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Specific values of partial Bell polynomials and series expansions for real powers of functions and for composite functions

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Abstract. Starting from Maclaurin's series expansions for positive integer powers of analytic functions, the authors derive an explicit formula for specific values of partial Bell polynomials, present a general term of Maclaurin's series expansions for real powers of analytic functions, obtain Maclaurin's series expansions of some composite functions, recover Maclaurin's series expansions for real powers of inverse sine function and sinc function, recover a combinatorial identity involving the falling factorials and the Stirling numbers of the second kind, deduce an explicit formula of the Bernoulli numbers of the second kind in terms of the Stirling numbers of the first kind, recover an explicit formula of the Bernoulli numbers in terms of the Stirling numbers of the second kind, recover an explicit formula of the Bell numbers in terms of the Stirling numbers of the second kind, reformulate three specific partial Bell polynomials in terms of central factorial numbers of the second kind, and present some Maclaurin's series expansions and identities related to the Euler numbers and their generating function.

1. Motivations

Let the function $f(z)$ be analytic at $z = 0$ and let

$$f(z) = \sum_{n=0}^{\infty} C_{1,n} \frac{z^n}{n!} \tag{1.1}$$

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be Maclaurin's series expansion of $f(z)$ around $z = 0$. For fixed positive integer $j \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$, let

$$f^j(z) = \sum_{n=0}^{\infty} C_{j,n} \frac{z^n}{n!} \tag{1.2}$$

be Maclaurin's series expansion of $f^j(z)$ at $z = 0$ with assumptions of

$$C_{0,0} = 1 \quad \text{and} \quad C_{0,n} = 0 \tag{1.3}$$

for $n \in \mathbb{N}$. For $\alpha \in \mathbb{R}$, if $f^\alpha(z)$ is analytic at $z = 0$, let

$$f^\alpha(z) = \sum_{n=0}^{\infty} C_{\alpha,n} \frac{z^n}{n!} \tag{1.4}$$

be Maclaurin's series expansion of $f^\alpha(z)$ around $z = 0$ with assumptions in (1.3). By common knowledge in mathematical analysis, we know that

$$C_{1,n} = \lim_{z\to0} \frac{\mathrm{d}\, f(z)}{\mathrm{d}\, z}, \quad C_{j,n} = \lim_{z\to0} \frac{\mathrm{d}^n\, f^j(z)}{\mathrm{d}\, z^n}, \quad C_{\alpha,n} = \lim_{z\to0} \frac{\mathrm{d}^n\, f^\alpha(z)}{\mathrm{d}\, z^n}.$$

Can one obtain an explicit formula for $C_{j,n}$ in terms of $C_{1,n}$ in Maclaurin's series expansion (1.1)? Can one obtain an explicit formula for $C_{\alpha,n}$ in terms of $C_{j,n}$?

These two problems have been specifically, explicitly, or recursively solved in the papers [5, 6, 8, 13, 16, 19] and many closely related references therein.

The first problem has been generally and recursively solved by the power series raised to powers

$$\left(\sum_{k=0}^{\infty} a_k x^k\right)^n = \sum_{k=0}^{\infty} c_k x^k,$$

where $c_0 = a_0^n$ and

$$c_m = \frac{1}{m a_0} \sum_{k=1}^{m} (kn - m + k) a_k c_{m-k}$$

for $m, n \in \mathbb{N}$. See [4, p. 18].

In this paper, we will give a general solution to the second problem. In other words, we will present a general formula for $C_{\alpha,n}$ in terms of the sequence $C_{j,n}$ by deriving an explicit formula for specific values

$$B_{n,k}\Big(f'(0), f''(0), \dots, f^{(n-k+1)}(0)\Big),$$

where $B_{n,k}(x_1, x_2, \dots, x_{n-k+1})$ for $n \ge k \ge 0$ denotes partial Bell polynomials or the Bell polynomials of the second kind in [2, Definition 11.2] and [3, p. 134, Theorem A]. Hereafter, we will obtain Maclaurin's series expansions of some composite functions, recover Maclaurin's series expansions for real powers of inverse sine function and sinc function, recover a combinatorial identity involving the falling factorials and the Stirling numbers of the second kind, deduce an explicit formula of the Bernoulli numbers of the second kind in terms of the Stirling numbers of the first kind, recover an explicit formula of the Bernoulli numbers in terms of the Stirling numbers of the second kind, recover an explicit formula of the Bell numbers in terms of the Stirling numbers of the second kind, reformulate three specific partial Bell polynomials in terms of central factorial numbers of the second kind, and present some Maclaurin's series expansions and identities related to the Euler numbers and their generating function.

