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ORTHOGONAL POLYNOMIAL SYSTEMS AND SOME APPLICATIONS

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Abstract. Orthogonal polynomial systems on the real line, the unit circle, the unit semicircle and a circular arc with respect to a given positive definite or to a non-Hermitian inner product are considered. The basic properties, recurrence relations, distributions of zeros and other characterizations of such polynomials are included. The classical and non-classical orthogonal polynomials on the real line, as well as some Sobolev type orthogonal polynomials are considered. An interpretation of s-orthogonality is also treated. Finally, some applications in numerical integration, numerical differentiation, moment-preserving spline approximation and summation of slowly convergent series are done.

1. Introduction and basic definitions

The theory of inner product spaces enables an introduction of orthogonal systems which play an important role in many branches of mathematics, physics and other applied and computational sciences. We start with the following definition:

Definition 1.1. Given a real linear space of functions X, an *inner product* (f,g) defined is a mapping of X^2 into $\mathbb{R}$ such that

(a) $(f+g,h)=(f,h)+(g,h)$ (Linearity),

(b) $(\alpha f,g)=\alpha(f,g)$ (Homogeneity),

(c) $(f,g)=(g,f)$ (Symmetry),

(d) $(f,f)>0,\ (f,f)=0 \Leftrightarrow f=0$ (Positivity),

where $f,g,h\in X$ and α is a real parameter. The space X will be called an *inner product space*.

A similar definition can be done for complex spaces. Namely, if X is a complex linear space, then the inner product $(f,g)\colon X^2\to\mathbb{C}$ is such that the condition (c) is replaced by

(c′) $(f,g)=\overline{(g,f)}$ (Hermitian Symmetry).

The bar in the above line designates the complex conjugate.

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Definition 1.2. A system S of elements of an inner product space is called *orthogonal* if $(f,g)=0$ for every $f \neq g$ $(f,g \in S)$. If $(f,f)=1$ for each $f \in S$, then the system is called *orthonormal*.

Starting from a linearly independent system of elements of an inner product space and using the well-known *Gram-Schmidt orthogonalizing process* we can construct the corresponding orthogonal (orthonormal) system. In this survey we consider only orthogonal (orthonormal) polynomial systems with respect to different inner product spaces.

Definition 1.3. A system of polynomials $\{\pi_k\}$, where

$$\begin{aligned} &\pi_k(t) = b_k t^k + c_k t^{k-1} + \text{ lower degree terms}, \quad b_k > 0, \\ &(\pi_k, \pi_m) = \delta_{km}, \quad k, m \geq 0, \end{aligned} \tag{1.1}$$

is called a system of orthonormal polynomials with respect to the inner product $(\,\cdot\,,\,\cdot\,)$.

Orthogonal polynomial systems are very useful in many problems in the approximation theory, mathematical and numerical analysis, and their applications (for example, Gaussian quadrature processes, least square approximation of functions, differential and difference equations, Fourier series, etc.).

This paper is organized as follows. In Section 2 we consider the orthogonality on the real line, with several classes of orthogonal polynomials such as classical, semi-classical and non-classical polynomials, as well as orthogonal polynomials with a Sobolev inner product. Also, an interpretation of s-orthogonality is included. Section 3 is devoted to some important applications of orthogonal polynomials on the real line as Gauss-Christoffel quadrature formulas, moment-preserving spline approximation and summation of slowly convergent series. The case of orthogonality on the unit circle is considered in Section 4, and the cases on the semicircle and a circular arc in Section 5. Finally, in Section 6 we deal with some applications of polynomials orthogonal on the semicircle.

2. Orthogonality on the real line

Let $\lambda\colon \mathbb{R} \to \mathbb{R}$ be a fixed non-decreasing function with infinitely many points of increase for which all moments $\mu_k = \int_{\mathbb{R}} t^k \, d\lambda(t)$, $k = 0, 1, \ldots$, exist and are finite.

Then the improper Stieltjes integral $\int_{\mathbb{R}} P(t)\, d\lambda(t)$ exists for every polynomial P. By the application of the Lebesgue-Stieltjes integral $\int_{\mathbb{R}} f(t)\, d\lambda(t)$ to characteristic functions of sets, the function λ engenders a Lebesgue-Stieltjes measure $d\lambda(t)$, which is known also as *m-distribution* (cf. Freud [26]). Moreover, if $t \mapsto \lambda(t)$ is an absolutely continuous function, then we say that $\lambda'(t) = w(t)$ is a *weight function*. In that case, the measure $d\lambda$ can be express as $d\lambda(t) = w(t)\, dt$, where the weight function $t \mapsto w(t)$ is a non-negative and measurable in Lebesgue's sense for which all moments exists and $\mu_0 = \int_{\mathbb{R}} w(t)\, dt > 0$.

In the general case the function λ can be written in the form $\lambda = \lambda_{\mathrm{ac}} + \lambda_{\mathrm{s}} + \lambda_{\mathrm{j}}$, where λ_{ac} is absolutely continuous, λ_{s} is singular, and λ_{j} is a jump function.

The support of the measure, i.e., the set of points of increase of $t \mapsto \lambda(t)$ we denote by $\mathrm{supp}(d\lambda)$. It is always an infinite and closed set. If $\mathrm{supp}(d\lambda)$ is bounded, then the smallest closed interval containing $\mathrm{supp}(d\lambda)$ we will denote by $\Delta(d\lambda)$. For example, if $\Delta(d\lambda) = [a, b]$ then we say that $d\lambda(t)$ lies in $[a, b]$. In that case we have $\lambda(t) = \lambda(-\infty)$ for $t < a$ and $\lambda(t) = \lambda(+\infty)$ for $t > b$. In addition, if $t \mapsto \lambda(t)$ is absolutely continuous, then the weight function $t \mapsto w(t)$ vanishes outside of $[a, b]$, or more generally, outside of $\mathrm{supp}(d\lambda)$.

2.1. General properties. For any m-distribution $d\lambda(t)$ there exists a unique system of orthonormal polynomials $\pi_k(\cdot) = \pi_k(\,\cdot\,; d\lambda)$, $k = 0, 1, \ldots$, defined by (1.1), where $b_k = b_k(d\lambda)$, $c_k = c_k(d\lambda)$ and the inner product is given by

$$(f, g) = \int_{\mathbb{R}} f(t) g(t)\, d\lambda(t) \qquad \big(f, g \in X = L^2(\mathbb{R}) \equiv L^2(\mathbb{R}; d\lambda)\big). \tag{2.1.1}$$

If we have an absolutely continuous function $t \mapsto \lambda(t)$, then instead of $\pi_k(\,\cdot\,; d\lambda)$, $b_k(d\lambda)$, $\mathrm{supp}(d\lambda)$, $\ldots$, we usually write $\pi_k(\,\cdot\,; w)$, $b_k(w)$, $\mathrm{supp}(w)$, $\ldots$, respectively, where $\lambda'(t) = w(t)$.

If we have $\mathrm{supp}(w) = [a, b]$, where $-\infty < a < b < +\infty$, we say that $\{\pi_k\}$ is a system of orthonormal polynomials in a finite interval $[a, b]$. For (a, b) we say that it is an *interval of orthogonality*.

Now we give a few basic properties of orthogonal polynomials:

Theorem 2.1.1. *The system of orthonormal polynomials $\{\pi_k\}$, associated with the distribution $d\lambda(t)$, satisfy a three-term recurrence relation*

$$t\pi_k(t) = u_{k+1}\pi_{k+1}(t) + v_k \pi_k(t) + u_k \pi_{k-1}(t) \quad (k \ge 0), \tag{2.1.2}$$

where $\pi_{-1}(t) = 0$ and the coefficients $u_k = u_k(d\lambda)$ and $v_k = v_k(d\lambda)$ are given by

$$u_k = \frac{b_{k-1}}{b_k} \qquad \textit{and} \qquad v_k = \int_{\mathbb{R}} t\pi_k(t)^2\, d\lambda(t).$$

Since $\pi_0(t) = b_0 = 1/\sqrt{\mu_0}$ and $b_{k-1} = u_k b_k$ we have that $b_k = b_0/(u_1 u_2 \cdots u_k)$. Notice that $u_k > 0$ for each k.

Contrary, for two given real sequences $\{u_k\}_{k\in\mathbb{N}}$ and $\{v_k\}_{k\in\mathbb{N}_0}$, where $u_k > 0$ for each $k \in \mathbb{N}$, one can construct a sequence of polynomials using the three-term recurrence relation (2.1.2), starting with initial values $\pi_{-1}(t) = 0$ and $\pi_0(t) = 1$. It is well-known by Favard's theorem (cf. Chihara [20]) that there exists a positive measure $d\sigma(t)$ on $\mathbb{R}$ such that

$$\int_{\mathbb{R}} \pi_k(t)\pi_m(t)\, d\sigma(t) = \delta_{km}, \qquad k, m \ge 0.$$

The measure $d\sigma(t)$ is not unique which depends of the fact whether or not the Hamburger moment problem is determined. A sufficient condition for a unique measure is the Carleman's condition given by $\sum\limits_{k=1}^{+\infty}(1/u_k) = +\infty$. Evidently, it holds if $\{u_k\}_{k\in\mathbb{N}}$ is a bounded sequence.

In many considerations and applications we use the monic orthogonal polynomials $\hat{\pi}_k(t) = \pi_k(t)/b_k = t^k +$ lower degree terms. Such polynomials satisfy the following three-term recurrence relation

$$\hat{\pi}_{k+1}(t) = (t-\alpha_k)\hat{\pi}_k(t) - \beta_k\hat{\pi}_{k-1}(t), \qquad k = 0,1,2,\dots, \tag{2.1.3}$$

where $\alpha_k = v_k$ and $\beta_k = u_k^2 > 0$.

Because of orthogonality, we have that

$$\alpha_k = \frac{(t\hat{\pi}_k, \hat{\pi}_k)}{(\hat{\pi}_k, \hat{\pi}_k)} \quad (k \geq 0), \quad \beta_k = \frac{(\hat{\pi}_k, \hat{\pi}_k)}{(\hat{\pi}_{k-1}, \hat{\pi}_{k-1})} \quad (k \geq 1).$$

The coefficient β_0, which multiplies $\hat{\pi}_{-1} = 0$ in three-term recurrence relation may be arbitrary. Sometimes, it is convenient to define it by $\beta_0 = \mu_0 = \int_{\mathbb{R}} d\lambda(t)$. Then the norm of $\hat{\pi}_k$ can be express in the form

$$\|\hat{\pi}_k\| = \sqrt{(\hat{\pi}_k, \hat{\pi}_k)} = \sqrt{\beta_0\beta_1\cdots\beta_k}\,. \tag{2.1.4}$$

We mention that the existence of a three-term recurrence relation for orthogonal polynomials is a consequence of the property $(tf,g) = (f,tg)$ of the inner product (2.1.1).

Theorem 2.1.2. *All zeros of* $t \mapsto \pi_n(t; d\lambda)$, $n \geq 1$, *are real and distinct and are located in the interior of the interval* $\Delta(d\lambda)$.

Let $\tau_k^{(n)}$, $k = 1,\dots,n$, denote the zeros of $\pi_n(t; d\lambda)$ in increasing order

$$\tau_1^{(n)} < \tau_2^{(n)} < \cdots < \tau_n^{(n)}.$$

Theorem 2.1.3. *The zeros of* $\pi_n(t; d\lambda)$ *and* $\pi_{n+1}(t; d\lambda)$ *interlace, i.e.,*

$$\tau_k^{(n+1)} < \tau_k^{(n)} < \tau_{k+1}^{(n+1)} \qquad (k = 1,\dots,n;\ n \in \mathbb{N}).$$

Taking $k = 0,1,\dots,n-1$ in (2.1.2), one can obtain the following system of equations

$$t\boldsymbol{p}_n(t) = J_n(d\lambda)\boldsymbol{p}_n(t) + u_n\pi_n(t)\boldsymbol{e}_n,$$

where

$$J_n(d\lambda) = \begin{bmatrix} v_0 & u_1 & & & \mathrm{O} \\ u_1 & v_1 & u_2 & & \\ & u_2 & v_2 & \ddots & \\ & & \ddots & \ddots & u_{n-1} \\ \mathrm{O} & & & u_{n-1} & v_{n-1} \end{bmatrix}, \quad \boldsymbol{p}_n(t) = \begin{bmatrix} \pi_0(t) \\ \pi_1(t) \\ \pi_2(t) \\ \vdots \\ \pi_{n-1}(t) \end{bmatrix}, \quad \boldsymbol{e}_n = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}.$$

The tridiagonal matrix $J_n = J_n(d\lambda)$ is known as the *Jacobi matrix*. It is clear that $\pi_n(t) = 0$ if and only if $t\boldsymbol{p}_n(t) = J_n\boldsymbol{p}_n(t)$, i.e., the zeros $\tau_k^{(n)}$ of $\pi_n(t)$ are the same as the eigenvalues of the Jacobi matrix J_n. Also, notice that the monic polynomial $\hat{\pi}_n(t)$ can be expressed in the following determinant form

$$\hat{\pi}_n(t) = \det(tI_n - J_n),$$

where I_n is the identity matrix of the order n.

Define the function $(t, x) \mapsto K_n(t, x)$ by

$$K_n(t, x) = \sum_{\nu=0}^{n-1} \pi_\nu(t)\pi_\nu(x) \qquad (n \geq 1), \tag{2.1.5}$$

which plays a fundamental role in the integral representation of partial sums of the orthogonal expansions. Namely, $K_n(t, x)$ represents the kernel of the n-th partial sum. Notice that $K_n(t, x) = K_n(x, t)$. Using the three-term recurrence relation from Theorem 2.1.1 we can prove:

Theorem 2.1.4. *Let $K_n(t, x)$ be defined by* (2.1.5). *Then we have*

$$K_n(t, x) = u_{n+1}\frac{\pi_n(t)\pi_{n-1}(x) - \pi_{n-1}(t)\pi_n(x)}{t - x}, \tag{2.1.6}$$

where u_{n+1} is defined in Theorem 2.1.1.

Letting $t \to x$ we find

$$K_n(x, x) = u_{n+1}(\pi_n'(x)\pi_{n-1}(x) - \pi_{n-1}'(x)\pi_n(x)).$$

Formula (2.1.6) is known as the *Christoffel-Darboux identity*.

We mention now an important extremal problem for monic polynomials (cf. [101, pp. 416–417]): *Among all polynomials of degree n, with leading coefficient unity, find the polynomial which deviates least from zero in L^2-norm*

$$\|P\| = \sqrt{(P, P)} = \left(\int_{\mathbb{R}} |P(t)|^2 \, d\lambda(t)\right)^{1/2}.$$

The solution is given in the following theorem:

Theorem 2.1.5. *Let* $P(t) = \sum_{\nu=0}^{n} a_\nu t^\nu$, *with* $a_n = 1$, *be an arbitrary monic polynomial of degree* n. *Then*

$$\|P\| \geq \|\hat{\pi}_n\| = b_n^{-1},$$

with equality only if $P(t) = \hat{\pi}_n(t) = \pi_n(t)/b_n$, *where* $\hat{\pi}_n$ *is the monic polynomial orthogonal with respect to the measure* $d\lambda(t)$ *on* $\mathbb{R}$.

A survey on characterization theorems for orthogonal polynomials on the real line was given recently by Al-Salam [6]. In next subsection we consider a special class of orthogonal polynomials so-called *classical orthogonal polynomials*. For some extensions of this polynomial class see Andrews and Askey [8], Askey and Wilson [9], and Atakishiyev and Suslov [10].

2.2. Classical orthogonal polynomials. A very important class of orthogonal polynomials on an interval of orthogonality $(a, b) \in \mathbb{R}$ is constituted by so-called the *classical orthogonal polynomials*. They are distinguished by several particular properties.

Let $\boldsymbol{\mathcal{P}}_n$ be the set of all algebraic polynomials P $(\not\equiv 0)$ of degree at most n and the inner product is given by

$$(f, g)_w = \int_a^b w(t) f(t) g(t)\, dt. \tag{2.2.1}$$

Since every interval (a, b) can be transformed by a linear transformation to one of following intervals: $(-1, 1)$, $(0, +\infty)$, $(-\infty, +\infty)$, we will restrict our consideration (without loss of generality) only to these three intervals.

Definition 2.2.1. The orthogonal polynomials $\{Q_k\}$ on (a, b) with respect to the inner product (2.2.1) are called the *classical orthogonal polynomials* if their weight functions $t \mapsto w(t)$ satisfy the differential equation

$$\frac{d}{dt}(A(t)w(t)) = B(t)w(t),$$

where

$$A(t) = \begin{cases} 1 - t^2, & \text{if } (a, b) = (-1, 1), \\ t, & \text{if } (a, b) = (0, +\infty), \\ 1, & \text{if } (a, b) = (-\infty, +\infty), \end{cases}$$

and $B(t)$ is a polynomial of the first degree. For such classical weights we will write $w \in CW$.

We note that if $w \in CW$, then $w \in C^1(a, b)$, and also the following property:

Theorem 2.2.1. *If $w \in CW$ then for each $m = 0, 1, \ldots$ we have*

$$\lim_{t \to a+} t^m A(t) w(t) = 0 \qquad \textit{and} \qquad \lim_{t \to b-} t^m A(t) w(t) = 0.$$

Based on the above definition, the classical orthogonal polynomials $\{Q_k\}$ on (a, b) can be specificated as the *Jacobi polynomials* $P_k^{(\alpha,\beta)}(t)$ $(\alpha, \beta > -1)$ on $(-1, 1)$, the *generalized Laguerre polynomials* $L_k^s(t)$ $(s > -1)$ on $(0, +\infty)$, and finally as the *Hermite polynomials* $H_k(t)$ on $(-\infty, +\infty)$. Their weight functions and the corresponding polynomials $A(t)$ and $B(t)$ are given in Table 2.2.1.

