it/\pi) \right| dt = 0$, *where $c = 2$ or $c = 1$, when we consider $S_+(f;m)$ or $S_-(f;m)$, respectively.* It was shown that

$$S_\pm(f;m) = \int_0^{+\infty} \Phi_\pm(m - \tfrac{1}{2}, \tfrac{1}{\pi}t) w_\pm(t)\, dt, \tag{1.4}$$

where

$$\Phi_+(x,y) = -\frac{1}{2}\left[F(x+iy) + F(x-iy)\right], \tag{1.5}$$

$$\Phi_-(x,y) = \frac{(-1)^m}{2i}\left[F(x+iy) - F(x-iy)\right], \tag{1.6}$$

and weights are the hyperbolic functions given by $w_+(t) = 1/\cosh^2 t$ and $w_-(t) = \sinh t/\cosh^2 t$. Numerical quadratures of Gaussian type with respect to these weights were constructed in [7]. The first $n = 40$ coefficients in the three-term recurrence relation for the corresponding orthogonal polynomials were obtained accurately to 30 decimal digits.

Numerical experiments show that is enough to use only the quadrature with respect to the first weight $w_+(t) = 1/\cosh^2 t$. Namely, in the series $S_-(f;m)$ we can include the hyperbolic sine as a factor in the corresponding integrand so that

$$S_-(f;m) = \int_0^{+\infty} \Phi_-(m - \tfrac{1}{2}, \tfrac{1}{\pi}t) \sinh(t) w_+(t)\, dt. \tag{1.7}$$

The paper is organized as follows. Section 2 discusses the summation of (1.1) and (1.2) using Gaussian quadratures with respect to the hyperbolic weight $w(t) = w_+(t) = 1/\cosh^2 t$. An application to the Riemann zeta function is given in §3. Finally, numerical examples are presented in §4.

## 2. Summation of $T_m(\nu, a, p)$ and $S_m(\nu, a, p)$

We consider the series (1.1) and (1.2) under conditions $\mathrm{Re}\, a \geq 0$, $0 < \nu < 1$, and $\mathrm{Re}\, p \geq 1$. In order to employ the method from [7], we need the indefinite integral $F$ of $f(x) = x^{\nu-1}(x+a)^{-p}$ which satisfies the decay conditions (C1) – (C2).

Let ${}_2F_1(a,b;c;z)$ be the Gauss hypergeometric function defined by the series

$$F(a,b;c;z) = {}_2F_1(a,b;c;z) = \sum_{n=0}^{+\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!},$$

which is convergent in the unit circle. This function is analytic for $|z| < 1$, and it can be analytically continued using the integral representation

$$F(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 u^{b-1}(1-u)^{c-b-1}(1-zu)^{-a}\, du,$$

where $\mathrm{Re}\, c > \mathrm{Re}\, b > 0$. Namely, this integral represents a one valued analytic function in the $z$-plane cut along the real axis from 1 to $\infty$ and hence it gives the analytic continuation of the Gauss hypergeometric function.

Since (cf. [9, p. 30])

$$\int_0^z \frac{x^{\nu-1}}{(x+a)^p}\,dx = \frac{z^\nu}{\nu a^p}\,{}_2F_1(p,\nu;1+\nu;-z/a) \qquad (\operatorname{Re}\nu>0)$$

and

$$\int_z^\infty \frac{x^{\nu-1}}{(x+a)^p}\,dx = \frac{z^{\nu-p}}{\nu-p}\,{}_2F_1(p,p-\nu;1+p-\nu;-a/z) \qquad (\operatorname{Re}\nu<\operatorname{Re}p)$$

we can take

$$F(z)=\begin{cases} \dfrac{z^{\nu-p}}{\nu-p}\,{}_2F_1(p,p-\nu;1+p-\nu;-a/z), & |z|>|a|,\\ \dfrac{z^{\nu}}{\nu a^p}\,{}_2F_1(p,\nu;1+\nu;-z/a)+C, & |z|<|a|,\end{cases}$$

where

$$C=a^{\nu-p}\Big\{\frac{1}{\nu-p}\,{}_2F_1(p,p-\nu;1+p-\nu;-1)-\frac{1}{\nu}\,{}_2F_1(p,\nu;1+\nu;-1)\Big\},$$

so that $F(\infty)=0$. In integral form we have, for example,

$$F(z)=-z^{\nu-p}\int_0^1 u^{p-\nu-1}\left(1+\frac{au}{z}\right)^{-p}du, \qquad \operatorname{Re}p>\operatorname{Re}\nu>0. \tag{2.1}$$