2. Specific values of partial Bell polynomials

As the first step to reach our aim, we derive an explicit formula for specific values

$$B_{n,k}\Big(f'(0), f''(0), \dots, f^{(n-k+1)}(0)\Big)$$

of partial Bell polynomials $B_{n,k}(x_1, x_2, \dots, x_{n-k+1})$ with $x_k = f^{(k)}(0)$ for $k \in \mathbb{N}$.

Theorem 2.1. *Let $f(z)$ is analytic at $z = 0$. For $j \in \mathbb{N}_0$, if the series expansion* (1.2) *is valid, then partial Bell polynomials $B_{n,k}(x_1, x_2, \dots, x_{n-k+1})$ for $n \ge k \ge 0$ satisfy*

$$B_{n,k}\Big(f'(0), f''(0), \dots, f^{(n-k+1)}(0)\Big) = \frac{(-1)^k}{k!} \sum_{q=0}^{k} (-1)^q \binom{k}{q} f^{k-q}(0) C_{q,n} \tag{2.1}$$

and the sequence $C_{j,n}$ satisfies the identity

$$\sum_{q=0}^{k} (-1)^q \binom{k}{q} f^{k-q}(0) C_{q,n} = 0, \quad 0 \le n < k. \tag{2.2}$$

Proof. In the last line of [3, p. 133], there exists the formula

$$\frac{1}{k!}\Bigg(\sum_{m=1}^{\infty} x_m \frac{t^m}{m!}\Bigg)^k = \sum_{n=k}^{\infty} B_{n,k}(x_1, x_2, \dots, x_{n-k+1}) \frac{t^n}{n!}$$

for $k \ge 0$. This means that

$$\Bigg(\frac{1}{t}\sum_{m=1}^{\infty} x_m \frac{t^m}{m!}\Bigg)^k = \sum_{n=0}^{\infty} \frac{B_{n+k,k}(x_1, x_2, \dots, x_{n+1})}{\binom{n+k}{k}} \frac{t^n}{n!}$$

for $k \ge 0$. Therefore, we arrive at

$$\frac{B_{n+k,k}(x_1, x_2, \dots, x_{n+1})}{\binom{n+k}{k}} = \lim_{t \to 0} \frac{\mathrm{d}^n}{\mathrm{d}\, t^n} \Bigg(\frac{1}{t}\sum_{m=1}^{\infty} x_m \frac{t^m}{m!}\Bigg)^k, \quad k, n \ge 0. \tag{2.3}$$

Substituting specific values $x_m = f^{(m)}(0)$ for $m = 1, 2, \dots$ into the formula (2.3) yields

$$\begin{aligned}
\frac{B_{n+k,k}\Big(f'(0), f''(0), \dots, f^{(n+1)}(0)\Big)}{\binom{n+k}{k}} &= \lim_{t \to 0} \frac{\mathrm{d}^n}{\mathrm{d}\, t^n} \Bigg[\frac{1}{t} \sum_{m=1}^{\infty} f^{(m)}(0) \frac{t^m}{m!}\Bigg]^k \\
&= (-1)^k \lim_{t \to 0} \frac{\mathrm{d}^n}{\mathrm{d}\, t^n} \Bigg[\frac{1}{t}\Bigg(f(0) - \sum_{m=0}^{\infty} f^{(m)}(0) \frac{t^m}{m!}\Bigg)\Bigg]^k \\
&= \lim_{t \to 0} \frac{\mathrm{d}^n}{\mathrm{d}\, t^n} \bigg[\frac{f(t) - f(0)}{t}\bigg]^k \\
&= \lim_{t \to 0} \frac{\mathrm{d}^n}{\mathrm{d}\, t^n} \frac{\sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} f^{k-j}(0) f^j(t)}{t^k}
\end{aligned}$$