TABLE 2.2.1
The Classification of the Classical Orthogonal Polynomials

(a,b)	$w(t)$	$A(t)$	$B(t)$	λ_k
$(-1,1)$	$(1-t)^\alpha(1+t)^\beta$	$1-t^2$	$\beta-\alpha-(\alpha+\beta+2)t$	$k(k+\alpha+\beta+1)$
$(0,+\infty)$	$t^s e^{-t}$	t	$s+1-t$	k
$(-\infty,+\infty)$	e^{-t^2}	1	$-2t$	$2k$

Special cases of the Jacobi polynomials are:

1° The *Legendre polynomials* $P_k(t)$ (for $\alpha = \beta = 0$);

2° The *Chebyshev polynomials of the first kind* $T_k(t)$ (for $\alpha = \beta = -1/2$);

3° The *Chebyshev polynomials of the second kind* $S_k(t)$ (for $\alpha = \beta = 1/2$);

4° The *Chebyshev polynomials of the third kind* $U_k(t)$ (for $\alpha = -\beta = -1/2$);

5° The *Chebyshev polynomials of the fourth kind* $V_k(t)$ (for $\alpha = -\beta = 1/2$);

6° The *Gegenbauer* or *ultraspherical polynomials* $C_k^\lambda(t)$ (for $\alpha = \beta = \lambda - 1/2$).

If $s = 0$, the generalized Laguerre polynomials reduces to the standard Laguerre polynomials $L_k(t)$.

There are many characterizations of the classical orthogonal polynomials. In sequel we give the basic common properties of these polynomials (cf. [101]).

Theorem 2.2.2. *The derivatives of the classical orthogonal polynomials $\{Q_k\}_{k \in \mathbb{N}_0}$ form also a sequence of the classical orthogonal polynomials.*

Applying the induction method we can prove a more general result:

Theorem 2.2.3. *The sequence $\{Q_k^{(m)}\}_{k=m,m+1,\ldots}$ is a classical orthogonal polynomial sequence on (a, b) with respect to the weight function $t \mapsto w_m(t) = A(t)^m w(t)$. The differential equation for this weight is $(A(t) w_m(t))' = B_m(t) w_m(t)$, where $B_m(t) = mA'(t) + B(t)$.*

Theorem 2.2.4. *The classical orthogonal polynomial* $t \mapsto Q_k(t)$ *is a particular solution of the second order linear differential equation of hyphergeometric type*

$$L[y] = A(t)y'' + B(t)y' + \lambda_k y = 0, \tag{2.2.2}$$

where

$$\lambda_k = -k\left(\tfrac{1}{2}(k-1)A''(0) + B'(0)\right). \tag{2.2.3}$$

The equation (2.2.2) can be written in the Sturm-Liouville form

$$\frac{d}{dt}\left(A(t)w(t)\frac{dy}{dt}\right) + \lambda_k w(t)y = 0. \tag{2.2.4}$$

The coefficients λ_k are also displayed in Table 2.2.1.

Similarly, the m-th derivative of Q_k satisfies the differential equation

$$\frac{d}{dt}\left(A(t)w_m(t)\frac{dy}{dt}\right) + \lambda_{k,m} w_m(t)y = 0,$$

where $\lambda_{k,m} = -(k-m)\left(\frac{1}{2}(k+m-1)A''(0) + B'(0)\right)$. We note that this expression for $\lambda_{k,m}$ reduces to (2.2.3) for $m = 0$, i.e., $\lambda_{k,0} = \lambda_k$.

Remark 2.2.1. The characterization of the classical orthogonal polynomials by differential equation (2.2.2), i.e. (2.2.4), was proved by Lesky [76], and conjectured by Aczél [2] (see also Bochner [15]). Such a differential equation appears in many mathematical models in atomic physics, electrodynamics and acoustics. As an example we mention the well-known Schrödinger equation.

The classical orthogonal polynomials possess a Rodrigues' type formula (cf. Bateman and Erdélyi [11], Tricomi [125], and Suetin [121]).

Theorem 2.2.5. *The classical orthogonal polynomial* $Q_k(t)$ *can be expressed in the form*

$$Q_k(t) = \frac{C_k}{w(t)} \cdot \frac{d^k}{dt^k}\bigl(A(t)^k w(t)\bigr), \tag{2.2.5}$$

where C_k *are constants different from zero.*

Using the Cauchy formula for k-th derivative of a regular function, (2.2.5) can be represented in the following integral form

$$Q_k(t) = \frac{C_k}{w(t)} \cdot \frac{k!}{2\pi i} \oint_{\Gamma} \frac{A(z)^k w(z)}{(z-t)^{k+1}}\, dz, \tag{2.2.6}$$

where Γ is a closed contour such that $t \in \operatorname{int} \Gamma$.

The constants C_k in (2.2.5) and (2.2.6) can be chosen in different way (for example, Q_k to be monic, orthonormal, etc.). A historical reason leads to

$$C_k = \begin{cases} \dfrac{(-1)^k}{2^k k!} & \text{for } P_k^{(\alpha,\beta)}(t), \\ \ \ 1 & \text{for } L_k^s(t), \\ (-1)^k & \text{for } H_k(t). \end{cases}$$

In addition, the Gegenbauer and the Chebyshev polynomials need

$$\begin{aligned} C_k^\lambda(t) &= \frac{(2\lambda)_k}{\left(\lambda+\frac{1}{2}\right)_k} P_k^{(\alpha,\alpha)}(t) \quad (\alpha = \lambda - 1/2), \\ T_k(t) &= \frac{k!}{\left(\frac{1}{2}\right)_k} P_k^{(-1/2,-1/2)}(t), \\ S_k(t) &= \frac{(k+1)!}{\left(\frac{3}{2}\right)_k} P_k^{(1/2,1/2)}(t), \end{aligned}$$

where $(s)_k$ is the standard notation for Pochhammer's symbol

$$(s)_k = s(s+1)\cdots(s+k-1) = \frac{\Gamma(s+k)}{\Gamma(s)} \quad (\Gamma \text{ is the gamma function}).$$

For such defined polynomials $Q_k(t) = a_k\bigl(t^k + r_k t^{k-1} + \cdots\bigr)$, we give the leading coefficient a_k, the coefficient r_k, and the norm $\|Q_k\|$.

1° *Jacobi case*:

$$a_k = \frac{(k+\alpha+\beta+1)_k}{2^k k!}, \qquad r_k = \frac{k(\alpha-\beta)}{2k+\alpha+\beta},$$

$$\|P_k^{(\alpha,\beta)}\|^2 = \frac{2^{\alpha+\beta+1}\Gamma(k+\alpha+1)\Gamma(k+\beta+1)}{k!(2k+\alpha+\beta+1)\Gamma(k+\alpha+\beta+1)};$$

2° *Gegenbauer case*:

$$a_k = \frac{2^k}{k!}(\lambda)_k, \quad r_k = 0, \quad \|C_k^\lambda\|^2 = \sqrt{\pi}\,\frac{(2\lambda)_k\Gamma(\lambda+1/2)}{(k+\lambda)k!\Gamma(\lambda)};$$

3° *Legendre case*:

$$a_k = \frac{(2k)!}{2^k (k!)^2}, \qquad r_k = 0, \qquad \|P_k\|^2 = \frac{2}{2k+1};$$

4° *Chebyshev case of the first kind*:

$$a_k = 2^{k-1}, \quad r_k = 0, \quad \|T_0\|^2 = \pi, \quad \|T_k\|^2 = \frac{\pi}{2} \quad (k \neq 0);$$

5° *Chebyshev case of the second kind*:

$$a_k = 2^k, \qquad r_k = 0, \qquad \|S_k\|^2 = \frac{\pi}{2}\,;$$

5° *Generalized Laguerre case*:

$$a_k = (-1)^k, \quad r_k = -k(k+s), \quad \|L_k^s\|^2 = k!\Gamma(k+s+1);$$

6° *Hermite case*:

$$a_k = 2^k, \qquad r_k = 0, \qquad \|H_k\|^2 = 2^k k!\sqrt{\pi}\,.$$

Putting $t = \cos\theta$, $-1 \le t \le 1$, Chebyshev polynomials of the first and second kind can be expressed in the forms

$$T_k(x) = T_k(\cos\theta) = \cos k\theta \quad \text{and} \quad S_k(x) = S_k(\cos\theta) = \frac{\sin(k+1)\theta}{\sin\theta},$$

respectively. Thus, trigonometric representations of these polynomials are

$$T_k(t) = \cos(k\arccos t) \quad \text{and} \quad S_k(x) = \frac{\sin\bigl((k+1)\arccos t\bigr)}{\sqrt{1-t^2}}\,.$$

Similarly to the well-known Landau inequality (see Landau [72]) for continuously-differentiable functions and other generalizations (cf. Djordjević and Milovanović [22], Gorny [59], Hille [64], Kolmogoroff [69], Milovanović [90], Schoenberg [115], Steckhin [119]), Agarwal and Milovanović [3] stated the following characterization of the classical orthogonal polynomials:

Theorem 2.2.6. *Let $\|f\|^2 = (f,f)_w$, where $w \in CW$. For all $P \in \boldsymbol{\mathcal{P}}_n$ the inequality*

$$(2\lambda_n + B'(0))\|\sqrt{A}P'\|^2 \le \lambda_n^2\|P\|^2 + \|AP''\|^2 \tag{2.2.7}$$

holds, with equality if only if $P(t) = cQ_n(t)$, where $Q_n(t)$ is the classical orthogonal polynomial on (a,b) with respect to the weight function $t \mapsto w(t)$, and c is an arbitrary real constant. λ_n, $A(t)$ and $B(t)$ are given in Table 2.2.1.

The equality case in (2.2.7) gives a characterization of the classical orthogonal polynomials. For $w(t) = e^{-t^2}$ on $(-\infty,+\infty)$, the inequality (2.2.7) reduces to Varma's result [127]

$$\|P'\|^2 \le \frac{1}{2(2n-1)}\|P''\|^2 + \frac{2n^2}{2n-1}\|P\|^2.$$

Recently, Guessab and Milovanović [62] have considered a weighted L^2 analogues of the well-known Bernstein's inequality, which can be stated in the following form (cf. [101]):

$$\|\sqrt{1-t^2}P'(t)\|_\infty \le n\|P\|_\infty \qquad (P \in \boldsymbol{\mathcal{P}}_n), \tag{2.2.8}$$

where $\|f\|_\infty = \max\limits_{-1\le x\le 1} |f(t)|$. Using the norm $\|f\|^2 = (f,f)_w$, $w \in CW$, they have considered the following problem connected with the Bernstein's inequality (2.2.8): *Determine the best constant* $C_{n,m}(w)$ $(1 \le m \le n)$ *such that the inequality*

$$\|A^{m/2}P^{(m)}\|_w \le C_{n,m}(w)\|P\|_w \tag{2.2.9}$$

holds for all $P \in \boldsymbol{\mathcal{P}}_n$.

Theorem 2.2.7. *For all* $P \in \boldsymbol{\mathcal{P}}_n$ *the inequality* (2.2.9) *holds with the best constant*

$$C_{n,m}(w) = \sqrt{\lambda_{n,0}\lambda_{n,1}\cdots\lambda_{n,m-1}}, \tag{2.2.10}$$

where $\lambda_{n,k} = -(n-k)\left(\frac{1}{2}(n+k-1)A''(0) + B'(0)\right)$. *The equality is attained in* (2.2.10) *if and only if* $P(t)$ *is a constant multiple of the classical polynomial* $Q_n(t)$ *orthogonal with respect to the weight function* $w \in CW$ *on* (a,b).

We list now the coefficients α_k $(k \ge 0)$ and β_k $(k \ge 1)$ in the three-term recurrence relation for the monic classical orthogonal polynomials $\hat{Q}_k(t)$ on (a,b),

$$\begin{aligned} &\hat{Q}_{k+1}(t) = (t-\alpha_k)\hat{Q}_k(t) - \beta_k\hat{Q}_{k-1}(t) \quad (k \ge 0),\\ &\hat{Q}_1(t) = 0, \quad \hat{Q}_0(t) = 1. \end{aligned} \tag{2.2.11}$$

The coefficients u_k and v_k in the corresponding recurrence relation for orthonormal polynomials (see (2.1.3)) are given by $u_k = \sqrt{\beta_k}$ and $v_k = \alpha_k$.

Also, we give the moment $\mu_0 = \int_a^b w(t)\,dt$ $(w \in CW)$.

1° *Jacobi case*: $\hat{P}_k^{(\alpha,\beta)}(t) = 2^k k!/\big((k+\alpha+\beta+1)_k\big)P_k^{(\alpha,\beta)}(t)$,

$$\begin{aligned} \mu_0 &= \frac{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)},\\ \alpha_k &= \frac{\beta^2-\alpha^2}{(2k+\alpha+\beta)(2k+\alpha+\beta+2)},\\ \beta_k &= \frac{4k(k+\alpha)(k+\beta)(k+\alpha+\beta)}{(2k+\alpha+\beta)^2\big((2k+\alpha+\beta)^2-1\big)}. \end{aligned}$$

2° *Gegenbauer case*: $\hat{C}_k^\lambda(t) = k!/\big(2^k(\lambda)_k\big)C_k^\lambda(t)$,

$$\mu_0 = \frac{\sqrt{\pi}\Gamma(\lambda+1/2)}{\Gamma(\lambda+1)}, \quad \alpha_k = 0, \quad \beta_k = \frac{k(k+2\lambda-1)}{4(k+\lambda)(k+\lambda-1)}.$$

3° *Legendre case*: $\hat{P}_k(t) = 2^k(k!)^2/\big((2k)!\big)P_k(t)$,

$$\mu_0 = 2, \quad \alpha_k = 0, \quad \beta_k = \frac{k^2}{4k^2-1}\,.$$

4° *Chebyshev case of the first kind*: $\hat{T}_0(t) = 1$, $\hat{T}_k(t) = 2^{1-k}T_k(t)$,

$$\mu_0 = \pi, \quad \alpha_k = 0, \quad \beta_1 = \frac{1}{2}\,, \quad \beta_k = \frac{1}{4} \ \ (k \ge 2).$$

5° *Chebyshev case of the second kind*: $\hat{S}_k(t) = 2^{-k}S_k(t)$,

$$\mu_0 = \frac{\pi}{2}\,, \quad \alpha_k = 0, \quad \beta_k = \frac{1}{4} \ \ (k \ge 1).$$

5° *Generalized Laguerre case*: $\hat{L}_k^s(t) = (-1)^k L_k^s(t)$,

$$\mu_0 = \Gamma(s+1), \quad \alpha_k = 2k+s+1, \quad \beta_k = k(k+s).$$

6° *Hermite case*: $\hat{H}_k(t) = 2^{-k}H_k(t)$,

$$\mu_0 = \sqrt{\pi}\,, \quad \alpha_k = 0, \quad \beta_k = \frac{k}{2}\,.$$

The norm of monic orthogonal polynomials can be calculated using (2.1.4).

In the case of classical orthogonal polynomials one can express $Q_k'(t)$ in terms of $Q_k(t)$ and $Q_{k-1}(t)$. Such a formula for the monic orthogonal polynomials is

$$A(t)\frac{d}{dt}\,\hat{Q}_k(t) = (e_k t + f_k)\hat{Q}_k(t) + \omega_k\beta_k\hat{Q}_{k-1}(t), \tag{2.2.12}$$

where $A(t)$ is given in Table 2.2.1 and the coefficient β_k is the same as in the three-term recurrence relation (2.2.11).

In the Jacobi case we have

$$e_k = -k, \qquad f_k = \frac{k(\alpha-\beta)}{2k+\alpha+\beta}\,, \qquad \omega_k = 2k+\alpha+\beta+1.$$

In the generalized Laguerre case and the Hermite case, formula (2.2.12) reduces to

$$t\frac{d}{dt}\hat{L}_k^s(t) = k\hat{L}_k^s(t) + k(k+s)\hat{L}_{k-1}^s(t) \quad \text{and} \quad \frac{d}{dt}\hat{H}_k(t) = k\hat{H}_{k-1}(t),$$

respectively.

2.3. Non-classical orthogonal polynomials. As we have seen in the previous subsection the monic Chebyshev polynomials of the second kind

$$\hat{S}_k(t) = \frac{1}{2^k} \cdot \frac{\sin(k+1)\theta}{\sin\theta}, \qquad t = \cos\theta,$$

have a very simple three-term recurrence relation (2.2.11) with $\alpha_k = 0$ and $\beta_k = 1/4$. A system of orthogonal polynomials for which the recursion coefficients satisfy

$$\lim_{k\to+\infty} \alpha_k = 0 \quad \text{and} \quad \lim_{k\to+\infty} \beta_k = \frac{1}{4} \tag{2.3.1}$$

will be said to be a perturbation of the polynomials $\hat{S}_k(t)$. Similarly, one can consider orthogonal polynomials with such coefficients that

$$\lim_{k\to+\infty} \alpha_k = a \quad \text{and} \quad \lim_{k\to+\infty} \beta_k = \frac{b^2}{4} > 0,$$

where $a, b \in \mathbb{R}$. These polynomials are perturbations of $t \mapsto b^k \hat{S}_k((t-a)/b)$ and are said to belong to the class $\boldsymbol{\mathcal{N}}(a,b)$, which was introduced and considered in detail by Nevai [106][1]. Evidently, (2.3.1) holds for polynomials from the class $\boldsymbol{\mathcal{N}}(0,1)$.

There are several classes of orthogonal polynomials which are in certain sense close to the classical orthogonal polynomials. For example, when the weight $t \mapsto W(t)$ is the product of a classical weight $t \mapsto w(t)$ times a polynomial, Ronveaux [112] found the second-order differential equation for the corresponding orthogonal polynomials. Ronveaux and Thiry [114] developed a REDUCE package giving such differential equations. The following cases have been studied by Ronveaux and Marcellán [113]:

1° *Rational Case.* $W(t) = R(t)w(t)$, where R is a rational function with poles and zeros outside the support of w;

2° δ *Dirac distribution.*

$$W(t) = w(t) + \sum_{k=1}^{m} \lambda_k \delta(t - t_k),$$

where the positive mass λ_k is located at t_k (t_k outside or inside the support of w).

In both cases, orthogonal polynomials are *semi-classical* (see Maroni [85]).

A nice survey about orthogonal polynomials and spectral theory was given by Everitt and Littlejohn [24].