Thus, using (1.4) and (1.7), we reduce the series $T_m(\nu,a,p)$ and $S_m(\nu,a,p)$ to the corresponding integrals, where $\Phi_+$ and $\Phi_-$ are given by (1.5) and (1.6), respectively. Formulas (1.4) and (1.7) suggest to apply Gaussian quadrature to the integrals on the right, using the weight function $w(t)=1/\cosh^2 t$.

In some important cases, $F(z)$ can be an elementary function. For some details see [6, Ch. 6] and [1, Ch. 15]. The Padé approximation for ${}_2F_1(1,\sigma;\varrho+1;-1/z)$ is given in [6].

We mention here a few special cases which will be treated in §4. For fixed numbers $\nu$ and $a$, we denote the function (2.1) by $F_p(z)=F_p(z;\nu,a)$.

$1^0$ For $\nu=1/2$ and $p=1,2,3,4$:

$$\begin{aligned}
F_1(z) &= F_1(z;1/2,a)=\frac{2}{\sqrt{a}}\left(\arctan\sqrt{\frac{z}{a}}-\frac{\pi}{2}\right),\\
F_2(z) &= \frac{1}{2a}F_1(z)+\frac{\sqrt{z}}{a(a+z)},\\
F_3(z) &= \frac{3}{8a^2}F_1(z)+\frac{\sqrt{z}}{4a^2(a+z)^2}(3z+5a),\\
F_4(z) &= \frac{5}{16a^3}F_1(z)+\frac{\sqrt{z}}{24a^3(a+z)^3}\left(15z^2+10(3a+1)z+33a^2\right).
\end{aligned}$$

$2^0$ For $\nu = 1/2$ and $p = 3/2$:

$$F_{3/2}(z) = F_{3/2}(z; 1/2, a) = \frac{2}{a}\left(\sqrt{\frac{z}{z+a}} - 1\right).$$

$3^0$ For $\nu = 2/3$ and $p = 1$:

$$\begin{aligned} F_1(z) &= F_1(z; 2/3, a) \\ &= \frac{1}{a^{1/3}}\left\{c\Big(\arctan\frac{-1+2w}{c} - \frac{\pi}{2}\Big) - \frac{1}{2}\log\frac{(1+w)^2}{1-w+w^2}\right\}, \end{aligned}$$

where $w = (z/a)^{1/3}$ and $c = \sqrt{3}$.

$4^0$ For $\nu = 3/4$ and $p = 1$:

$$\begin{aligned} F_1(z) = F_1(z; 3/4, a) = \; & \frac{c}{a^{1/4}}\Big\{-\pi + \frac{1}{2}\log\frac{1-cw+w^2}{1+cw+w^2} \\ & + \arctan(-1+cw) + \arctan(1+cw)\Big\}, \end{aligned}$$

where $w = (z/a)^{1/4}$ and $c = \sqrt{2}$.

## 3. Riemann Zeta Function

The well-known Riemann $\zeta$-function is defined as

$$\zeta(z) = \sum_{k=1}^{+\infty} \frac{1}{k^z}, \tag{3.1}$$

and the Dirichlet series of (3.1) converges for any $z$ with $\operatorname{Re} z > 1$, uniformly, for any fixed $\sigma > 1$, in any subset of $\operatorname{Re} z \geq \sigma$, which establishes that $\zeta(z)$ is an analytic function in $\operatorname{Re} z > 1$. By means of analytic continuation, it is known that $\zeta(z)$ is analytic for any complex $z$, except for $z = 1$, which is a simple pole of $\zeta(z)$ with residue 1. This function satisfies the functional equation