for $k, n \ge 0$. Further considering the series (1.2) leads to

$$\frac{B_{n+k,k}\Big(f'(0), f''(0), \dots, f^{(n+1)}(0)\Big)}{\binom{n+k}{k}} = \lim_{t \to 0} \frac{\mathrm{d}^n}{\mathrm{d}\, t^n} \frac{\sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} f^{k-j}(0) \sum_{\ell=0}^{\infty} C_{j,\ell} \frac{t^\ell}{\ell!}}{t^k}$$

$$
\begin{aligned}
&= \lim_{t\to 0} \frac{\mathrm{d}^n}{\mathrm{d}\, t^n} \frac{\sum_{\ell=0}^{\infty} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} f^{k-j}(0) C_{j,\ell} \frac{t^\ell}{\ell!}}{t^k} \\
&= \lim_{t\to 0} \frac{\mathrm{d}^n}{\mathrm{d}\, t^n} \sum_{\ell=0}^{\infty} \frac{\sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} f^{k-j}(0) C_{j,\ell}}{\ell!} t^{\ell-k}.
\end{aligned}
$$

This must imply that

$$
\sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} f^{k-j}(0) C_{j,\ell} = 0, \quad 0 \le \ell < k
$$

and

$$
\begin{aligned}
\frac{B_{n+k,k}\big(f'(0), f''(0), \dots, f^{(n+1)}(0)\big)}{\binom{n+k}{k}} &= \lim_{t\to 0} \frac{\mathrm{d}^n}{\mathrm{d}\, t^n} \sum_{\ell=k}^{\infty} \frac{\sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} f^{k-j}(0) C_{j,\ell}}{\ell!} t^{\ell-k} \\
&= \lim_{t\to 0} \frac{\mathrm{d}^n}{\mathrm{d}\, t^n} \sum_{\ell=0}^{\infty} \frac{\sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} f^{k-j}(0) C_{j,\ell+k}}{(\ell+k)!} t^{\ell} \\
&= \lim_{t\to 0} \sum_{\ell=n}^{\infty} \frac{\sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} f^{k-j}(0) C_{j,\ell+k}}{(\ell+k)!} \langle \ell \rangle_n t^{\ell-n} \\
&= \frac{n!}{(n+k)!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} f^{k-j}(0) C_{j,n+k}
\end{aligned}
$$

for $k, n \ge 0$, where the falling factorial of a complex number $\lambda \in \mathbb{C}$ is defined by

$$
\langle \lambda \rangle_m = \prod_{k=0}^{m-1} (\lambda - k) = \begin{cases} 1, & m = 0; \\ \lambda(\lambda-1)\cdots(\lambda-m+1), & m \in \mathbb{N}. \end{cases}
$$

Accordingly, we derive

$$
B_{n+k,k}\big(f'(0), f''(0), \dots, f^{(n+1)}(0)\big) = \frac{1}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} f^{k-j}(0) C_{j,n+k} = \sum_{j=0}^{k} (-1)^{k-j} \frac{f^{k-j}(0)}{(k-j)!} \frac{C_{j,n+k}}{j!}
$$

for $k, n \ge 0$. Replacing $n+k$ by n results in

$$
\begin{aligned}
B_{n,k}\big(f'(0), f''(0), \dots, f^{(n-k+1)}(0)\big) &= \sum_{j=0}^{k} (-1)^{k-j} \frac{f^{k-j}(0)}{(k-j)!} \frac{C_{j,n}}{j!} \\
&= \sum_{m=0}^{k} (-1)^m \frac{f^m(0)}{m!} \frac{C_{k-m,n}}{(k-m)!} = \frac{(-1)^k}{k!} \sum_{q=0}^{k} (-1)^q \binom{k}{q} f^{k-q}(0) C_{q,n}
\end{aligned}
$$

for $n \ge k \ge 0$. The required results in Theorem 2.1 are thus proved. □

Example 2.1. *In the papers [5, 6], the following conclusions were proved.*

1. *For $m \in \mathbb{N}$ and $|t| < 1$, the function $\left(\frac{\arcsin t}{t}\right)^m$, whose value at $t = 0$ is regarded as 1, has Maclaurin's series expansion*