In many applications of orthogonal polynomials it is very important to know the recursion coefficients α_k and β_k. If $d\lambda(t)$ is one of the classical measures, then α_k

[1] Originally, Nevai defined this class for orthonormal polynomials.

and β_k are known explicitly. Furthermore, there are certain non-classical measures when we know also these coefficients. In sequel we mention only a few of them:

1° *Generalized Gegenbauer weight* $w(t) = |t|^\mu(1-t^2)^\alpha$, $\mu, \alpha > -1$, on $[-1,1]$. The (monic) generalized Gegenbauer polynomials $W_k^{(\alpha,\beta)}(t)$, $\beta = (\mu-1)/2$, were introduced by Lascenov [74] (see, also, Chihara [20, pp. 155–156]). These polynomials can be expressed in terms of the Jacobi polynomials,

$$\begin{aligned} W_{2k}^{(\alpha,\beta)}(t) &= \frac{k!}{(k+\alpha+\beta+1)_k} P_k^{\alpha,\beta)}(2t^2-1), \\ W_{2k+1}^{(\alpha,\beta)}(t) &= \frac{k!}{(k+\alpha+\beta+2)_k} x P_k^{\alpha,\beta+1)}(2t^2-1). \end{aligned}$$

Notice that $W_{2k+1}^{(\alpha,\beta)}(t) = tW_{2k}^{(\alpha,\beta+1)}(t)$. Their three-term recurrence relation is

$$\begin{aligned} &W_{k+1}^{(\alpha,\beta)}(t) = tW_k^{(\alpha,\beta)}(t) - \beta_k W_{k-1}^{(\alpha,\beta)}(t), \quad k = 0,1,\ldots, \\ &W_{-1}^{(\alpha,\beta)}(t) = 0, \quad W_0^{(\alpha,\beta)}(t) = 1, \end{aligned}$$

where

$$\beta_{2k} = \frac{k(k+\alpha)}{(2k+\alpha+\beta)(2k+\alpha+\beta+1)}, \quad \beta_{2k-1} = \frac{(k+\beta)(k+\alpha+\beta)}{(2k+\alpha+\beta-1)(2k+\alpha+\beta)},$$

for $k = 1,2,\ldots$, except when $\alpha+\beta = -1$; then $\beta_1 = (\beta+1)/(\alpha+\beta+2)$. Some applications of these polynomials in numerical quadratures and least square approximation with constraint were given in [70] and [98], respectively.

2° *The hyperbolic weight* $w(t) = 1/\cosh t$ on $(-\infty,+\infty)$. The coefficients in three-term recurrence relation are given by

$$\alpha_k = 0, \qquad \beta_0 = \pi, \quad \beta_k = \frac{\pi^2 k^2}{4} \quad (k \ge 1).$$

For details and generalizations see Chihara [20, pp. 191–193].

3° *The logistic weight* $w(t) = e^{-t}/(1+e^{-t})^2$ on $(-\infty,+\infty)$. Here we have

$$\alpha_k = 0, \qquad \beta_0 = 1, \quad \beta_k = \frac{\pi^2 k^4}{4k^2-1} \quad (k \ge 1).$$

A system of orthogonal polynomials for which the recursion coefficients are not known explicitly will be said to be *strong non-classical* orthogonal polynomials. In such cases there are a few known approaches to compute the first n coefficients α_k, β_k, $k = 0,1,\ldots,n-1$. These then allow us to compute all orthogonal polynomials of degree $\le n$ by a straightforward application of the three-term recurrence relation (2.2.11).

One of approaches for numerical construction of the monic orthogonal polynomials $\{\hat{\pi}_k\}$ is the *method of moments*, or precisely, *Chebyshev* or *modified Chebyshev algorithm*.

The second method makes use of explicit representations

$$\alpha_k = \frac{(t\hat{\pi}_k, \hat{\pi}_k)}{(\hat{\pi}_k, \hat{\pi}_k)} \quad (k \ge 0), \quad \beta_0 = (\hat{\pi}_0, \hat{\pi}_0), \quad \beta_k = \frac{(\hat{\pi}_k, \hat{\pi}_k)}{(\hat{\pi}_{k-1}, \hat{\pi}_{k-1})} \quad (k \ge 1),$$

in terms of the inner product $(\,.\,,\,.\,)$. The method is known as the *Stieltjes procedure*. Using a discretization of the inner product by some appropriate quadrature

$$(f,g) \approx (f,g)_N = \sum_{k=1}^{N} w_k f(x_k) g(x_k), \quad w_k > 0,$$

the corresponding method is called the *discretized Stieltjes procedure*.

For details in numerical construction of orthogonal polynomials see papers of Gautschi [32], [39], [41], [44]. In Section 3 we will mention a few non-classical weights for which the recursion coefficients were constructed numerically as well as the corresponding Gaussian formulas.

2.4. On s-orthogonal polynomials. In this subsection we give a short account of so-called *s-orthogonal polynomials* which are connected with Gauss-Turán quadrature formulas (see Turán [126]).

2.4.1. Turán quadratures and s-orthogonal polynomials. In 1950 P. Turán investigated numerical quadratures with multiple nodes,

$$\int_{-1}^{1} f(t)\,dt = \sum_{\nu=1}^{n} \sum_{i=0}^{k-1} A_{i,\nu} f^{(i)}(\tau_\nu) + R_{n,k}(f), \tag{2.4.1}$$

where

$$A_{i,\nu} = \int_{-1}^{1} \ell_{\nu,i}(t)\,dt \qquad (\nu = 1, \dots, n;\ i = 0, 1, \dots, k-1)$$

and $\ell_{\nu,i}(t)$ are the fundamental functions of Hermite interpolation. The $A_{i,\nu}$ are Cotes number of higher order. This formula is exact if f is a polynomial at most $kn-1$ and the points $-1 \le \tau_1 < \tau_2 < \cdots < \tau_n \le 1$ are arbitrary.

For $k = 1$ this formula can be exact for all polynomials of degree $\le 2n-1$ if the nodes τ_ν are the zeros of the Legendre polynomial P_n. That is the well-known Gauss-Legendre quadrature.

Because of the theorem of Gauss it is natural to ask whether knots τ_ν can be chosen so that the quadrature formula (2.4.1) will be exact for polynomials of degree not exceeding $(k+1)n-1$. Turán [126] showed that the answer is negative for $k = 2$, and for $k = 3$ it is positive. He proved that the knots τ_ν should be chosen as the zeros of the monic polynomial $\pi^*(t) = t^n + \cdots$, which minimizes the following integral

$$\int_{-1}^{1} \pi_n(t)^4 \, dt,$$

where $\pi_n(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0$.

More generally, the answer is negative for even, and positive for odd k, and then τ_ν are the zeros of the polynomial minimizing

$$\int_{-1}^{1} \pi_n(t)^{k+1} \, dt. \tag{2.4.2}$$

For $k = 1$, π_n^* is the monic Legendre polynomial $\hat{P}_n$.

Because of the above, we put $k = 2s + 1$. Instead of (2.4.1) it is interesting to investigate the analogous formula with some positive measure $d\lambda(t)$ on the real line $\mathbb{R}$,

$$\int_{\mathbb{R}} f(t) \, d\lambda(t) = \sum_{i=0}^{2s} \sum_{\nu=1}^{n} A_{i,\nu} f^{(i)}(\tau_\nu) + R(f), \tag{2.4.3}$$

which must be exact for all polynomials of degree at most $2(s+1)n - 1$. The role of the integral (2.4.2) is taken over

$$F_s = \int_{\mathbb{R}} \pi_n(t)^{2s+2} \, d\lambda(t),$$

where $F_s \equiv F_s(a_0, \ldots, a_{n-1})$, $\pi_n(t) = \sum_{k=0}^{n} a_k t^k$, $a_n = 1$. In order to minimize F_s we must have

$$\int_{\mathbb{R}} \pi_n(t)^{2s+1} t^k \, d\lambda(t) = 0, \qquad k = 0, 1, \ldots, n-1. \tag{2.4.4}$$

Usually, instead of $\pi_n(t)$ we write $P_{n,s}(t)$.

The case $d\lambda(t) = w(t)\, dt$ on $[a, b]$ has been considered by the Italian mathematicians Ossicini [109], Ghizzetti and Ossicini [55], Guerra [60–61]. Is known that there exists a unique polynomial $P_{n,s} \in \boldsymbol{\mathcal{P}}_n$ satisfying the conditions (2.4.4), and whose zeros are real, distinct and located in the support interval of the measure $d\lambda(t)$. These polynomials are known as *s-orthogonal* (or *s-self associated*) *polynomials* with respect to the measure $d\lambda(t)$ on $\mathbb{R}$.

For $s = 0$ we have the standard case of orthogonal polynomials on the real line and the generalized Gauss-Turán formula (2.4.3) reduces to the Gauss-Christoffel quadrature (cf. Section 3).

In the case of the Chebyshev measure $d\lambda(t) = (1-t^2)^{-1/2}\,dt$ on $(-1,1)$, in 1930 Bernstein [14] showed that the monic Chebyshev polynomial $\hat{T}_n(t) = 2^{1-n}T_n(t)$ minimizes the integral F_s for any $s \ge 0$. Therefore, in this case, the nodes in (2.4.3) are the Chebyshev points $\tau_\nu = \cos[(2\nu-1)\pi/2n]$, $\nu = 1,\ldots,n$, for any $s \ge 0$. For other measures the case when $s > 0$ is very difficult.

2.4.2. Construction of s-orthogonal polynomials. G. Vincenti [128] has considered an iterative process to compute the coefficients of s-orthogonal polynomials in a special case when $d\lambda(t) = w(t)\,dt$ with an even weight function on a symmetric interval, say $[-b,b]$. Then $P_{n,s}(-t) = (-1)^n P_{n,s}(t)$. Vincenti applied his process to the Legendre case. When n and s increase, the process becomes ill-conditioned. A stable procedure for such a purpose was given by Milovanović [91]. The main idea was an interpretation of the "orthogonality conditions" (2.4.4) as orthogonality with respect to the positive measure $d\mu(t) = d\mu_{n,s}(t) = (\pi_n^{n,s}(t))^{2s}\,d\lambda(t)$. Thus, these conditions can be interpreted as

$$\int_{\mathbb{R}} \pi_\nu^{n,s}(t)t^i\,d\mu(t) = 0, \quad i = 0,1,\ldots,\nu-1,$$

where $\{\pi_\nu^{n,s}\}$ is a sequence of monic orthogonal polynomials with respect to the new measure $d\mu(t)$. Of course, $P_{n,s}(\,\cdot\,) = \pi_n^{n,s}(\,\cdot\,)$. As we can see, the polynomials $\pi_\nu^{n,s}$, $\nu = 0,1,\ldots$, are implicitly defined, because the measure $d\mu(t)$ depends of $\pi_n^{n,s}(t)$ $\big(= \pi_n(t)\big)$. The general class of such polynomials was introduced by H. Engels (see [23, pp. 214–226]).

We will write only π_ν instead of $\pi_\nu^{n,s}(\cdot)$. These polynomials satisfy a three-term recurrence relation

$$\begin{aligned}
&\pi_{\nu+1}(t) = (t-\alpha_\nu)\pi_\nu(t) - \beta_\nu\pi_{\nu-1}(t), \quad \nu = 0,1,\ldots,\\
&\pi_{-1}(t) = 0, \quad \pi_0(t) = 1,
\end{aligned}$$

where, because of orthogonality,

$$\begin{aligned}
\alpha_\nu &= \alpha_\nu(n,s) = \frac{(t\pi_\nu,\pi_\nu)}{(\pi_\nu,\pi_\nu)} = \frac{\int_{\mathbb{R}} t\pi_\nu^2(t)\,d\mu(t)}{\int_{\mathbb{R}} \pi_\nu^2(t)\,d\mu(t)},\\
\beta_\nu &= \beta_\nu(n,s) = \frac{(\pi_\nu,\pi_\nu)}{(\pi_{\nu-1},\pi_{\nu-1})} = \frac{\int_{\mathbb{R}} t\pi_\nu^2(t)\,d\mu(t)}{\int_{\mathbb{R}} \pi_{\nu-1}^2(t)\,d\mu(t)},
\end{aligned}$$

and, for example, $\beta_0 = \int_{\mathbb{R}} d\mu(t)$.

Finding the coefficients α_ν, β_ν ($\nu = 0,1,\ldots,n-1$) gives us access to the first $n+1$ orthogonal polynomials π_0, π_1, $\ldots$, π_n. Of course, we are interested only in the last of them, i.e., $\pi_n \equiv \pi_n^{n,s}$. Thus, for $n = 0,1,\ldots$, the diagonal (boxed) elements in Table 2.4.1 are our s-orthogonal polynomials $\pi_n^{n,s}$.

Table 2.4.1

n	$d\mu^{n,s}(t)$	Orthogonal Polynomials			
0	$(\pi_0^{0,s}(t))^{2s}\,d\lambda(t)$	$\boxed{\pi_0^{0,s}}$			
1	$(\pi_1^{1,s}(t))^{2s}\,d\lambda(t)$	$\pi_0^{1,s}$	$\boxed{\pi_1^{1,s}}$		
2	$(\pi_2^{2,s}(t))^{2s}\,d\lambda(t)$	$\pi_0^{2,s}$	$\pi_1^{2,s}$	$\boxed{\pi_2^{2,s}}$	
3	$(\pi_3^{3,s}(t))^{2s}\,d\lambda(t)$	$\pi_0^{3,s}$	$\pi_1^{3,s}$	$\pi_2^{3,s}$	$\boxed{\pi_3^{3,s}}$
$\vdots$					

A stable scheme, based on Stieltjes' procedure, that computes $\pi_n^{n,s}$ in an iterative fashion, was developed by Milovanović [91].

2.5. Sobolev type orthogonal polynomials. Sobolev type orthogonal polynomials appeared in works of Lewis [77], Althammer [7], Brenner [16], and others. In general, the inner product can be introduced by

$$\langle f, g\rangle = \sum_{k=0}^{m} \int_{\mathbb{R}} f^{(k)}(t) g^{(k)}(t)\, d\lambda_k(t), \tag{2.5.1}$$

where $d\lambda_k(t)$, $k = 0, 1, \ldots, m$ are given positive measures on $\mathbb{R}$.

Let $\{s_n\}$ denote a set of polynomials orthogonal with respect to the inner product (2.5.1). In the case $m = 0$, i.e., when $\langle f, g\rangle$ reduces to the inner product $(f, g) = \int_{\mathbb{R}} f(t)g(t)\, d\lambda_0(t)$, let $\{\pi_k\}$ be the corresponding set of orthogonal polynomials. It is clear that the all zeros of $\pi_n(t)$ lie in the interior of the interval $\Delta(d\lambda_0)$ (see Theorem 2.1.2).

Althammer [7] pointed out that the position of the zeros of $s_n(t)$ can be different from those of $\pi_n(t)$. For example for $m = 1$, if $d\lambda_0(t)$ is the Legendre measure, then the measure $d\lambda_1(t)$ on $(-1, 1)$ can be chosen in such a way that $s_n(t)$ has a zero outside $(-1, 1)$. Brenner [16] also obtained a similar result for $m = 1$ and the Laguerre measure $d\lambda_0(t) = \exp(-t)\, dt$ on $(0, +\infty)$. However, in both cases the measure $d\lambda_1(t)$ can be chosen that $s_n(t)$ has all simple zeros inside of interval of orthogonality. Zero distribution and behaviour of orthogonal polynomials in such cases were considered by Cohen [21].

Recently several authors ([4], [67–68], [81–82], [84], [86–88]) studied polynomials orthogonal with respect to the inner product (2.5.1), where the measures $d\lambda_k(t)$, $k \geq 1$, are concentrated in one point $t = c$ (discrete Sobolev inner product),

$$\langle f, g\rangle = \int_{\mathbb{R}} f(t)g(t)\, d\lambda(t) + \sum_{k=0}^{m} M_k f^{(k)}(c) g^{(k)}(c), \tag{2.5.2}$$

where $M_k \geq 0$, $k = 0, 1, \ldots, m$. Taking $d\lambda(t) = \big(t^\alpha \exp(-t)/\Gamma(\alpha+1)\big)\, dt$, $\alpha > -1$, on $(0, +\infty)$ R. Koekoek [67] proved that the coefficients A_k, $k = 0, 1, \ldots, m+1$, can be chosen in such a way that the polynomials $\big\{L_n^{\alpha, M_0, M_1, \ldots, M_m}(t)\big\}_{n=0}^{+\infty}$, defined by

$$L_n^{\alpha, M_0, M_1, \ldots, M_m}(t) = \sum_{k=0}^{m+1} A_k\, D^k L_n^{(\alpha)}(t),$$

are orthogonal with respect to the inner product (2.5.2). These polynomials satisfy a $(2m+3)$-terms recurrence relation of the form

$$t^{m+1} L_n^{\alpha, M_0, M_1, \ldots, M_m}(t) = \sum_{k=n-m-1}^{n+m+1} E_k^{(n)}\, L_k^{\alpha, M_0, M_1, \ldots, M_m}(t).$$

Also some cases of two symmetric points $t = \pm c$ were investigated by Bavinck and Meijer [12–13].

For more details see papers [4], [65], [81–82], [110], as well as a survey of the results and a complete list of references [80].

3. Some applications of orthogonal polynomials on the real line

This section is devoted to some important applications of orthogonal polynomials on the real line as Gauss-Christoffel quadrature formulas, moment-preserving spline approximation and summation of slowly convergent series.

3.1. Gaussian type of quadratures. One of the important uses of orthogonal polynomials is in the construction of quadrature formulas of maximum, or nearly maximum, algebraic degree of exactness for integrals involving a positive measure $d\lambda(t)$.

The n-point *Gaussian quadrature formula*

$$\int_{\mathbb{R}} f(t) d\lambda(t) = \sum_{\nu=1}^{n} \lambda_\nu^{(n)} f(\tau_\nu^{(n)}) + R_n(f) \tag{3.1.1}$$

has maximum algebraic degree of exactness $2n-1$, in the sense that $R_n(f) = 0$ for all $f \in \mathcal{P}_{2n-1}$. In formula (3.1.1), $\tau_\nu = \tau_\nu^{(n)}$ are the *Gauss nodes*, and $\lambda_\nu = \lambda_\nu^{(n)}$ the *Gauss weights* or *Christoffel numbers*. This formula is also known as *Gauss-Christoffel quadrature formula*. A nice survey on that was given by Gautschi [31].