$$\zeta(z) = 2^z \pi^{z-1} \sin\frac{\pi z}{2}\Gamma(1-z)\zeta(1-z).$$

As we mentioned before, in our notation, $\zeta(z) = T_1(1, 0, z)$. Using (1.4) and (1.5), after some calculations we can express the Riemann function in the integral form

$$\zeta(z+1) = \sum_{k=1}^{m} \frac{1}{k^{z+1}} + \frac{1}{z}\Big(m + \frac{1}{2}\Big)^{-z} \int_0^{+\infty} f(\beta_m t; z) w(t)\, dt, \tag{3.2}$$

for $\operatorname{Re} z > 0$, where

$$f(t; z) = \exp\Big(-\frac{z}{2}\log(1+t^2)\Big)\cos(z \arctan t), \qquad w(t) = \frac{1}{\cosh^2 t}$$

and $\beta_m = 2/((2m+1)\pi)$, $m \in \mathbb{N}_0$. However, (3.2) holds for every $z \neq 0$ as an analytic continuation of the Riemann zeta function. Formula (3.2) suggests an application of a Gaussian formula with respect to the hyperbolic weight $w(t)$, i.e.,

$$\zeta(z+1) = \sum_{k=1}^{m} \frac{1}{k^{z+1}} + \frac{1}{z}\Big(m + \frac{1}{2}\Big)^{-z} \sum_{\nu=1}^{n} \lambda_\nu f(\beta_m \tau_\nu; z) + R_n(f), \tag{3.3}$$

where $\tau_\nu = \tau_\nu^{(n)}$ and $\lambda_\nu = \lambda_\nu^{(n)}$, $\nu = 1, \ldots, n$, are the corresponding nodes and weights, and $R_n(f)$ is the remainder term.

## 4. Numerical Examples

In this section we illustrate the previous method taking the series

$$T_1(\nu, a, p) = \sum_{k=1}^{+\infty} \frac{k^{\nu-1}}{(k+a)^p} \quad \text{and} \quad S_1(\nu, a, p) = \sum_{k=1}^{+\infty} \frac{(-1)^k k^{\nu-1}}{(k+a)^p},$$

with $a = \alpha e^{i\theta}$, where $\theta = 0, \pi/4, \pi/2$ and $\alpha = \alpha_k = 2^{k-1}$, $k = 0(1)7$. The parameters $\nu$ and $p$ are taken as in $1^0$–$4^0$ (cf. Section §2).

Thus,

$$T_1(\nu, a, p) = \sum_{k=1}^{m-1} \frac{k^{\nu-1}}{(k+a)^p} + \sum_{\nu=1}^{n} \lambda_\nu \Phi_+(m - \tfrac{1}{2}, \tfrac{1}{\pi}\tau_\nu) + R_n(\widetilde{\Phi}_+)$$

and

$$S_1(\nu, a, p) = \sum_{k=1}^{m-1} \frac{(-1)^k k^{\nu-1}}{(k+a)^p} + \sum_{\nu=1}^{n} \lambda_\nu \Phi_-(m - \tfrac{1}{2}, \tfrac{1}{\pi}\tau_\nu) \sinh(\tau_\nu) + R_n(\widetilde{\Phi}_-),$$

where

$$t \mapsto \widetilde{\Phi}_+(t) = \Phi_+(m - \tfrac{1}{2}, \tfrac{1}{\pi}t) \quad \text{and} \quad t \mapsto \widetilde{\Phi}_(t) = \Phi_-(m - \tfrac{1}{2}, \tfrac{1}{\pi}t) \sinh(t),$$

and $\Phi_\pm$ defined by (1.5) and (1.6). In each example we calculate the relative errors in Gaussian approximations for $n = 5(5)40$ and $m = 1(1)5$. Some of these results are presented below. All computations were done in Q-arithmetic on the MICROVAX 3400 computer (machine precision, m.p.$\approx 1.93 \times 10^{-34}$).

**Example 4.1.** Consider $T_1(1/2, a, p)$ and $S_1(1/2, a, p)$, for $p = 1, 2, 3, 4$ (see $1^0$ in §2). These series were calculated by Gautschi [2, §5], using the Laplace transform method for $a = \alpha$ and $a = i\alpha$, and $\alpha = .5$, 1., 2., 4., 8., 16., and 32. In this case, the function $g_0$ in (1.3) includes Dawson's integral (for $a = \alpha$) and the Fresnel integrals $C(x)$ and $S(x)$ (for $a = i\alpha$). As $\alpha$ increases, the convergence of the Gauss quadrature formula slows down considerably. In order to achieve better accuracy, when $\alpha$ is large, Gautschi [2] used "stratified" summation.