The nodes τ_ν are the zeros of the n-th orthogonal polynomial $\pi_n(\cdot, d\lambda)$, and the weights λ_ν, which are all positive, can be also expressed in terms of the same orthogonal polynomials. As we have seen in 2.1, the nodes τ_ν are the eigenvalues of the n-th order Jacobi matrix

$$J_n(d\lambda) = \begin{bmatrix} \alpha_0 & \sqrt{\beta_1} & & & \mathrm{O} \\ \sqrt{\beta_1} & \alpha_1 & \sqrt{\beta_2} & & \\ & \sqrt{\beta_2} & \alpha_2 & \ddots & \\ & & \ddots & \ddots & \sqrt{\beta_{n-1}} \\ \mathrm{O} & & & \sqrt{\beta_{n-1}} & \alpha_{n-1} \end{bmatrix},$$

where α_ν and β_ν are the coefficients in three-term recurrence relation for the monic orthogonal polynomials $\pi_n(\cdot, d\lambda)$. The weights λ_ν are given by

$$\lambda_\nu = \beta_0 v_{\nu,1}^2, \qquad \nu = 1, \dots, n,$$

where $\beta_0 = \int_{\mathbb{R}} d\lambda(t)$ and $v_{\nu,1}$ is the first component of the normalized eigenvector $\boldsymbol{v}_\nu$ corresponding to the eigenvalue τ_ν (cf. Golub and Welsch [56], and Gautschi [29]),

$$J_n(d\lambda)\boldsymbol{v}_\nu = \tau_\nu \boldsymbol{v}_\nu, \qquad \boldsymbol{v}_\nu^T \boldsymbol{v}_\nu = 1, \qquad \nu = 1, \dots, n.$$

There are well-known and efficient algorithms, such as the QR algorithm with shifts, to compute eigenvalues and eigenvectors of symmetric tridiagonal matrices (cf. the routine GAUSS in the package ORTHPOL given by Gautschi [44]). There are many methods for estimating the remainder term $R_n(f)$ in (3.1.1). Error bounds in the class of analytic functions were investigated by Gautschi and Varga [50].

A simple modification of the previous method can be applied to the construction of Gauss-Radau and Gauss-Lobatto quadrature formulas.

In sequel we mention a few non-classical measures $d\lambda(t) = w(t)\,dt$ for which recursion coefficients $\alpha_k(d\lambda)$, $\beta_k(d\lambda)$, $k = 0, 1, \dots, n-1$, have been tabulated in the literature and used in the construction of Gaussian quadratures.

1° *One-side Hermite weight* $w(t) = \exp(-t^2)$ on $[0, c]$, $0 < c \le +\infty$. This distribution $w(t)\,d\lambda(t)$ is known as the *Maxwell* (*velocity*) *distribution*. The cases $c = 1$, $n = 10$ and $c = +\infty$, $n = 15$ were considered by Steen, Byrne and Gelbard [120] (see also Gautschi [41]).

2° *Logarithmic weight* $w(t) = t^\alpha \log(1/t)$, $\lambda > -1$ on $(0, 1)$. Piessens and Branders [111] considered cases when $\alpha = 0, \pm 1/2, \pm 1/3, -1/4, -1/5$ (see also Gautschi [39]).

3° *Airy weight* $w(t) = \exp(-t^3/3)$ on $(0, +\infty)$. The inhomogeneous Airy functions $\mathrm{Hi}(x)$ and $\mathrm{Gi}(x)$, arise in theoretical chemistry (e.g. in harmonic oscillator models for large quantum numbers) and their integral representations (see Lee [75]) are given by

$$\begin{aligned} \mathrm{Hi}(x) &= \frac{1}{\pi} \int_0^{+\infty} w(t) e^{tx}\, dt, \\ \mathrm{Gi}(x) &= -\frac{1}{\pi} \int_0^{+\infty} w(t) e^{-tx/2} \cos\Big(\frac{\sqrt{3}}{2}\, tx + \frac{2\pi}{3}\Big)\, dt. \end{aligned}$$

These functions can be effectively evaluated by Gaussian quadrature relative to the Airy weight $w(t)$. It needs orthogonal polynomials with respect to this weight. Gautschi [35] computed the recursion coefficients for $n = 15$ with 16 decimal digits after the decimal point (D).

3° *Reciprocal gamma function* $w(t) = 1/\Gamma(t)$ on $(0, +\infty)$. Gautschi [34] determined the recursion coefficients for $n = 40$ with 20 significant decimal digits (S). This function could be useful as a probability density function in reliability theory (see Fransén [25]).

4° *Einstein's* and *Fermi's weight functions* on $(0, +\infty)$,

$$w_1(t) = \varepsilon(t) = \frac{t}{e^t - 1} \qquad \text{and} \qquad w_2(t) = \varphi(t) = \frac{1}{e^t + 1}\,. \tag{3.1.2}$$

These functions arise in solid state physics. Integrals with respect to the measure $d\lambda(t) = \varepsilon(t)^r\, dt$, $r = 1$ and $r = 2$, are widely used in phonon statistics and lattice specific heats and occur also in the study of radiative recombination processes. Similarly, integrals with $\varphi(t)$ are encountered in the dynamics of electrons in metals. For $w_1(t)$, $w_2(t)$, $w_3(t) = \varepsilon(t)^2$ and $w_4(t) = \varphi(t)^2$, Gautschi and Milovanović [46] determined the recursion coefficients α_k and β_k, for $n = 40$ with 25 S, and gave an application of the corresponding Gauss-Christoffel quadratures to summation of slowly convergent series.

5° *The hyperbolic weights* on $(0, +\infty)$,

$$w_1(t) = \frac{1}{\cosh^2 t} \qquad \text{and} \qquad w_2(t) = \frac{\sinh t}{\cosh^2 t}\,. \tag{3.1.3}$$

The recursion coefficients α_k, β_k, for $n = 40$ with 30 S, were obtained by Milovanović [95]. The discretization was based on the Gauss-Laguerre quadrature rule.

3.2. Moment-preserving spline approximation. In this subsection we give some applications of orthogonal polynomials and Gauss-Christoffel and generalized Gauss-Turán quadratures to the moment-preserving spline approximation of functions. Such a problem of approximation was appeared in physics (see Laframboise and Stauffer [71] and Calder and Laframboise [18]). Namely, it was the problem of finding splines that reproduce as many as possible of the initial moments of a given spherically symmetric function $t \mapsto f(t)$, $t = \|\boldsymbol{x}\|$, $0 \le t < \infty$, in $\mathbb{R}^d$, $d \ge 1$. Gautschi [36] solved this problem of approximating by a piecewise constant function

$$t \mapsto s_n(t) = \sum_{\nu=1}^{n} a_\nu H(\tau_\nu - t) \qquad (a_\nu \in \mathbb{R},\ 0 < \tau_1 < \cdots < \tau_n < +\infty),$$

where H is the Heaviside step function. Also, he considered an approximation by a linear combination of Dirac delta functions. The approximation was to preserve as many moments of f as possible. This work was extended to spline approximation of arbitrary degree by Gautschi and Milovanović [49]. Namely, they considered a spline function of degree $m \geq 0$ on $[0, +\infty)$, vanishing at $t = +\infty$, with $n \geq 1$ positive knots τ_ν $(\nu = 1, \ldots, n)$, which can be written in the form

$$s_{n,m}(t) = \sum_{\nu=1}^{n} a_\nu (\tau_\nu - t)_+^m \qquad (a_\nu \in \mathbb{R},\ 0 \leq t < +\infty), \tag{3.2.1}$$

where the plus sign on the right is the cutoff symbol, $u_+ = u$ if $u > 0$ and $u_+ = 0$ if $u \leq 0$. Given a function $t \mapsto f(t)$ on $[0, +\infty)$, they determined $s_{n,m}$ such that

$$\int_0^{+\infty} s_{n,m}(t) t^j \, dV = \int_0^{+\infty} f(t) t^j \, dV \qquad (j = 0, 1, \ldots, 2n-1), \tag{3.2.2}$$

where dV is the volume element depending on the geometry of the problem. (For example, $dV = Ct^{d-1}\,dt$ if $d > 1$, where C is some constant, and $dV = dt$ if $d = 1$ were used in [49]. For some details see Gautschi [40].) In any case, the spline $s_{n,m}$ is such to faithfully reproduce the first $2n$ moments of f. Under suitable assumptions on f, it was shown that the problem has a unique solution if and only if certain Gauss-Christoffel quadratures exist corresponding to a moment functional or weight distribution depending on f. Existence, uniqueness and pointwise convergence of such approximation were analyzed. We mention two main results (Gautschi and Milovanović [49]) in the case when $dV = dt$.

Theorem 3.2.1. *Let $f \in C^{m+1}[0, +\infty]$ and*

$$\int_0^{+\infty} t^{2n+m+1} |f^{(m+1)}(t)| \, dt < +\infty \, .$$

Then a spline function $s_{n,m}$ of the form (3.2.1) *with positive knots τ_ν, that satisfies* (3.2.2)*, exists and is unique if and only if the measure*

$$d\lambda(t) = \frac{(-1)^{m+1}}{m!} t^{m+1} f^{(m+1)}(t) \, dt \quad \text{on} \quad [0, +\infty) \tag{3.2.3}$$

admits an n-point Gauss-Christoffel quadrature formula

$$\int_0^{+\infty} g(t) \, d\lambda(t) = \sum_{\nu=1}^{n} \lambda_\nu^{(n)} g(\tau_\nu^{(n)}) + R_n(g; d\lambda), \tag{3.2.4}$$

with distinct positive nodes $\tau_\nu^{(n)}$, where $R_n(g; d\lambda) = 0$ for all $g \in \boldsymbol{\mathcal{P}}_{2n-1}$. In that event, the knots τ_ν and weights a_ν in (3.2.1) *are given by*

$$\tau_\nu = \tau_\nu^{(n)}, \quad a_\nu = \tau_\nu^{-(m+1)} \lambda_\nu^{(n)} \qquad (\nu = 1, \ldots, n).$$

Theorem 3.2.2. *Given f as in Theorem 3.2.2, assume that the measure $d\lambda$ in (3.2.3) admits the n-point Gauss-Christoffel quadrature formula (3.2.4) with distinct positive nodes $\tau_\nu = \tau_\nu^{(n)}$ and the remainder term $R_n(g; d\lambda)$. Define*

$$\sigma_t(x) = x^{-(m+1)}(x-t)_+^m\,.$$

Then, for any $t > 0$, we have for the error of the spline approximation (3.2.1), (3.2.2),

$$f(t) - s_{n,m}(t) = R_n(\sigma_t; d\lambda).$$

For example, if f is completely monotonic on $[0, +\infty)$ then $d\lambda(t)$ in (3.2.3) is a positive measure for every m and the Gauss-Christoffel quadrature formula exists uniquely, with n distinct and positive nodes $\tau_\nu^{(n)}$. Theorem 3.2.2 shows that $s_{n,m}$ converges pointwise to f as $n \to +\infty$ if the Gauss-Christoffel quadrature formula (3.2.4) converges for the particular function $x \mapsto g(x) = \sigma_t(x)$ $(x > 0)$.

Similarly, we can consider an approximation of a given function $t \mapsto f(t)$ on $[0, +\infty)$ by defective splines. A spline function of degree $m \geq 2$ and defect k on the interval $0 \leq t < +\infty$, vanishing at $t = +\infty$, with $n \geq 1$ positive knots τ_ν $(\nu = 1, \ldots, n)$, can be written in the form

$$s_{n,m}(t) = \sum_{\nu=1}^{n} \sum_{i=m-k+1}^{m} a_{i,\nu}(\tau_\nu - t)_+^i \tag{3.2.5}$$

where $a_{i,\nu}$ are real numbers.

Under suitable assumptions on f and $k = 2s+1$, Milovanović and Kovačević [99–100] showed that the approximation problem has a unique solution if and only if certain generalized Turán quadratures exist corresponding to a measure depending on f (cf. Subsection 2.4).

Theorem 3.2.3. *Let $f \in C^{m+1}[0, +\infty]$ and*

$$\int_0^{+\infty} t^{2(s+1)n+m+1} |f^{(m+1)}(t)|\, dt < +\infty\,.$$

Then a spline function $s_{n,m}$ of the form (3.2.5) with $k = 2s+1$ and positive knots τ_ν, that satisfies (3.2.2), with $j = 0, 1, \ldots, 2(s+1)n-1$, exists and is unique if and only if the measure

$$d\lambda(t) = \frac{(-1)^{m+1}}{m!} t^{m+1} f^{(m+1)}(t)\, dt \quad \text{on} \quad [0, +\infty)$$

admits a generalized Gauss-Turán quadrature formula

$$\int_0^{+\infty} g(t)\, d\lambda(t) = \sum_{\nu=1}^{n} \sum_{i=0}^{2s} A_{i,\nu}^{(n)} g^{(i)}(\tau_\nu^{(n)}) + R_n^G(g; d\lambda), \tag{3.2.6}$$

with distinct positive nodes $\tau_\nu^{(n)}$*, where* $R_n^G(g;d\lambda)=0$ *for all* $g\in\boldsymbol{\mathcal{P}}_{2(s+1)n-1}$*. The knots* τ_ν *in* (3.2.5) *are given by* $\tau_\nu=\tau_\nu^{(n)}$*, and coefficients* $a_{i,\nu}$ *by the following triangular system*

$$A_{i,\nu}^{(n)}=\sum_{j=i}^{2s}\frac{(m-j)!}{m!}\binom{j}{i}\left[D^{j-i}t^{m+1}\right]_{t=\tau_\nu}a_{m-j,\nu}\quad(i=0,1,\dots,2s),$$

where D *is the standard differentiation operator.*

Theorem 3.2.4. *Given* f *as in Theorem* 3.2.3*, assume that the measure* $d\lambda(t)$ *admits the* n*-point generalized Gauss-Turán quadrature formula* (3.2.6) *with distinct positive nodes* $\tau_\nu=\tau_\nu^{(n)}$ *and the remainder term* $R_n^G(g;d\lambda)$*. Then the error of the spline approximation is given by*

$$f(t)-s_{n,m}(t)=R_n^G(\sigma_t;d\lambda)\qquad(t>0),$$

where $x\mapsto\sigma_t(x)=x^{-(m+1)}(x-t)_+^m$*.*

Again, if f is completely monotonic on $[0,+\infty)$ then $d\lambda(t)$ is a positive measure for every m and the generalized Gauss-Turán quadrature formula exists uniquely, with n distinct and positive nodes $\tau_\nu^{(n)}$.

Frontini, Gautschi and Milovanović [27] and Frontini and Milovanović [28] considered analogous problems on an arbitrary finite interval, which can be standardized to $[a,b]=[0,1]$. If the approximations exist, they can be represented in terms of generalized Gauss-Lobatto and Gauss-Radau quadrature formulas relative to appropriate measures depending on f, when the defect $k=1$. Using defective splines with odd defect $k=2s+1$, approximation problems reduce to certain generalized Gauss-Turán-Lobatto and Gauss-Turán-Radau quadrature formulas. A more general case with variable defects was considered by Gori and Santi [57]. In that case, approximation problems reduce to Gauss-Turán-Stancu type of quadratures and σ-orthogonal polynomials (cf. Gautschi [31], Gori, Lo Cascio and Milovanović [58]).

Further extensions of the moment-preserving spline approximation on $[0,1]$ are given by Micchelli [89]. He relates this approximation to the theory of the monosplines.

3.3. Summation of slowly convergent series. We consider convergent series of the type

$$T=\sum_{k=1}^{+\infty}a_k\qquad\text{and}\qquad S=\sum_{k=1}^{+\infty}(-1)^k a_k\tag{3.3.1}$$

and introduce the notation: $T=T^{(m-1)}+T_m^{(\infty)}$, $S=S^{(m-1)}+S_m^{(\infty)}$,

$$T_m^{(n)}=\sum_{k=m}^{n}a_k,\qquad S_m^{(n)}=\sum_{k=m}^{n}(-1)^k a_k,$$

where $T^{(m-1)}$ and $S^{(m-1)}$ are the corresponding partial sums of (3.3.1).

Some methods of summation these series can be found, for example, in the books of Henrici [63], Lindelöf [78], and Mitrinović and Kečkić [104].

Recently, a few new summation/integration procedures for slowly convergent series are developed (see [46], [42–43], [95–97]). Here we give a short account of these methods.

3.3.1. Laplace transform method. Suppose that the general term of T (and S) is expressible in terms of the derivative of a Laplace transform, or in terms of the Laplace transform itself. Namely, let $a_k = F'(k)$, where

$$F(p) = \int_0^{+\infty} e^{-pt} f(t)\,dt, \qquad \operatorname{Re} p \geq 1.$$

Then

$$\sum_{k=1}^{+\infty} F'(k) = -\sum_{k=1}^{+\infty} \int_0^{+\infty} te^{-kt} f(t)\,dt = -\int_0^{+\infty} \frac{t}{e^t - 1}\, f(t)\,dt.$$

Similarly, for "alternating" series, one obtains

$$\sum_{k=1}^{+\infty} (-1)^k F'(k) = \int_0^{+\infty} \frac{t}{e^t + 1}\, f(t)\,dt$$

and

$$\sum_{k=1}^{+\infty} (-1)^k F(k) = -\int_0^{+\infty} \frac{1}{e^t + 1}\, f(t)\,dt.$$

In a joint paper with Gautschi [46] we considered the construction of Gaussian quadrature formulas on $(0, +\infty)$,

$$\int_0^{+\infty} g(t)w(t)\,dt = \sum_{\nu=1}^{n} \lambda_\nu g(\tau_\nu) + R_n(g), \tag{3.3.2}$$

with respect to the weight functions given by (3.1.2). If the series T and S are slowly convergent and the respective function f on the right of the equalities above is smooth, then low-order Gaussian quadrature (3.3.2) applied to the integrals on the right, provides a possible summation procedure. Numerical examples show fast convergence of this procedure (see [46, §4]). A problem which arises with this procedure (*Laplace transform method*) is the determination of the original function f for a given series. For some other applications see [42–43].

3.3.2. Contour Integration Over a Rectangle. Suppose now that $a_k = f(k)$, where $z \mapsto f(z)$ is a holomorphic function in the region

$$G_m = \bigl\{z \in \mathbb{C} \;\big|\; \operatorname{Re} z \geq \alpha,\ m-1 < \alpha < m\bigr\}, \quad m \in \mathbb{N}.$$

In [95] we derived an alternative summation/integration method for the series (3.3.1) which requires the indefinite integral F of f chosen so as to satisfy the following decay conditions:

(C1) *F is a holomorphic function in the region G_m;*

(C2) $\displaystyle\lim_{|t|\to+\infty} e^{-c|t|} F\left(x + it/\pi\right) = 0$*, uniformly for $x \geq \alpha$;*

(C3) $\displaystyle\lim_{x\to+\infty} \int_{-\infty}^{+\infty} e^{-c|t|} \,\bigl|F\left(x+it/\pi\right)\bigr|\, dt = 0,$

where $c = 2$ or $c = 1$, when we consider $T_m^{(n)}$ or $S_m^{(n)}$, respectively.

Namely, taking $\Gamma = \partial G$ and

$$G = \Bigl\{z \in \mathbb{C} \mid \alpha \leq \operatorname{Re} z \leq \beta,\, |\operatorname{Im} z| \leq \frac{\delta}{\pi}\Bigr\},$$

where $m-1 < \alpha < m$, $n < \beta < n+1$ $(m, n \in \mathbb{Z}, m \leq n)$, we obtain that

$$T_m^{(n)} = \frac{1}{2\pi i} \oint_\Gamma f(z) \frac{\pi}{\tan \pi z}\, dz \quad \text{and} \quad S_m^{(n)} = \frac{1}{2\pi i} \oint_\Gamma f(z) \frac{\pi}{\sin \pi z}\, dz.$$

After integration by parts, these formulas reduce to

$$T_m^{(n)} = \frac{1}{2\pi i} \oint_\Gamma \Bigl(\frac{\pi}{\sin \pi z}\Bigr)^2 F(z)\, dz \tag{3.3.3}$$

and

$$S_m^{(n)} = \frac{1}{2\pi i} \oint_\Gamma \Bigl(\frac{\pi}{\sin \pi z}\Bigr)^2 \cos \pi z\, F(z)\, dz, \tag{3.3.4}$$

where $z \mapsto F(z)$ is an integral of $z \mapsto f(z)$.

Taking $\alpha = \alpha_m = m - 1/2$, $\beta = \beta_n = n + 1/2$, and letting $\delta \to +\infty$ and $n \to +\infty$, under conditions (C1) – (C3), the integrals in (3.3.3) and (3.3.4) over Γ reduce to integrals along the line $z = \alpha_m + iy$ $(-\infty < y < +\infty)$.

After some calculations, we reduce T and S to a problem of quadrature on $(0, +\infty)$ with respect to the hyperbolic weight functions given by (3.1.3). Thus,

$$T = T^{(m-1)} + \int_0^{+\infty} \Phi\left(\alpha_m, t/\pi\right) w_1(t)\, dt$$

and

$$S = S^{(m-1)} + \int_0^{+\infty} \Psi\left(\alpha_m, t/\pi\right) w_2(t)\, dt,$$

where w_1 and w_2 are defined in (3.1.3) and

$$\Phi(x, y) = -\frac{1}{2}\left[F(x+iy) + F(x-iy)\right],$$

$$\Psi(x, y) = \frac{(-1)^m}{2i}\left[F(x+iy) - F(x-iy)\right].$$

Numerical experiments shows that is enough to use only the quadrature with respect to the first weight $w_1(t) = 1/\cosh^2 t$. Namely, in the series S we can include the hyperbolic sine as a factor in the corresponding integrand so that

$$S = S^{(m-1)} + \int_0^{+\infty} \Psi\left(\alpha_m, t/\pi\right) \sinh(t) w_1(t)\, dt.$$

Using this approach we gave an appropriate method for calculating values of the Riemann zeta function $\zeta(z) = \sum\limits_{k=1}^{+\infty} k^{-z}$, which can be transformed to a weighted integral on $(0, +\infty)$ of the function $t \mapsto \exp\bigl(-(z/2)\log(1+\beta_m^2 t^2)\bigr)\cos(z \arctan(\beta_m t))$, $\beta_m = 2/((2m+1)\pi)$, $m \in \mathbb{N}_0$, involving the hyperbolic weight $w(t) = 1/\cosh^2 t$ (see [96]).

Also some other methods for series with irrational terms were given in [97].

4. Orthogonality on the unit circle

Another type of orthogonality is *orthogonality on the unit circle*. The polynomials orthogonal on the unit circle with respect to a given weight function have been introduced and studied by Szegő [122–123] and Smirnov [116–117]. A more general case was considered by Achieser and Kreĭn [1], Geronimus [51–52], P. Nevai [107–108], Alfaro and Marcellán [5], Marcellán and Sansigre [83], etc. These polynomials are linked with many questions in the theory of time series, digital filters, statistics, image processing, scattering theory, control theory and so on.

In the next three sections we give some basic definitions and properties of polynomials orthogonal on the unit circle.

4.1. Basic definitions and properties of orthogonal polynomials. Let an inner product defined by

$$(f, g) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\overline{g(e^{i\theta})}\, d\mu(\theta), \tag{4.1.1}$$

where $d\mu(\theta)$ is a finite positive measure on the interval $[0,2\pi]$ whose support is an infinite set. In that case there is a unique system of orthonormal polynomials $\{\varphi_k\}_{k\in\mathbb{N}_0}$ such that

$$(\varphi_k,\varphi_m)=\delta_{km},\quad \varphi_k(z)=b_kz^k+\cdots,\quad b_k>0\ \ (k=0,1,\dots). \tag{4.1.2}$$

If $\theta\mapsto\mu(\theta)$ is an absolutely continuous function on $[0,2\pi]$, then we say that $\mu'(\theta)=w(\theta)$ is a *weight function.*

The monic polynomials orthogonal on the unit circle with respect to the inner product (4.1.1) we will denote by $\phi_k(z)$. According to (4.1.2) we have that $\phi_k(z)=\varphi_k(z)/b_k$.

Theorem 4.1.1. *The monic orthogonal polynomials $\{\phi_k\}$ on the unit circle $|z|=1$ satisfy the recurrence relations*

$$\phi_{k+1}(z)=z\phi_k(z)+\phi_{k+1}(0)\phi_k^*(z),\quad \phi_{k+1}^*(z)=\phi_k^*(z)+\overline{\phi_{k+1}(0)}z\phi_k^*(z), \tag{4.1.3}$$

for $k=0,1,\dots,$ where $\phi_k^*(z)=z^k\bar{\phi}_k(1/z)$.

In order to prove the first recurrence relation in (4.1.3) we put

$$\frac{\phi_k(z)-z^k}{z^{k-1}}=\sum_{\nu=0}^{n-1}c_\nu^{(k)}\bar{\phi}_\nu(1/z). \tag{4.1.4}$$

Multiplying this equality by $\phi_m(z)$, where $0\le m\le k-1$, putting $z=e^{i\theta}$ and integrating, we obtain

$$\frac{1}{2\pi}\int_0^{2\pi}\phi_k(e^{i\theta})\frac{\phi_m(e^{i\theta})}{e^{i(k-1)\theta}}\,d\mu(\theta)-\frac{1}{2\pi}\int_0^{2\pi}e^{i\theta}\phi_m(e^{i\theta})\,d\mu(\theta)=c_m^{(k)}\|\phi_m\|^2,$$

where

$$\|\phi_m\|^2=\frac{1}{2\pi}\int_0^{2\pi}\phi_\nu(e^{i\theta})\overline{\phi_m(e^{i\theta})}\,d\mu(\theta).$$

Because of orthogonality, the first integral on the left hand side in the previous equality vanish, and we have

$$c_m^{(k)}=\frac{1}{2\pi\|\phi_m\|^2}\int_0^{2\pi}e^{i\theta}\phi_m(e^{i\theta})\,d\mu(\theta)\qquad(m=0,1,\dots,n-1).$$

We note that $c_m^{(k)}$ does not depend on k and therefore we write $c_m^{(k)}=c_m$. Using (4.1.4) we have

$$\phi_k(z)=z^k+z^{k-1}\sum_{\nu=0}^{n-1}c_\nu\bar{\phi}_\nu(1/z),$$

wherefrom we obtain

$$\phi_{k+1}(z) = z\phi_k(z) + z^k c_k \bar{\phi}_k(1/z) = z\phi_k(z) + c_k\phi_k^*(z). \tag{4.1.5}$$

Since $\phi_k^*(0) = 1$, setting $z = 0$ we find $c_k = \phi_{k+1}(0)$. Thus, (4.1.5) reduces to the first relation in (4.1.3).

As we can see recurrence relations (4.1.3) are not three-term relations like (2.3). The values $\phi_k(0)$ are called reflection parameters or Szegő parameters. Defining a sequence of parameters $\{a_k\}$ by $a_k = -\overline{\phi_{k+1}(0)}$, $k = 0, 1, \ldots$, and using (4.1.3), Geronimus [53, Chapter VIII] derived the following three-term recurrence relations:

$$\begin{aligned}
\bar{a}_{k-1}\phi_{k+1}(z) &= (\bar{a}_{k-1}z + \bar{a}_k)\phi_k(z) - \bar{a}_k z(1 - |a_{k-1}|^2)\phi_{k-1}(z),\\
a_{k-1}\phi_{k+1}^*(z) &= (a_{k-1}z + a_k)\phi_k^*(z) - a_k z(1 - |a_{k-1}|^2)\phi_{k-1}^*(z),
\end{aligned}$$

where $k = 1, 2, \ldots$, and $\phi_0(z) = 1$, $\phi_1(z) = z - \bar{a}_0$.

There is a relation between parameters $\{a_k\}$ and moments $\{c_k\}$, defined by

$$c_k = \frac{1}{2\pi}\int_0^{2\pi} e^{-ik\theta}\, d\mu(\theta), \qquad k = 0, 1, \ldots\ .$$

Introducing Toeplitz determinants $\Delta_k = |c_{i-j}|_0^k$, $c_{-k} = \bar{c}_k$, $k = 0, 1, \ldots$, then one can express $\phi_k(z)$ in the explicit form

$$\phi_k(z) = \frac{1}{\Delta_{k-1}}\begin{vmatrix}
c_0 & c_{-1} & c_{-2} & \cdots & c_{-k}\\
c_1 & c_0 & c_{-1} & & c_{-k+1}\\
\vdots & & & & \\
c_{k-1} & c_{k-2} & c_{k-3} & & c_{-1}\\
1 & z & z^2 & & z^k
\end{vmatrix} = z^k + \text{ lower degree terms},$$

where $k = 0, 1, \ldots$ and $\Delta_{-1} = 1$. According to the previous relations, one can get the following relation

$$\frac{\Delta_k}{\Delta_{k-1}} = c_0 \prod_{\nu=0}^{k-1}\left(1 - |a_\nu|^2\right) \qquad (n \in \mathbb{N}). \tag{4.1.6}$$

For a given sequence $\{a_k\}$ such that

$$|a_k| < 1 \qquad (k \in \mathbb{N}_0), \tag{4.1.7}$$

one can construct a sequence of (monic) polynomials $\{\phi_k(z)\}$ and determine a non-decreasing bounded function $\mu(\theta)$ with infinitely many points of increase (see Geronimus [53]). Namely, (4.1.7) is equivalent to the condition $\Delta_k > 0$ for every $k \in \mathbb{N}_0$, which is the necessary and sufficient condition for the existence of $\mu(\theta)$.

For orthonormal polynomials defined in (4.1.2) one can introduce the *kernel polynomial*

$$K_n(z,w) = \sum_{\nu=0}^{n} \overline{\varphi_\nu(z)}\,\varphi_\nu(w),$$

which can be expressed in the form

$$K_n(z,w) = \frac{\overline{\varphi_{n+1}^*(z)}\varphi_{n+1}^*(w) - \overline{\varphi_{n+1}(z)}\varphi_{n+1}(w)}{1-\bar{z}w}\,.$$

This analogue of the Christoffel-Darboux formula was proved by Szegő (cf. Freud [26, p. 196]). Some characteristic values of $K_n(z,w)$ are

$$K_n(0,z) = b_n\varphi_n^*(z), \qquad K_n(0,0) = b_n^2 = \Delta_{n-1}/\Delta_n,$$

where b_n is the coefficient of z^n in $\varphi(z)$.

4.2. Extremal properties of orthogonal polynomials. An analogue of Theorem 2.1.5 is the following result:

Theorem 4.2.1. *Let* $P(z) = \sum\limits_{\nu=0}^{n} a_\nu z^\nu$, *with* $a_n = 1$, *be an arbitrary monic polynomial of degree* n. *Then*

$$\frac{1}{2\pi}\int_0^{2\pi} |P(z)|^2\, d\mu(\theta) \geq \|\phi_n\|^2 = b_n^{-2} = \frac{\Delta_n}{\Delta_{n-1}}, \quad z = e^{i\theta},$$

with equality only if $P(z) = \phi_n(z) = \varphi_n(z)/b_n$. *The ratio* Δ_n/Δ_{n-1} *is given by* (4.1.6).

We also mention the following statement (cf. Szegő [124, p. 290]):

Theorem 4.2.2. *Let* $P \in \boldsymbol{\mathcal{P}}_n$ *and*

$$\frac{1}{2\pi}\int_0^{2\pi} |P(z)|^2\, d\mu(\theta) = 1, \qquad z = e^{i\theta}.$$

For a fixed value of a, *the maximum of* $|P(a)|^2$ *is attained if*

$$P(z) = \frac{\varepsilon K_n(a,z)}{\sqrt{K_n(a,a)}}\,, \qquad |\varepsilon| = 1.$$

This maximum is given by $K_n(a,a)$.

4.3. Zeros of orthogonal polynomials. For the zeros of orthogonal polynomials on the unit circle one can prove the following result:

Theorem 4.3.1. *For $n \geq 1$, all the zeros of $\phi_n(z)$ lie in $|z| < 1$.*

A short proof of this theorem can be given in the following way (see Landau [73]): Let γ be an arbitrary zero of $\phi_n(z)$, so that $\phi_n(z) = (z-\gamma)Q_{n-1}(z)$, or

$$\phi_n(z) + \gamma Q_{n-1}(z) = zQ_{n-1}(z), \tag{4.3.1}$$

with some $Q_{n-1} \in \boldsymbol{\mathcal{P}}_{n-1}$. Then, since ϕ_n is orthogonal to $\boldsymbol{\mathcal{P}}_{n-1}$, on taking norms in (4.3.1) and using the fact that $(zf, zg) = (f, g)$ we find

$$\|\phi_n\|^2 + |\gamma|^2 \|Q_{n-1}\|^2 = \|zQ_{n-1}\|^2 = \|Q_{n-1}\|^2,$$

whence $1 - |\gamma|^2 = \|\phi_n\|^2 / \|Q_{n-1}\|^2 > 0$, as required.

For zeros of the kernel polynomials $z \mapsto K_n(a, z)$ we have (see Szegő [124, p. 292]):

Theorem 4.3.2. *For $|a| < 1$ the zeros of $K_n(a, z)$ lie in $|z| > 1$, for $|a| > 1$ in $|z| < 1$, and for $|a| = 1$ on $|z| = 1$.*

At the end of this section, we mention that classical and semiclassical orthogonal polynomials on the unit circle can also be considered (cf. Marcellán [79]). Similarly, orthogonal polynomials on a rectifiable curve or arc lying in the complex plane can be considered (cf. Geronimus [53], Szegő [124]).

5. Orthogonality on the semicircle and a circular arc

Polynomials orthogonal on the semicircle $\Gamma_0 = \{z \in \mathbb{C} \mid z = e^{i\theta},\ 0 \leq \theta \leq \pi\}$ have been introduced by Gautschi and Milovanović [47–48]. The inner product is given by

$$(f, g) = \int_\Gamma f(z) g(z) (iz)^{-1}\, dz,$$

where Γ is the semicircle $\Gamma = \{z \in \mathbb{C} \mid z = e^{i\theta}, 0 \leq \theta \leq \pi\}$. Alternatively,

$$(f, g) = \int_0^\pi f(e^{i\theta}) g(e^{i\theta})\, d\theta.$$

This inner product is not Hermitian, but the corresponding (monic) orthogonal polynomials $\{\pi_k\}$ exist uniquely and satisfy a three-term recurrence relation of the form

$$\begin{aligned} &\pi_{k+1}(z) = (z - i\alpha_k)\pi_k(z) - \beta_k \pi_{k-1}(z), \qquad k = 0, 1, 2, \ldots, \\ &\pi_{-1}(z) = 0, \quad \pi_0(z) = 1. \end{aligned}$$

Notice that the inner product possesses the property $(zf, g) = (f, zg)$.

The general case of complex polynomials orthogonal with respect to a *complex weight function* was considered by Gautschi, Landau and Milovanović [45]. A generalization of such polynomials on a circular arc was given by M.G. de Bruin [17], and further investigations were done by Milovanović and Rajković [103].

5.1. Orthogonality on the semicircle. Let $w\colon(-1,1) \mapsto \mathbb{R}_+$ be a weight function which can be extended to a function $w(z)$ holomorphic in the half disc $D_+ = \{z \in \mathbb{C} \,|\, |z| < 1,\, \operatorname{Im} z > 0\}$, and

$$(f,g) = \int_\Gamma f(z)g(z)w(z)(iz)^{-1}\,dz = \int_0^\pi f(e^{i\theta})g(e^{i\theta})w(e^{i\theta})\,d\theta. \tag{5.1.1}$$

Together with (5.1.1) consider the inner product

$$[f,g] = \int_{-1}^1 f(x)\overline{g(x)}w(x)\,dx, \tag{5.1.2}$$

which is positive definite and therefore generates a unique set of real (monic) orthogonal polynomials $\{p_k\}$:

$$[p_k,p_m] = 0 \quad \text{for} \quad k \neq m \qquad \text{and} \qquad [p_k,p_m] > 0 \quad \text{for} \quad k = m \qquad (k,m \in \mathbb{N}_0).$$

On the other hand, the inner product (5.1.1) is not Hermitian; the second factor g is not conjugated and the integration is not with respect to the measure $|w(e^{i\theta})|\,d\theta$. The existence of corresponding orthogonal polynomials, therefore, is not guaranteed.