In [7, Example 4.3] we showed that our method applied to $T_1(1/2, \alpha, 1)$ is very efficient. Moreover, its convergence is slightly faster if the parameter $\alpha$ is larger.

Now, as typical results, we present the relative errors in the real and imaginary part of Gaussian approximations for $T_1(1/2, i\alpha, 1)$ and $S_1(1/2, i\alpha, 1)$, $\alpha = 16$, in Tables 4.1 and 4.2, respectively. (Numbers in parentheses indicate decimal exponents.)

TABLE 4.1
*Relative errors in the real and imaginary part of Gaussian approximations for $T_1(1/2, 16i, 1)$ for $m = 1(1)3$*

| $n$ | $m=1$ | | $m=2$ | | $m=3$ | |
|---|---|---|---|---|---|---|
| 5 | 2.1(−9) | 1.1(−7) | 8.0(−12) | 6.8(−10) | 3.4(−13) | 4.3(−11) |
| 10 | 8.3(−12) | 5.9(−9) | 8.6(−16) | 3.0(−13) | 2.0(−18) | 4.5(−16) |
| 15 | 9.2(−14) | 1.2(−10) | 9.3(−19) | 3.6(−16) | 1.0(−23) | 4.1(−20) |
| 20 | 3.6(−15) | 4.8(−12) | 1.5(−21) | 1.4(−19) | 2.4(−26) | 2.5(−23) |
| 25 | 2.5(−16) | 2.5(−13) | 2.1(−23) | 1.2(−20) | 1.2(−28) | 1.1(−25) |
| 30 | 1.5(−17) | 6.8(−15) | 8.5(−26) | 1.4(−22) | 1.0(−30) | 4.5(−28) |
| 35 | 1.3(−19) | 1.3(−15) | 1.2(−27) | 1.1(−24) | 1.0(−33) | 1.1(−30) |
| 40 | 1.5(−19) | 2.5(−16) | 6.3(−29) | 1.1(−27) | m.p. | 2.2(−32) |

TABLE 4.2
*Relative errors in the real and imaginary part of Gaussian approximations for $S_1(1/2, 16i, 1)$ for $m = 1(1)3$*

| $n$ | $m=1$ | | $m=2$ | | $m=3$ | |
|---|---|---|---|---|---|---|
| 5 | 1.2(−3) | 4.1(−4) | 1.5(−3) | 3.6(−4) | 1.7(−3) | 3.2(−4) |
| 10 | 8.2(−8) | 5.6(−8) | 9.1(−8) | 1.3(−8) | 1.0(−7) | 1.2(−8) |
| 15 | 3.5(−11) | 1.5(−9) | 4.1(−12) | 3.6(−13) | 4.4(−12) | 3.2(−13) |
| 20 | 1.4(−12) | 5.8(−11) | 2.2(−16) | 5.0(−18) | 1.9(−16) | 6.2(−18) |
| 25 | 9.3(−14) | 3.0(−12) | 1.5(−20) | 1.5(−19) | 5.5(−21) | 1.8(−22) |
| 30 | 5.6(−15) | 7.8(−14) | 1.4(−21) | 1.7(−21) | 1.3(−23) | 3.6(−24) |
| 35 | 2.6(−17) | 1.7(−14) | 2.4(−22) | 1.6(−23) | 3.1(−24) | 1.3(−25) |
| 40 | 5.6(−17) | 3.0(−15) | 5.6(−23) | 2.5(−25) | 2.5(−25) | 1.4(−26) |

The exact sums $T_1(1/2, i\alpha, 1)$ and $S_1(1/2, i\alpha, 1)$ (to 30 significant digits), as determined by Gaussian quadrature, are displayed in Table 4.3.