We call a system of complex polynomials $\{\pi_k\}$ *orthogonal on the semicircle* if

$$(\pi_k,\pi_m) = 0 \quad \text{for} \quad k \neq m \qquad \text{and} \qquad (\pi_k,\pi_m) > 0 \quad \text{for} \quad k = m \qquad (k,m \in \mathbb{N}_0).$$

where we assume that π_k is monic of degree k.

The existence of the orthogonal polynomials $\{\pi_k\}$ can be established assuming only that

$$\operatorname{Re}(1,1) = \operatorname{Re} \int_0^\pi w(e^{i\theta})\,d\theta \neq 0. \tag{5.1.3}$$

5.1.1. Existence and representation of π_k. Assume that the weight function w is positive on $(-1,1)$, holomorphic in D_+ and such that the integrals in (5.1.1) and (5.1.2) exist for smooth f and g (possibly) as improper integrals. We also assume that the condition (5.1.3) is satisfied.

Let C_ε, $\varepsilon > 0$, denote the boundary of D_+ with small circular parts of radius ε and centres at ± 1 spared out and let $\boldsymbol{\mathcal{P}}$ be the set of all algebraic polynomials. Then, by Cauchy's theorem, for any $g \in \boldsymbol{\mathcal{P}}$ we have

$$\begin{aligned} 0 &= \int_{C_\varepsilon} g(z)w(z)\,dz \\ &= \left(\int_{\Gamma_\varepsilon} + \int_{C_{\varepsilon,-1}} + \int_{C_{\varepsilon,+1}}\right) g(z)w(z)\,dz + \int_{-1+\varepsilon}^{1-\varepsilon} g(x)w(x)\,dx, \end{aligned} \tag{5.1.4}$$

where Γ_ε and $C_{\varepsilon,\pm 1}$ are the circular parts of C_ε (with radii 1 and ε respectively). We assume that w is such that for all $g \in \boldsymbol{\mathcal{P}}$,

$$\lim_{\varepsilon\to 0} \int_{C_{\varepsilon,\pm 1}} g(z)w(z)\, dz = 0. \tag{5.1.5}$$

Then, if $\varepsilon \to 0$ in (5.1.4), we obtain

$$0 = \int_C g(z)w(z)\, dz = \int_\Gamma g(z)w(z)\, dz + \int_{-1}^{1} g(x)w(x)\, dx, \quad g \in \boldsymbol{\mathcal{P}}. \tag{5.1.6}$$

The (monic, real) polynomials $\{p_k\}$, orthogonal with respect to the inner product (5.1.2), as well as the associated polynomials of the second kind,

$$q_k(z) = \int_{-1}^{1} \frac{p_k(z) - p_k(x)}{z - x} w(x)\, dx \qquad (k = 0, 1, 2, \ldots),$$

are known to satisfy a three-term recurrence relation of the form

$$y_{k+1} = (z - a_k)y_k - b_k y_{k-1} \qquad (k = 0, 1, 2, \ldots), \tag{5.1.7}$$

where

$$y_{-1} = 0,\ y_0 = 1 \ \text{ for } \{p_k\} \qquad \text{and} \qquad y_{-1} = -1,\ y_0 = 0 \ \text{ for } \{q_k\}. \tag{5.1.8}$$

Denote by m_k and μ_k the moments associated with the inner products (5.1.1) and (5.1.2), respectively,

$$\mu_k = (z^k, 1), \qquad m_k = [x^k, 1], \qquad k \geq 0,$$

where, in view of (5.1.8), $b_0 = m_0$.

In [45], Gautschi, Landau and Milovanović proved the following result:

Theorem 5.1.1. *Let w be a weight function, positive on $(-1, 1)$, holomorphic in $D_+ = \{z \in \mathbb{C} \mid |z| < 1,\ \mathrm{Im}\, z > 0\}$, and such that* (5.1.5) *is satisfied and the integrals in* (5.1.6) *exist* (*possibly*) *as improper integrals. Assume in addition that*

$$\mathrm{Re}(1, 1) = \mathrm{Re} \int_0^\pi w(e^{i\theta})\, d\theta \neq 0.$$

Then there exists a unique system of (*monic, complex*) *orthogonal polynomials* $\{\pi_k\}$ *relative to the inner product* (5.1.1). *Denoting by* $\{p_k\}$ *the* (*monic, real*) *orthogonal polynomials relative to the inner product* (5.1.2), *we have*

$$\pi_k(z) = p_k(z) - i\theta_{k-1}p_{k-1}(z) \qquad (k = 0, 1, 2, \ldots), \tag{5.1.9}$$

where

$$\theta_{k-1} = \frac{\mu_0 p_k(0) + i q_k(0)}{i\mu_0 p_{k-1}(0) - q_{k-1}(0)} \qquad (k = 0, 1, 2, \dots). \tag{5.1.10}$$

Alternatively,

$$\theta_k = i a_k + \frac{b_k}{\theta_{k-1}} \quad (k = 0, 1, 2, \dots); \qquad \theta_{-1} = \mu_0, \tag{5.1.11}$$

where a_k, b_k *are the recursion coefficients in* (5.1.7) *and* $\mu_0 = (1, 1)$. *In particular, all* θ_k *are real* (*in fact, positive*) *if* $a_k = 0$ *for all* $k \geq 0$. *Finally,*

$$(\pi_k, \pi_k) = \theta_{k-1}[p_{k-1}, p_{k-1}] \neq 0 \quad (k = 1, 2, \dots), \qquad (\pi_0, \pi_0) = \mu_0. \tag{5.1.12}$$

As we can see, relation (5.1.9), with (5.1.10), gives a connection between orthogonal polynomials on the semicircle and the standard polynomials orthogonal on $[-1, 1]$ with respect to the same weight function w. The norms of these polynomials are in relation (5.1.12).

We mention now two interesting examples from [45]:

EXAMPLE 5.1.1. Let $w(z) = 1+z$. Since $\mu_0 = (1, 1) = \pi+2i$, $\mathrm{Re}\,\mu_0 \neq 0$, the orthogonal polynomials $\{\pi_k\}$ exist. Furthermore, $b_0 = m_0 = 2$, $a_k = (2k+1)^{-1}(2k+3)^{-1}$ for $k \geq 0$, and $b_k = k(k+1)(2k+1)^{-2}$ for $k \geq 1$, so that by (5.1.11),

$$\theta_0 = \frac{\pi - 4i}{3(2 - i\pi)}, \quad \theta_1 = \frac{3\pi + 8i}{5(4 + i\pi)}, \quad \dots .$$

Then, from (5.1.7) and (5.1.8), we find

$$p_0(z) = 1, \quad p_1(z) = z - \frac{1}{3}, \quad p_2(z) = z^2 - \frac{2}{3}z - \frac{1}{5}, \quad \dots ,$$

and by (5.1.9),

$$\pi_0(z) = 1, \quad \pi_1(z) = z - \frac{2}{2 - i\pi}, \quad \pi_2 = z^2 - \frac{i\pi}{4 + i\pi}z - \frac{4}{3(4 + i\pi)}, \quad \dots .$$

EXAMPLE 5.1.2. Consider now $w(z) = z^2$. Here $\mu_0 = \int_0^\pi e^{2i\theta}\, d\theta = 0$ so that (5.1.3) is violated and thus the polynomials $\{\pi_k\}$ do not exist, even though $w(x) \geq 0$ on $[-1, 1]$ and the polynomials $\{p_k\}$ do exist. It is easily seen that $q_k(0) = 0$ when k is even, so that θ_{k-1} is zero for k even, and undefined for k odd. For an explanation of this example see Theorem 5.1.3.

5.1.2. Three-term recurrence relation for $\pi_k(z)$. We assume that

$$\mathrm{Re}\,(1, 1) = \mathrm{Re} \int_0^\pi w(e^{i\theta})\, d\theta \neq 0, \tag{5.1.13}$$

so that orthogonal polynomials $\{\pi_k\}$ exist. Since $(zf, g) = (f, zg)$, it is known that they must satisfy a three-term recurrence relation

$$\begin{aligned} &\pi_{k+1}(z) = (z - i\alpha_k)\pi_k(z) - \beta_k \pi_{k-1}(z), \qquad k = 0, 1, 2, \dots , \\ &\pi_{-1}(z) = 0, \quad \pi_0(z) = 1. \end{aligned} \tag{5.1.14}$$

Using the representation (5.1.9), we can find a connection between the coefficients in (5.1.14) and the corresponding coefficients in the three-term recurrence relation (5.1.7) for polynomials $\{p_k\}$ (see [45]):

Theorem 5.1.2. *Under the assumption* (5.1.13)*, the* (*monic, complex*) *polynomials* $\{\pi_k\}$ *orthogonal with respect to the inner product* (5.1.1) *satisfy the recurrence relation* (5.1.14)*, where the coefficients* α_k*,* β_k *are given by*

$$\alpha_k = \theta_k - \theta_{k-1} - ia_k, \qquad \beta_k = \frac{\theta_{k-1}}{\theta_{k-2}} b_{k-1} = \theta_{k-1}(\theta_{k-1} - ia_{k-1}),$$

for $k \geq 1$ *and* $\alpha_0 = \theta_0 - ia_0$*, with the* θ_k *defined in Theorem* 5.1.1.

Alternatively, the coefficients α_k can be expressed in the form

$$\alpha_k = -\theta_{k-1} + \frac{b_k}{\theta_{k-1}}, \quad k \geq 1, \qquad \alpha_0 = \frac{b_0}{\theta_{-1}} = \frac{m_0}{\mu_0}.$$

5.1.3. Distribution of zeros. It follows from (5.1.14) that the zeros of $\pi_n(z)$ are the eigenvalues of the (complex, tridiagonal) matrix

$$J_n = \begin{bmatrix} i\alpha_0 & 1 & & & O \\ \beta_1 & i\alpha_1 & 1 & & \\ & \beta_2 & i\alpha_2 & \ddots & \\ & & \ddots & \ddots & 1 \\ O & & & \beta_{n-1} & i\alpha_{n-1} \end{bmatrix}, \tag{5.1.15}$$

where α_k and β_k are given in Theorem 5.1.2.

If the weight w is symmetric, i.e.,

$$w(-z) = w(z), \qquad w(0) > 0, \tag{5.1.16}$$

then $\mu_0 = (1,1) = \pi w(0) > 0$, $a_k = 0$, $\theta_k > 0$, for all $k \geq 0$, and

$$\alpha_0 = \theta_0, \quad \alpha_k = \theta_k - \theta_{k-1}, \quad \beta_k = \theta_{k-1}^2, \quad k \geq 1.$$

In that case J_n can be transformed into a real nonsymmetric tridiagonal matrix

$$A_n = -iD_n^{-1} J_n D_n = \begin{bmatrix} \alpha_0 & \theta_0 & & & O \\ -\theta_0 & \alpha_1 & \theta_1 & & \\ & -\theta_1 & \alpha_2 & \ddots & \\ & & \ddots & \ddots & \theta_{n-2} \\ O & & & -\theta_{n-2} & \alpha_{n-1} \end{bmatrix},$$

where $D_n = \operatorname{diag}(1, i\theta_0, i^2\theta_0\theta_1, i^3\theta_0\theta_1\theta_2, \dots) \in \mathbb{C}^{n\times n}$. The eigenvalues η_ν, $\nu = 1, \dots, n$, of A_n can be calculated using the EISPACK subroutine HQR (see [118]). Then all the zeros ζ_ν, $\nu = 1, \dots, n$, of $\pi_n(z)$ are given by $\zeta_\nu = i\eta_\nu$, $\nu = 1, \dots, n$.

In [45] we proved the following result for a symmetric weight (5.1.16):

Theorem 5.1.3. *All zeros of π_n are located symmetrically with respect to the imaginary axis and contained in $D_+ = \{z \in \mathbb{C} \mid |z| < 1,\ \operatorname{Im} z > 0\}$, with the possible exception of a single (simple) zero on the positive imaginary axis.*

If we define the half strip $S_+ = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0,\ -\xi_n \le \operatorname{Re} z \le \xi_n\}$, where ξ_n is the largest zero of the real polynomial p_n, then we can prove that all zeros of π_n are also in S_+ (see [45] and [48]). Thus, all zeros are contained in $D_+ \cap S_+$.

For the Gegenbauer weight $w(z) = (1 - z^2)^{\lambda - 1/2}$, $\lambda > -1/2$, the exceptional case from Theorem 5.1.3 can only arise if $n = 1$ and $-1/2 < \lambda \le 0$. Likewise, no exceptional cases seem to occur for Jacobi weights $w(z) = (1 - z)^\alpha (1 + z)^\beta$, $\alpha, \beta > -1$, if $n \ge 2$, as was observed by several numerical computations (see [45]). However, in a general case, Gautschi [38] exhibited symmetric functions w for which $\pi_n(\,\cdot\,; w)$, for arbitrary fixed n, has a zero iy with $y \ge 1$.

5.2. Orthogonality on a circular arc. Recently M. G. de Bruin [17] considered the polynomials $\{\pi_k^R\}$ orthogonal on a circular arc with respect to the complex inner product

$$(f, g) = \int_\varphi^{\pi - \varphi} f_1(\theta) g_1(\theta) w_1(\theta)\, d\theta, \tag{5.2.1}$$

where $\varphi \in (0, \pi/2)$, and for $f(z)$ the function $f_1(\theta)$ is defined by

$$f_1(\theta) = f\bigl(-iR + e^{i\theta}\sqrt{R^2 + 1}\bigr), \qquad R = \tan\varphi.$$

Alternatively, the inner product (5.2.1) can be expressed in the form

$$(f, g) = \int_{\varGamma_R} f(z) g(z) w(z) (iz - R)^{-1}\, dz, \tag{5.2.2}$$

where $\varGamma_R = \{z \in \mathbb{C} \,|\, z = -iR + e^{i\theta}\sqrt{R^2 + 1},\ \varphi \le \theta \le \pi - \varphi,\ \tan\varphi = R\}$.

Under suitable integrability conditions on the weight function w, which is positive on $(-1, 1)$ and is holomorphic in the moon-shaped region

$$M_+ = \Bigl\{ z \in \mathbb{C} \;\Big|\; |z + iR| < \sqrt{R^2 + 1},\ \operatorname{Im} z > 0 \Bigr\},$$

where $R > 0$, the polynomials $\{\pi_k^R\}$ orthogonal on the circular arc $\varGamma_R$ with respect to the complex inner product (5.2.1) always exist and have similar properties like polynomials orthogonal on the semicircle.

For $R = 0$ the arc $\varGamma_R$ reduces to the semicircle $\varGamma$, and polynomials $\{\pi_k^R\}$ to $\{\pi_k\}$. It is easy to prove that the condition

$$\operatorname{Re} \int_{\varGamma_R} w(z)(iz - R)^{-1}\, dz = \operatorname{Re} \int_\varphi^{\pi - \varphi} w_1(\theta)\, d\theta \ne 0$$

is automatically satisfied for $R > 0$ in contrast to the case $R = 0$ (see condition (5.1.13)).

Quite analogous results to Theorems 5.1.1–5.1.4 were proved by de Bruin [17]. For example, for polynomials $\{\pi_k\}$ (the upper index R is omitted) equalities (5.1.9) and (5.1.12), as well as the three-term recurrence relation (5.1.14) hold, where now the θ_k is given by

$$\theta_k = -R + ia_k + \frac{b_k}{\theta_{k-1}} \quad (k = 0, 1, 2, \dots); \qquad \theta_{-1} = \mu_0,$$

instead of (5.1.11). Also, for the symmetric weight, $w(z) = w(-z)$, all zeros of π_n are contained in M_+ with the possible exception of just one simple zero situated on the positive imaginary axis.

5.2.1. Dual orthogonal polynomials. Let $\{\pi_n\}$ be the set of polynomials orthogonal on the circular arc Γ_R, with respect to the inner product (5.2.1), i.e., (5.2.2). Milovanović and Rajković [103] introduced the polynomials $\{\pi_n^*\}$ orthogonal on the symmetric down circular arc Γ_R^* with respect to the inner product defined by

$$(f, g)^* = \int_{\Gamma_R^*} f(z)g(z)w(z)(iz + R)^{-1}\, dz, \tag{5.2.3}$$

where $\Gamma_R^* = \{z \in \mathbb{C} \,|\, z = iR + e^{-i\theta}\sqrt{R^2 + 1},\ \varphi \le \theta \le \pi - \varphi,\ \tan\varphi = R\}$. Such polynomials are called *dual orthogonal polynomials* with respect to polynomials $\{\pi_n\}$.

Let M be a lentil-shaped region with the boundary $\partial M = \Gamma_R \cup \Gamma_R^*$, i.e.,

$$M = \bigl\{z \in \mathbb{C} \,|\, |z \pm iR| < \sqrt{R^2 + 1}\bigr\},$$

where $R > 0$.

We assume that w is a weight function, positive on $(-1, 1)$, holomorphic in M, and such that the integrals in (5.2.2), (5.2.3), and (5.1.2) exist for smooth functions f and g (possibly) as improper integrals. Under the same additional conditions on w and f, like previous, we have

$$0 = \int_{\Gamma} f(z)w(z)\, dz + \int_{-1}^{1} f(x)w(x)\, dx,$$

where $\Gamma = \Gamma_R$ or Γ_R^*. Then both systems of the orthogonal polynomials $\{\pi_n\}$ and $\{\pi_n^*\}$ exist uniquely.