Table 4.3

*The exact sums $T_1(1/2, i\alpha, 1)$ and $S_1(1/2, i\alpha, 1)$*

| $\alpha$ | $T_1(1/2, i\alpha, 1)$ | $S_1(1/2, i\alpha, 1)$ |
|---|---|---|
| .5 | 2.3821813228551716829321975015 4 | −.58207978630142323536535213832 4 |
| | −.56425932522086830488967191023 7 | .33811053960395960258647989988 8 |
| 1. | 2.0061526552274142694399024448 4 | −.32169608782058206306680278744 4 |
| | −.79648812356984802409461716206 5 | .39749652835842654275911079278 4 |
| 2. | 1.5182315903661535830181749151 1 | −.99667443759451044598204054131 4(−1) |
| | −.84398104769701669822497009682 8 | .28077889438622834558158595216 5 |
| 4. | 1.0976937744094845693474333672 9 | −.24141921449940753257872660884 0(−1) |
| | −.74603482811338204999202209054 3 | .14925010609389453411780565962 0 |
| 8. | .78214784984207491638985340410 2 | −.59611382908720505231102558017 3(−2) |
| | −.60290376240912468855526392278 6 | .75377476448811853860863239186 1(−1) |
| 16. | .55454818156053687252697051264 5 | −.14861330467281749342147083915 8(−2) |
| | −.46409443668759260091217335427 9 | .37777097885217433030856081522 3(−1) |
| 32. | .39249605968188663342632998876 5 | −.37128000576225843086841211180 2(−3) |
| | −.34706378117749456558844014547 8 | .18899457884746990665740781568 2(−1) |
| 64. | .27762942965430951463171540226 8 | −.92804264247604991954191483202 1(−4) |
| | −.25486224165721468124590151053 1 | .94510884823642201398082527410 4(−2) |

As we can see from Tables 4.1 an 4.2, the convergence in the second case is slightly slower. Very similar results are obtained for $p = 2, 3, 4$ in both cases $T_p(1/2, a, 1)$ and $S_p(1/2, a, 1)$, where $a = \alpha e^{i\theta}$.

**Example 4.2.** Consider

$$T_1(\tfrac{1}{2}, a, \tfrac{3}{2}) = \sum_{k=1}^{+\infty} \frac{1}{k^{1/2}(k+a)^{3/2}} \quad \text{and} \quad S_1(\tfrac{1}{2}, a, \tfrac{3}{2}) = \sum_{k=1}^{+\infty} \frac{(-1)^k}{k^{1/2}(k+a)^{3/2}}\,.$$

This is a case when $p$ is not an integer. Here, $f(x) = 1/\sqrt{x(x+a)^3}$, and $F(z) = F_{3/2}(z)$ is given by $2^0$ in §2.

In Table 4.4 we show only the results for $a = 2$; those for other $a$ are similar. We can see a rapidly increasing of convergence of the summation process as $m$ increases.

Table 4.4

*Relative errors in Gaussian approximations of $T_1(1/2, a, 3/2)$ and $S_1(1/2, a, 3/2)$, $a = 2$, for $m = 1(1)3$*

| $a = 2$ | $T_1(1/2, a, 3/2)$ | | | $S_1(1/2, a, 3/2)$ | | |
|---|---|---|---|---|---|---|
| $n$ | $m = 1$ | $m = 2$ | $m = 3$ | $m = 1$ | $m = 2$ | $m = 3$ |
| 5 | 7.1(−7) | 4.2(−9) | 2.5(−10) | 1.2(−4) | 9.9(−5) | 8.1(−5) |
| 10 | 3.0(−8) | 1.5(−12) | 2.4(−15) | 1.1(−7) | 1.5(−9) | 1.4(−9) |
| 15 | 5.9(−10) | 1.8(−15) | 2.0(−19) | 2.3(−9) | 2.9(−14) | 2.2(−14) |
| 20 | 2.4(−11) | 5.5(−19) | 1.2(−22) | 9.3(−11) | 4.8(−18) | 3.3(−19) |
| 25 | 1.2(−12) | 6.2(−20) | 5.4(−25) | 4.8(−12) | 2.3(−19) | 7.2(−24) |
| 30 | 3.4(−14) | 7.0(−22) | 2.3(−27) | 1.3(−13) | 2.8(−21) | 8.2(−27) |
| 35 | 6.5(−15) | 5.3(−24) | 5.2(−30) | 2.6(−14) | 2.4(−23) | 2.4(−29) |
| 40 | 1.2(−15) | 1.0(−26) | 1.1(−31) | 4.8(−15) | 2.7(−26) | 4.0(−31) |

**Example 4.3.** For series

$$T_1(\tfrac{2}{3},4,1)=\sum_{k=1}^{+\infty}\frac{1}{k^{1/3}(k+4)} \quad \text{and} \quad S_1(\tfrac{2}{3},4,1)=\sum_{k=1}^{+\infty}\frac{(-1)^k}{k^{1/3}(k+4)}$$

the indefinte integral is given by $3^0$ in §2. As to accuracy, a similar situation prevails as in the previous examples. For instance, with $m=3$ and $n=30$ we obtain the sums with the relative errors 1.9(−28) and 3.8(−27), respectively.

**Example 4.4.** Using (3.3) we can calculate the Riemann zeta function $\zeta(z)$ in any complex $z\neq 1$. In Table 4.6 we present the results for $z=1+i0.01$, when we take $m=0$ and $n=5(5)40$. In each entry the first digit in error is underlined.

Table 4.6
*Gaussian approximations of $\zeta(1+i0.01)$ and relative errors for $m=0$ and $n=5(5)30$*

| $n$ | Real part | Rel. err. | Imaginary part | Rel. err. |
|---|---|---|---|---|
| 5 | $.5772\underline{0}648$ | 1.7(−5) | $-99.99927\underline{2}40$ | 5.6(−9) |
| 10 | $.577215\underline{5}862$ | 5.0(−7) | $-99.9992718\underline{5}01$ | 8.9(−11) |
| 15 | $.57721615\underline{5}679$ | 1.3(−8) | $-99.99927184\underline{0}94$ | 2.6(−12) |
| 20 | $.577216149\underline{1}117$ | 5.3(−10) | $-99.999271841 2\underline{1}14$ | 1.1(−13) |
| 25 | $.5772161494\underline{3}4$ | 2.3(−11) | $-99.9992718412024\underline{3}0$ | 4.3(−15) |
| 30 | $.577216149420\underline{7}42$ | 1.4(−13) | $-99.99927184120284\underline{2}9$ | 1.5(−16) |

The exact values (to 30 significant digits), as determined by Gaussian quadrature, are

$$\mathrm{Re}\,\zeta(1+i0.01)= \quad 0.577216149420661408748004242512,$$

$$\mathrm{Im}\,\zeta(1+i0.01)=-99.9992718412028581571383971188.$$

Taking $m>0$ the summation process becomes more efective, giving full accuracy (30 decimals) with only $n=20$ when $m=5$. Table 4.7 shows the corresponding relative errors for $m=1(1)3$.

Table 4.1
*Relative errors in the real and imaginary part of Gaussian approximations of $\zeta(1+i0.01)$ for $m=1(1)3$*

| $n$ | $m=1$ | | $m=2$ | | $m=3$ | |
|---|---|---|---|---|---|---|
| 5 | 1.4(−8) | 5.9(−13) | 1.8(−9) | 1.8(−13) | 6.7(−11) | 6.6(−15) |
| 10 | 1.9(−11) | 2.8(−15) | 2.4(−14) | 3.1(−18) | 1.1(−17) | 1.7(−21) |
| 15 | 2.3(−14) | 3.5(−18) | 3.7(−18) | 6.8(−22) | 4.9(−21) | 8.0(−25) |
| 20 | 5.0(−17) | 1.3(−20) | 2.9(−21) | 5.9(−25) | 5.1(−25) | 4.1(−29) |
| 25 | 9.7(−19) | 1.6(−22) | 9.8(−24) | 1.8(−27) | 6.7(−28) | 8.8(−32) |
| 30 | 1.6(−20) | 3.3(−24) | 3.1(−26) | 4.7(−30) | 6.9(−31) | 1.6(−33) |

We mention also that for $z=-2$ (a trivial zero of the Riemann function) this method, for $n=2$ and $m=0$, gives the machine zero.

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