The inner products in (5.2.2) and (5.2.3) define the moment functionals

$$\mathcal{L}z^k = \mu_k, \qquad \mu_k = (z^k, 1) = \int_{\Gamma_R} z^k w(z)(iz - R)^{-1}\, dz$$

and

$$\mathcal{L}^* z^k = \mu_k^*, \qquad \mu_k^* = (z^k, 1)^* = \int_{\Gamma_R^*} z^k w(z)(iz+R)^{-1}\,dz,$$

respectively. Using the moment determinants, we can express the (monic) polynomials π_k and π_k^* as

$$\pi_k(z) = \frac{1}{\Delta_k}\begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_k \\ \mu_1 & \mu_2 & & \mu_{k+1} \\ \vdots & & & \\ \mu_{k-1} & \mu_k & & \mu_{2k-1} \\ 1 & z & & z^k \end{vmatrix}, \quad \pi_k^*(z) = \frac{1}{\Delta_k^*}\begin{vmatrix} \mu_0^* & \mu_1^* & \cdots & \mu_k^* \\ \mu_1^* & \mu_2^* & & \mu_{k+1}^* \\ \vdots & & & \\ \mu_{k-1}^* & \mu_k^* & & \mu_{2k-1}^* \\ 1 & z & & z^k \end{vmatrix},$$

respectively, where

$$\Delta_k = \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_{k-1} \\ \mu_1 & \mu_2 & & \mu_k \\ \vdots & & & \\ \mu_{k-1} & \mu_k & & \mu_{2k-2} \end{vmatrix}, \qquad \Delta_k^* = \begin{vmatrix} \mu_0^* & \mu_1^* & \cdots & \mu_{k-1}^* \\ \mu_1^* & \mu_2^* & & \mu_k^* \\ \vdots & & & \\ \mu_{k-1}^* & \mu_k^* & & \mu_{2k-2}^* \end{vmatrix}.$$

In [103] we proved that $\pi_k^*(\overline{z}) = \overline{\pi_k(z)}$, as well as the relation

$$\pi_k^*(z) = p_k(z) - i\theta_{k-1}^* p_{k-1}(z), \qquad k = 0, 1, 2, \ldots,$$

where

$$\theta_{k-1}^* = \frac{(\pi_k^*, \pi_k^*)^*}{[p_{k-1}, p_{k-1}]}, \quad k = 1, 2, \ldots, \qquad \theta_{-1}^* = \mu_0^*.$$

Here, $\theta_{k-1}^* = -\overline{\theta}_{k-1}$, where θ_{k-1} is the corresponding coefficient in the polynomial π_k.

Also, the following theorem holds:

Theorem 5.2.1. *The dual* (*monic*) *orthogonal polynomials* $\{\pi_k^*\}$ *satisfy the three-term recurrence relation*

$$\begin{aligned} &\pi_{k+1}^*(z) = (z - i\alpha_k^*)\pi_k^*(z) - \beta_k^*\pi_{k-1}^*(z), \qquad k = 0, 1, 2, \ldots, \\ &\pi_{-1}^*(z) = 0, \quad \pi_0^*(z) = 1, \end{aligned}$$

with $\alpha_k^* = -\overline{\alpha}_k$ *and* $\beta_k^* = \overline{\beta}_k$, *where* α_k *and* β_k *are the coefficients in the corresponding recurrence relation for the polynomials* $\{\pi_k\}$.

Using dual polynomials we can give a short proof of the following result stated in [17]:

Theorem 5.2.2. *Let $w(z) = w(-z)$. Then $\theta_{k-1} > 0$, for $k \geq 0$.*

Namely, since $(\pi_k, \pi_k) = \theta_{k-1}[p_{k-1}, p_{k-1}]$ it is enough to prove that $(\pi_k, \pi_k) > 0$. In this *symmetric case*, θ_{k-1} is real and we have $\theta_{k-1}^* = -\theta_{k-1}$ and

$$(\pi_k, \pi_k) = (\pi_k, \pi_k^*) = \int_{\Gamma_R} G(z)w(z)(iz-R)^{-1}\,dz = -\int_{-1}^{1} G(x)\frac{w(x)}{ix-R}\,dx,$$

where $G(z) = p_k(z)^2 + \theta_{k-1}^2 p_{k-1}(z)^2$. Then

$$(\pi_k, \pi_k) = R\int_{-1}^{1} G(x)\frac{w(x)}{R^2+x^2}\,dx + i\int_{-1}^{1} xG(x)\frac{w(x)}{R^2+x^2}\,dx.$$

Since $x \mapsto G(x)$ is an even positive function, the second integral on the right-hand side vanishes and $(\pi_k, \pi_k) > 0$.

5.2.2. Differential equation for the Jacobi weight. We consider the Jacobi weight function

$$w(z) = w^{\alpha,\beta}(z) = (1-z)^{\alpha}(1+z)^{\beta}, \qquad \alpha, \beta > -1,$$

where fractional powers are understood in terms of their principal branches.

The corresponding (monic) polynomials $\{\pi_k^R\}$ orthogonal on the circular arc Γ_R, with respect to the inner product (5.2.1), i.e., (5.2.2), where $w(z) = w^{\alpha,\beta}(z)$, can be expressed in the form

$$\pi_k^R(z) = \pi_k(z) = p_k(z) - i\theta_{k-1}p_{k-1}(z),$$

where $p_k(z) = \hat{P}_k^{\alpha,\beta}(z)$ are the monic Jacobi polynomials and $\theta_{k-1} = \theta_{k-1}^{\alpha,\beta}$ is given by

$$\theta_{k-1} = \frac{1}{i}\frac{\varrho_k(-iR)}{\varrho_{k-1}(-iR)}, \qquad k \geq 1,$$

where

$$\varrho_k(z) = \int_{-1}^{1} \frac{p_k(x)}{z-x}\,w(x)\,dx, \qquad k \geq 0.$$

The monic polynomials $p_k(z)$ satisfy Jacobi's differential equation

$$A(z)u'' + B(z)u' + \lambda_k u = 0 \qquad \text{(cf. (2.2.2))}$$

and differentiation formula

$$A(z)p_k'(z) = [(k+\alpha+\beta+1)z + v_k]p_k(z) - (2k+\alpha+\beta+1)p_{k+1}(z),$$

where (see Table 2.2.1)

$$A(z) = 1-z^2, \quad B(z) = \beta-\alpha-(\alpha+\beta+2)z, \quad \lambda_k = k(k+\alpha+\beta+1)$$

and

$$v_k = (\alpha-\beta)\frac{k+\alpha+\beta+1}{2k+\alpha+\beta+2}.$$

Solving the problem: *Find a function $z \mapsto \Omega(z)$ in the form*

$$\Omega(z) = (z-1)^{r_k - it_k}(z+1)^{s_k + it_k} \qquad (r_k, s_k, t_k \in \mathbb{R}), \tag{5.2.4}$$

such that $(z^2-1)[\Omega(z)u(z)]' = \Omega(z)v(z)$, for $u(z) = p_{n-1}(z)$ and $v(z) = \gamma_k \pi_k^R(z)$, where γ_k is a constant, Milovanović and Rajković [103] proved the following result:

Theorem 5.2.3. *The polynomial* $\pi_k^R(z)$ *satisfies the differential equation*

$$P(z)y'' + Q(z)y' + R(z)y = 0, \tag{5.2.5}$$

with polynomial coefficients

$$\begin{aligned} P(z) &= -A(z)^2\Omega(z)b(z),\\ Q(z) &= A(z)^2\Omega(z)\big(b'(z) - a(z)b(z)\big),\\ R(z) &= A(z)\big[A(z)\Omega(z)\big(a(z)b'(z) - a'(z)b(z)\big) + \Omega(z)^2 b(z)^2\big], \end{aligned} \tag{5.2.6}$$

where $\Omega(z)$ *defined by* (5.2.4) *with parameters*

$$r_k = \frac{1}{2}\left(k + \alpha + \beta + v_{k-1}\right),\ s_k = \frac{1}{2}\left(k + \alpha + \beta - v_{k-1}\right),\ t_k = \frac{1}{2}(2k+\alpha+\beta-1)\theta_{k-1},$$

and $a(z)$ *and* $b(z)$ *are given by*

$$\begin{aligned} a(z) &= \frac{1}{A(z)}(g(z) + B(z) - A'(z)),\\ b(z) &= -\frac{1}{A(z)\Omega(z)}\Big((\lambda_{k-1} + g'(z))A(z) + g(z)(g(z) + B(z) - A'(z))\Big), \end{aligned} \tag{5.2.7}$$

and $g(z) = (k + \alpha + \beta)z + c_{k-1}$, $c_{k-1} = v_{k-1} - i(2k + \alpha + \beta - 1)\theta_{k-1}$.

Notice that all coefficients in (5.2.6) have the factor $A(z)\Omega(z)$, but it is because of this factor that the coefficients turn out to be polynomials.

Using (5.2.6) and (5.2.7) we find $P(z) = (1 - z^2)C(z)$, where $z \mapsto C(z)$ is the following polynomial of the first degree

$$C(z) = k(k + \alpha + \beta) + (\beta - \alpha + c_{k-1})c_{k-1} - i(2k + \alpha + \beta)(2k + \alpha + \beta - 1)\theta_{k-1}z,$$

where

$$c_{k-1} = (\alpha - \beta)\frac{k + \alpha + \beta}{2k + \alpha + \beta} - i(2k + \alpha + \beta - 1)\theta_{k-1}.$$

After some calculation, $C(z)$ can be expressed in the form

$$C(z) = \gamma_0 + i\gamma_1\theta_{k-1} + \gamma_2\theta_{k-1}^2 - \eta_k z, \tag{5.2.8}$$

where

$$\begin{aligned} \gamma_0 &= \frac{4k(k + \alpha)(k + \beta)(k + \alpha + \beta)}{(2k + \alpha + \beta)^2}, \quad \gamma_1 = (\beta^2 - \alpha^2)\frac{2k + \alpha + \beta - 1}{2k + \alpha + \beta},\\ \gamma_2 &= -(2k + \alpha + \beta - 1)^2, \quad \eta_k = (2k + \alpha + \beta)(2k + \alpha + \beta - 1)i\theta_{k-1}. \end{aligned}$$

We remark that the differential equation (5.2.5) has regular singular points at 1, -1, ∞, and an additional regular singular point ζ_k which depends on k and is given by

$$\zeta_k = \frac{\gamma_1 - i(\gamma_2\theta_{k-1} + \gamma_0/\theta_{k-1})}{(2k+\alpha+\beta)(2k+\alpha+\beta-1)}.$$

The polynomials P, Q, and R in (5.2.6) can be expressed by the coefficients $A(z)$, $B(z)$, and λ_k in differential equation (2.2.2). Namely,

$$\begin{aligned} P(z) &= A(z)C(z), \\ Q(z) &= B(z)C(z) + A(z)\eta_k, \\ R(z) &= \lambda_k C(z) + A(z)a(z)\eta_k, \end{aligned} \tag{5.2.9}$$

where

$$a(z) = \frac{1}{A(z)}\Big(kz + \frac{k(\beta-\alpha)}{2k+\alpha+\beta} - i(2k+\alpha+\beta-1)\theta_{k-1}\Big)$$

and $C(z)$ is given by (5.2.8).

We can see that Q and R are the complex polynomials of degree two and one, respectively.

When $\alpha = \beta = \lambda - 1/2$ $(\lambda > -1/2)$ we obtain the Gegenbauer case, which is considered for $R = 0$ by Gautschi, Landau and Milovanović in [45].

It is interesting to consider a case when $R \to +\infty$, i.e., when Γ_R reduces to the interval $[-1, 1]$. Since $\lim\limits_{R\to+\infty} \theta_{k-1} = 0$, we have

$$\lim_{R\to+\infty} \eta_k = 0, \qquad \lim_{R\to+\infty} C(z) = \gamma_0, \qquad \lim_{R\to+\infty} a(z) = \frac{1}{A(z)}\Big(kz + \frac{k(\beta-\alpha)}{2k+\alpha+\beta}\Big).$$

Thus, the limit case of (5.2.9) gives

$$\lim_{R\to+\infty} P(z) = \gamma_0 A(z), \qquad \lim_{R\to+\infty} Q(z) = \gamma_0 B(z), \qquad \lim_{R\to+\infty} R(z) = \gamma_0 \lambda_k.$$

In that case, dividing (5.2.5) by γ_0 we obtain Jacobi's differential equation, which is, in fact, the result expected.

5.2.3. Functions of the second kind and some associated polynomials. Let the inner product $(\cdot,\cdot)$ be given by (5.2.1), i.e., (5.2.2). In connection with polynomials $\{\pi_k^R\}$ orthogonal with respect to $(\cdot,\cdot)$ on Γ_R, Milovanović and Rajković [103] introduced the functions, so-called *functions of the second kind*,

$$\varrho_k^R(z) = \int_{\Gamma_R} \frac{\pi_k^R(\zeta)}{z-\zeta} \cdot \frac{w(\zeta)}{i\zeta - R}\, d\zeta, \qquad k = 0, 1, 2, \ldots\,. \tag{5.2.10}$$

It is easily seen that they also satisfy the same recurrence relation as the polynomials π_k^R. Indeed, from the recurrence relation

(5.2.10) $$\pi_{k+1}^R(z) = (z - i\alpha_k)\pi_k^R(z) - \beta_k \pi_{k-1}^R(z), \qquad k = 0, 1, 2, \dots ,$$

for $z = \zeta$, multiplying by $w(\zeta)/((i\zeta - R)(z - \zeta))$ and integrating, we obtain

$$\varrho_{k+1}^R(z) = (z - i\alpha_k)\varrho_k^R(z) - \int_{\Gamma_R} \pi_k^R(\zeta)\frac{w(\zeta)}{i\zeta - R}\, d\zeta - \beta_k \varrho_{k-1}^R(z).$$

By orthogonality, the integral on the right side in the above equality vanishes if $k \geq 1$, and equals μ_0 if $k = 0$. If we define $\varrho_{-1}^R(z) = 1$ (and $\beta_0 = \mu_0$), we have

(5.2.12) $$\varrho_{k+1}^R(z) = (z - i\alpha_k)\varrho_k^R(z) - \beta_k \varrho_{k-1}^R(z), \qquad k = 0, 1, 2, \dots .$$

For $|z|$ sufficiently large, we can prove (see [103])

$$\varrho_k^R(z) = \frac{\|\pi_k^R\|^2}{z^{k+1}}\Big(1 + O\big(\frac{1}{z}\big)\Big),$$

where $\|\pi_k^R\|^2 = (\pi_k^R, \pi_k^R)$. Based on an idea by Stieltjes (see Monegato [105], Gautschi [31]), Milovanović and Rajković [103] considered an expansion of $1/\varrho_k^R(z)$ into descending powers of z, and obtained

$$\frac{1}{\varrho_k^R(z)} = \frac{z^{k+1}}{\|\pi_k^R\|^2}\left(1 + c_1 z^{-1} + c_2 z^{-2} + \cdots\right) = E_{k+1}^R(z) + d_1 z^{-1} + d_2 z^{-2} + \cdots ,$$

where

$$E_{k+1}^R(z) = \frac{1}{\|\pi_k^R\|^2}\left(z^{k+1} + c_1 z^k + \cdots + c_{k+1}\right)$$

and $d_k = c_{k+k+1}/\|\pi_k^R\|^2$, $k = 1, 2, \dots$. We call E_{k+1}^R the *Stieltjes polynomial associated with polynomials* $\{\pi_k^R\}$ orthogonal with respect to $(\cdot, \cdot)$ on Γ_R. By a residue calculation, this polynomial of exact degree $k + 1$, can be expressed in the form

$$E_{k+1}^R(z) = \frac{1}{2\pi i}\oint_C \frac{d\zeta}{(\zeta - z)\varrho_k^R(\zeta)},$$

where C is a sufficiently large contour with z in its interior.

Theorem 5.2.4. *Stieltjes' polynomial E_{k+1}^R is orthogonal to all lower-degree polynomials with respect to the complex measure $d\lambda(z) = \pi_k^R(z)w(z)(iz - R)^{-1}dz$, i.e.,*

$$\int_{\Gamma_R} E_{k+1}^R(z)p(z)\pi_k^R(z)\frac{w(z)}{iz - R}\, dz = 0 \qquad \big(p \in \boldsymbol{\mathcal{P}}_k\big),$$

where $\boldsymbol{\mathcal{P}}_k$ is the set of all polynomials of degree at most k.

The quantities $\varrho_k^R(z)/\pi_k^R(z)$, $|z|>1$, are important in getting error bounds for Gaussian quadrature formulas over Γ_R, applied to analytic functions (cf. Gautschi and Varga [50]). Stieltjes' polynomials appear in quadrature formulas of Gauss-Kronrod's type (cf. [37]).

We can also introduce the polynomials

$$\sigma_k^R(z)=\int_{\Gamma_R}\frac{\pi_k^R(z)-\pi_k^R(\zeta)}{z-\zeta}\cdot\frac{w(\zeta)}{i\zeta-R}\,d\zeta,\qquad k=0,1,2,\dots,$$

which are called the *polynomials associated with the orthogonal polynomials* π_k^R. It is easy to see that

$$\varrho_k^R(z)=\pi_k^R(z)\varrho_0^R(z)-\sigma_k^R(z).$$

The polynomials $\{\sigma_k^R\}$ satisfy the same three-term recurrence relation

$$\sigma_{k+1}^R(z)=(z-i\alpha_k)\sigma_k^R(z)-\beta_k\sigma_{k-1}^R(z),\qquad k=0,1,2,\dots, \tag{5.2.13}$$

where $\sigma_0^R(z)=0,\ \sigma_1^R(z)=\mu_0$. If we define $\sigma_{-1}^R(z)=-1$ and $\beta_0=\mu_0$, we can note that (5.2.13) also holds for $k=0$ (see Gautschi [30]).

Using the recurrence relations for $\{\pi_k^R\}$, $\{\varrho_k^R\}$, and $\{\sigma_k^R\}$ ((5.2.11), (5.2.12), and (5.2.13), respectively), where

1° $\pi_{-1}^R(z)=0,\quad \pi_0^R(z)=1\,;$

2° $\varrho_{-1}^R(z)=1,\quad \varrho_0^R(z)=F(z)$ (defined by (5.2.10));

3° $\sigma_{-1}^R(z)=-1,\quad \sigma_0^R(z)=0\,,$

Milovanović and Rajković [103] proved the following identity of Christoffel-Darboux type:

Theorem 5.2.5. *Let $\{f_k\}$ and $\{g_k\}$ satisfy the three-term recurrence relation of the form* (5.2.13)*, and $S_k(z,w)=f_{k+1}(z)g_k(w)-g_{k+1}(w)f_k(z)$. Then the identity*

$$(z-w)\sum_{k=0}^{n}\frac{f_k(z)g_k(w)}{\beta_0\beta_1\cdots\beta_k}=\frac{S_n(z,w)}{\beta_0\beta_1\cdots\beta_n}-S_{-1}(z,w)$$

holds, where β_k $(k=0,1,2,\dots)$ are the recursion coefficients in (5.2.13)*. Under conditions* 1°–3°*, we have the following special cases*

(a) $f_k:=\pi_k^R,\quad g_k:=\pi_k^R,\quad S_{-1}=0\,;$

(b) $f_k:=\pi_k^R,\quad g_k:=\varrho_k^R,\quad S_{-1}=1\,;$

(c) $f_k:=\pi_k^R,\quad g_k:=\sigma_k^R,\quad S_{-1}=-1\,;$

(d) $f_k:=\varrho_k^R,\quad g_k:=\varrho_k^R,\quad S_{-1}=F(z)-F(w)\,;$

(e) $f_k:=\varrho_k^R,\quad g_k:=\sigma_k^R,\quad S_{-1}=-F(z)\,;$

(f) $f_k:=\sigma_k^R,\quad g_k:=\sigma_k^R,\quad S_{-1}=0\,.$

5.2.4. Geronimus' version of orthogonality on a contour. In the paper [66], J. W. Jayne considered the Geronimus' concept of orthogonality for recursively generated polynomials. Ya. L. Geronimus proved that a sequence of polynomials $\{p_k\}$, which is orthogonal on a finite interval on real line, is also orthogonal in the sense that there is a weight function $z \mapsto \chi(z)$ having one or more singularities inside a simple curve C and such that

$$\langle p_k, p_m \rangle = \frac{1}{2\pi i} \oint_C p_k(z) p_m(z) \chi(z)\, dz = \begin{cases} 0, & k \neq m, \\ h_m, & k = m. \end{cases} \tag{5.2.14}$$

Following Geronimus [54] and Jayne [66], Milovanović and Rajković [102] determined such a complex weight function $z \mapsto \chi(z)$, for (monic) polynomials $\{\pi_k\}$ orthogonal on the semicircle $\mathit{\Gamma}$, and also for the corresponding polynomials $\{\pi_k^R\}$ orthogonal on the circular arc $\mathit{\Gamma}_R$ $(R > 0)$.

Denoting by C any positively oriented simple closed contour surrounding some circle $|z| = r > 1$, we assume that

$$\chi(z) = \sum_{k=1}^{\infty} \omega_k z^{-k}, \quad \omega_1 = 1, \tag{5.2.15}$$

for $|z| > 1$, and express z^n as a linear combination of the monic polynomials π_m, $m = 0, 1, \dots, n$, which are orthogonal on the semicircle $\mathit{\Gamma}$, with respect to the inner product (5.1.1). Thus,

$$z^n = \sum_{m=0}^{n} \gamma_{n,m} \pi_m(z), \tag{5.2.16}$$

where $(z^n, \pi_m) = \gamma_{n,m}(\pi_m, \pi_m)$, $m = 0, 1, \dots, n$. Using the inner product (5.2.14) and the representation (5.2.15), we obtain

$$\langle z^n, 1 \rangle = \frac{1}{2\pi i} \oint_C z^n \chi(z)\, dz = \frac{1}{2\pi i} \oint_C \sum_{k=1}^{\infty} \omega_k z^{n-k}\, dz = \omega_{n+1}.$$

On the other hand, because of (5.2.16) and the orthogonality condition (5.2.14), we find

$$\langle z^n, 1 \rangle = \langle \sum_{m=0}^{n} \gamma_{n,m} \pi_m(z), 1 \rangle = \sum_{m=0}^{n} \gamma_{n,m} \langle \pi_m, 1 \rangle,$$

i.e., $\langle z^n, 1 \rangle = \gamma_{n,0} \langle \pi_0, \pi_0 \rangle = \gamma_{n,0} h_0$. Thus, we have $w_{n+1} = \gamma_{n,0} h_0 = \gamma_{n,0}$, because $h_0 = \omega_1 = 1$.

Finally, using the moments $\mu_n = (z^n, 1)$, we obtain $\omega_{n+1} = \mu_n / \mu_0$, $n \geq 0$, and

$$\chi(z) = \frac{1}{\mu_0} \sum_{k=1}^{\infty} \mu_{k-1} z^{-k}, \qquad |z| > 1, \tag{5.2.17}$$

where we need the convergence of this series for $|z| > r > 1$.

Suppose that w be a weight function, nonnegative on $(-1,1)$, holomorphic in $D_+ = \{z \in \mathbb{C} \,|\, |z| < 1, \operatorname{Im} z > 0\}$, integrable over ∂D_+, and such that (5.1.13) is satisfied. Then the moments μ_k can be expressed in the form

$$\mu_0 = \int_\Gamma w(z)(iz)^{-1}\,dz = \frac{1}{i}\left(i\pi w(0) - \text{ v.p.}\int_{-1}^{1}\frac{w(x)}{x}\,dx\right)$$

and

$$\mu_k = \int_\Gamma z^k w(z)(iz)^{-1}\,dz = i\int_{-1}^{1} x^{k-1}w(x)\,dx, \quad k \geq 1.$$

These moments are included in the series (5.2.17).

Supposing that the weight function w has such moments μ_k, which provide the convergence of the series (5.2.17), for all z outside some circle $|z| = r > 1$ lying interior to C, Milovanović and Rajković [102] proved:

Theorem 5.2.6. *The monic polynomials $\{\pi_k\}$, which are orthogonal on the semicircle Γ with respect to the inner product* (5.1.1)*, are also orthogonal in the sense of* (5.2.14)*, where*

$$\chi(z) = \frac{1}{z}\left(1 + \frac{i}{\mu_0}\int_{-1}^{1}\frac{w(x)}{z-x}\,dx\right), \qquad |z| > r > 1,$$

and

$$\mu_0 = \pi w(0) + i\,\text{v.p.}\int_{-1}^{1}\frac{w(x)}{x}\,dx.$$

In Gegenbauer case they obtained the following result:

Corollary 5.2.7. *Let $w(z) = (1-z^2)^{\lambda-1/2}$, $\lambda > -1/2$. The monic polynomials $\{\pi_k\}$, which are orthogonal on the unit semicircle with respect to the inner product* (5.1.1)*, are also orthogonal in the sense of* (5.2.14)*, where*

$$\chi(z) = \frac{1}{z} + \frac{i}{\sqrt{\pi}\,z^2}\cdot\frac{\Gamma(\lambda+\frac{1}{2})}{\Gamma(\lambda+1)}\,F\Big(1, \frac{1}{2}, \lambda+1; \frac{1}{z^2}\Big),$$

where F is the Gauss hypergeometric series and Γ is the gamma function.

Example 5.2.1. In Legendre case ($\lambda = 1/2$) we have

$$\chi(z) = \frac{1}{z} + \frac{i}{\pi z}\log\frac{z+1}{z-1},$$

where the interval from -1 to 1 on the real axis is considered as a branch cut.

The corresponding complex weight for polynomials $\{\pi_k^R\}$ $(R > 0)$ orthogonal on the circular arc Γ_R was also derived in [102] in the form

$$\chi(z) = \frac{1}{\mu_0}\int_{-1}^{1}\frac{(R+ix)w(x)}{(R^2+x^2)(z-x)}\,dx, \qquad |z| > r > 1,$$

where

$$\mu_0 = \int_{-1}^{1}\frac{R+ix}{R^2+x^2}\,w(x)\,dx.$$

Several interesting properties of such polynomials and applications in numerical analysis, especially for Gegenbauer weight, were given in [48] and [93]. Also, differentiation formulas for higher derivatives of analytic functions, using quadratures on the semicircle, were considered in [19].

6.1. Gaussian quadratures. In this section we consider Gauss-Christoffel quadrature formula over the semicircle $\Gamma = \{z \in \mathbb{C} \mid z = e^{i\theta}, 0 \le \theta \le \pi\}$,

$$\int_0^\pi f(e^{i\theta}) w(e^{i\theta})\, d\theta = \sum_{\nu=1}^{n} \sigma_\nu f(\zeta_\nu) + R_n(f), \tag{6.1.1}$$

with Gegenbauer weight $w(z) = (1-z^2)^{\lambda-1/2}$, $\lambda > -1/2$, which is exact for all algebraic polynomials of degree at most $2n-1$.

In this case, (5.1.13) reduces to $\mathrm{Re}(1,1) = \pi \neq 0$, so that the corresponding orthogonal polynomials exist and they can be expressed in terms of monic Gegenbauer polynomials $\hat{C}_k^\lambda(z)$,

$$\pi_k(z) = \hat{C}_k^\lambda(z) - i\theta_{k-1}\hat{C}_{k-1}^\lambda(z),$$

where the sequence $\{\theta_k\}$ is given by

$$\theta_k = \frac{1}{\lambda + k} \cdot \frac{\Gamma((k+2)/2)\Gamma(\lambda + (k+1)/2)}{\Gamma((k+1)/2)\Gamma(\lambda + k/2)}, \qquad k \ge 0.$$

It was shown [45, Sect. 6.3] that all zeros of $\pi_n(z)$, $n \ge 2$, are simple and contained in the upper unit half disc $D_+ = \{z \in \mathbb{C} \,|\, |z| < 1, \mathrm{Im}\, z > 0\}$. The nodes $\zeta_\nu = \zeta_\nu^{(n)}$ in (6.1.1) are precisely the zeros of the polynomial π_n, i.e., the eigenvalues of the Jacobi matrix J_n given by (5.1.15). The weights $\sigma_\nu = \sigma_\nu^{(n)}$ can be obtained by an adaptation of the procedure of Golub and Welsch [56] (see [31] and [33]).

Following [50], we gave error bounds for the Gaussian quadratures (6.1.1), applied to analytic functions, using a contour integral representation of the remainder term (see [94]).

The Gaussian quadrature (6.1.1) can be applied to calculation of the Cauchy principal value integral

$$I_\lambda(\xi; f) = \text{ v.p.} \int_{-1}^{1} \frac{w(t) f(t)}{t - \xi}\, dt,$$

where $-1 < \xi < 1$ and $w(t) = (1-t^2)^{\lambda-1/2}$, $\lambda > -1/2$. Firstly, using the linear fractional transformation $t = (x+\xi)/(x\xi+1)$ we find

$$I_\lambda(\xi; f) = w(\xi) \text{ v.p.} \int_{-1}^{1} w(x) \frac{g(\xi; x)}{x}\, dx,$$

where $g(\xi;x)=f\Big(\dfrac{x+\xi}{x\xi+1}\Big)/(x\xi+1)^{2\lambda}$.

Let f be a meromorphic function with poles p_ν, $\nu=1,\dots,m$, in D_+ and $\psi(z)=w(z)g(\xi;z)/z$. In [93] we proved that

$$I_\lambda(\xi;f)\approx w(\xi)\ \mathrm{Im}\Big\{\sum_{\nu=1}^{n}\sigma_\nu g(\xi;\zeta_\nu)-2\pi\sum_{\nu=1}^{m}\operatorname*{Res}_{z=p_\nu}\psi(z)\Big\}.$$

6.2. Numerical differentiation. Let f be an analytic function on some domain containing the point a and a circular neighborhood of a with radius r. Using the central difference operator δ_h defined by

$$\delta_h f(a)=\frac{1}{h}\Big(f\Big(a+\frac{h}{2}\Big)-f\Big(a-\frac{h}{2}\Big)\Big),$$

we can find $\delta_h^m f(a)=\delta_h\left(\delta_h^{m-1}f(a)\right)$, i.e.,

$$\delta_h^m f(a)=\frac{1}{h^m}\sum_{k=0}^{m}(-1)^k\binom{m}{k}f\Big(a+\frac{m-2k}{2}h\Big). \tag{6.2.1}$$

Putting $he^{i\theta}$ instead of h, where h is such that $|a+\frac{mh}{2}e^{i\theta}|<r$, and integrating (6.2.1) over the semicircle, we obtain

$$\int_0^\pi \delta_{he^{i\theta}}^m f(a)w(e^{i\theta})\,d\theta=\pi f^{(m)}(a). \tag{6.2.2}$$

Applying the Gauss-Christoffel quadrature formula on the semicircle (6.1.1) to the integral on the left side in (6.2.2), we obtain the following differentiation formula to higher derivatives

$$f^{(m)}(a)\approx D_{n,h}^m f(a)=\frac{1}{\pi}\sum_{\nu=1}^{n}\sigma_\nu\delta_{h\zeta_\nu}^m f(a)\,,$$

i.e.,

$$D_{n,h}^m f(a)=\frac{1}{\pi h^m}\sum_{\nu=1}^{n}\frac{\sigma_\nu}{\zeta_\nu^m}\sum_{k=0}^{m}(-1)^k\binom{m}{k}f\Big(a+\frac{m-2k}{2}h\zeta_\nu\Big). \tag{6.2.3}$$

Regarding to the truncation error we can give the following result (see [19]):

Theorem 6.2.1. *The error of the differentiation formula* (6.2.3) *for analytical functions is given by*

$$R_{n,h}^m f(a)=f^{(m)}(a)-D_{n,h}^m f(a)=\frac{1}{\pi}\sum_{p=n}^{\infty}\frac{f^{(m+2p)}(a)}{(m+2p)!}S_{m+2p}^{(m)}R_n(z^{2p})h^{2p},$$

where

$$S_j^{(m)} = \sum_{k=0}^{m} (-1)^k \binom{m}{k} \left(\frac{m-2k}{2}\right)^j$$

and $R_n(z^{2p})$ *is defined by* (6.1.1). *The dominant error term is*

$$\frac{S_{m+2n}^{(m)}}{\pi(m+2n)!} \left(\frac{\Gamma\left((n+1)/2\right)\Gamma\left(\lambda+n/2\right)}{\Gamma(\lambda+n)}\right)^2 f^{(m+2n)}(a)h^{2n}.$$

Some considerations for $n = 2$ and $m = 1$ regarding to λ are given in [93].

For real-valued analytic functions the formula (6.2.3) can be simplified. Namely, when n is even and $\mathrm{Re}\,\zeta_\nu > 0$, for $\nu = 1, 2, \ldots, n/2$, one finds

$$\text{(6.2.4)} \qquad D_{n,h}^m f(a) = \frac{2}{\pi h^m} \sum_{\nu=1}^{n/2} \mathrm{Re}\left\{ \frac{\sigma_\nu}{\zeta_\nu^m} \sum_{k=0}^{m} (-1)^k \binom{m}{k} f\left(a + \frac{m-2k}{2} h\zeta_\nu\right) \right\}.$$

In the simplest case when $n = 2$, we have

$$\zeta_{1,2} = \frac{1}{4}\left(\pm\sqrt{3} + i\right), \qquad \sigma_{1,2} = \frac{\pi}{2}\left(1 \pm i\frac{\sqrt{3}}{3}\right).$$

Then, the corresponding differentiation formula (6.2.4) reduces to

$$D_{2,h}^m f(a) = \frac{2}{\pi h^m} \mathrm{Re}\left\{ \frac{\sigma_1}{\zeta_1^m} \sum_{k=0}^{m} (-1)^k \binom{m}{k} f\left(a + \frac{m-2k}{2} h\zeta_1\right) \right\}.$$

Its error is $O(h^4)$.

The formula (6.2.3) for real-valued analytic functions can be improved with a little change. Namely, if we put $he^{i\alpha}$ instead of h in (6.2.2), where α is an arbitrary real parameter, and applying again Gauss-Christoffel formula (6.1.1), we obtain the following differentiation formula

$$f^{(m)}(a) \approx D_{n,h,\alpha}^m f(a) = \frac{1}{\pi} \sum_{\nu=1}^{n} \sigma_\nu \delta_{he^{i\alpha}\zeta_\nu}^m f(a).$$

Similar to the above investigation we find an expression for the error, depending on the real parameter α

$$R_{n,h,\alpha}^m f(a) = f^{(m)}(a) - D_{n,h,\alpha}^m f(a) = \frac{1}{\pi} \sum_{p=n}^{\infty} \frac{f^{(m+2p)}(a)}{(m+2p)!} S_{m+2p}^{(m)} R_n(z^{2p}) e^{i2p\alpha} h^{2p}.$$

Since the derivative $f^{(m)}(a)$ is real for real a and real-valued functions the parameter α can be chosen such that the dominant error term in the last expressions be purely imaginary. Then, for such functions, the dominant error term in $R_{n,h,\alpha}^m f(a)$, i.e.,

$$\frac{1}{\pi(m+2n)!} f^{(m+2n)}(a) S_{m+2n}^{(m)} R_n(z^{2n}) e^{i2n\alpha} h^{2n},$$

becomes purely imaginary. This can be achieved for $\alpha = \pi/4n$. In that case, the dominant error term for real-valued functions becomes the real part of the term in $R_{n,h,\alpha}^m f(a)$ for $p = n+1$. So we have the following result:

Theorem 6.2.2. *The dominant error term of the differentiation formula*

$$f^{(m)}(a) \approx \mathrm{Re}\left\{ D^m_{n,h,\pi/4n} f(a) \right\}, \qquad a \in \mathbb{R},$$

for real-valued analytic functions is given by

$$-\frac{\sin(\pi/2n)}{\pi(m+2n+2)!} S^{(m)}_{m+2n+2} R_n(z^{2n+2}) f^{(m+2n+2)}(a) h^{2n+2}, \tag{6.2.5}$$

where $R_n(g)$ is defined in (6.1.1).

Remark 6.2.1. With $\alpha = 3\pi/4n$ we also obtain a rule of degree precision $2n+2$. Then, in the dominant error term (6.2.5), the factor $-\sin(\pi/2n)$ should be replaced by $\sin(3\pi/2n)$.

Several numerical experiments were done in [19], [48], and [92–93].

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