$$
\left(\frac{\arcsin t}{t}\right)^m = 1 + \sum_{k=1}^{\infty} (-1)^k \frac{Q(m, 2k; 2)}{\binom{m+2k}{m}} \frac{(2t)^{2k}}{(2k)!}, \tag{2.4}
$$

where

$$Q(m,k;\alpha)=\sum_{\ell=0}^{k}\binom{m+\ell-1}{m-1}s(m+k-1,m+\ell-1)\biggl(\frac{m+k-\alpha}{2}\biggr)^{\ell} \tag{2.5}$$

for $m,k\in\mathbb{N}$*, the constant* $\alpha\in\mathbb{R}$ *such that* $m+k\ne\alpha$*, and the Stirling numbers of the first kind* $s(m+k-1,m+\ell-1)$ *are analytically generalized by*

$$\frac{[\ln(1+x)]^k}{k!}=\sum_{n=k}^{\infty}s(n,k)\frac{x^n}{n!},\quad |x|<1. \tag{2.6}$$

Maclaurin's series expansion (2.4) *was also recovered in [16, Section 6]. See also [1, Section 1.2].*

2. *For* $k,n\ge0$ *and* $x_m\in\mathbb{C}$ *with* $m\in\mathbb{N}$*, we have*

$$B_{2n+1,k}\biggl(0,x_2,0,x_4,\dots,\frac{1+(-1)^k}{2}x_{2n-k+2}\biggr)=0. \tag{2.7}$$

For $k,n\in\mathbb{N}$ *such that* $2n\ge k\in\mathbb{N}$*, we have*

$$B_{2n,k}\biggl(0,\frac{1}{3},0,\frac{9}{5},0,\frac{225}{7},\dots,\frac{1+(-1)^{k+1}}{2}\frac{[(2n-k)!!]^2}{2n-k+2}\biggr)\\
=(-1)^{n+k}\frac{(4n)!!}{(2n+k)!}\sum_{k=1}^{k}(-1)^k\binom{2n+k}{k-k}Q(k,2n;2), \tag{2.8}$$

where $Q(k,2n;2)$ *is given by* (2.5).

For $t\in(-1,1)$*, let*

$$f(t)=\begin{cases}1, & t=0;\\ \dfrac{\arcsin t}{t}, & t\ne0.\end{cases} \tag{2.9}$$

Since

$$\frac{\arcsin t}{t}=\sum_{\ell=0}^{\infty}\frac{[(2\ell-1)!!]^2}{2\ell+1}\frac{t^{2\ell}}{(2\ell)!}=1+\frac{1}{3}\frac{t^2}{2!}+\frac{9}{5}\frac{t^4}{4!}+\frac{225}{7}\frac{t^6}{6!}+1225\frac{t^8}{8!}+\cdots,$$

we see that

$$f^{(m)}(0)=\begin{cases}0, & m=2\ell+1;\\ \dfrac{[(2\ell-1)!!]^2}{2\ell+1}, & m=2\ell,\end{cases}$$

where $\ell\in\mathbb{N}_0$ *and* $(-1)!!=1$*. On the other hand, the series expansion* (2.4) *implies that*

$$C_{m,n}=\begin{cases}0, & n=2k-1\\ (-1)^k2^{2k}\dfrac{Q(m,2k;2)}{\binom{m+2k}{m}}, & n=2k\end{cases} \tag{2.10}$$

for $k\in\mathbb{N}$*. Applying the formula* (2.1) *in Theorem 2.1 gives*

$$B_{2\ell-1,k}\biggl(0,\frac{1}{3},0,\frac{9}{5},0,\frac{225}{7},0,1225,\dots\biggr)=\sum_{m=0}^{k}(-1)^m\frac{f^m(0)}{m!}\frac{C_{k-m,2\ell-1}}{(k-m)!}=0$$

for $2\ell-1\ge k\ge 0$ *and*

$$\begin{aligned}
B_{2\ell,k}\biggl(0,\frac{1}{3},0,\frac{9}{5},0,\frac{225}{7},0,1225,\dotsc\biggr)&=\sum_{m=0}^{k}(-1)^m\frac{f^m(0)}{m!}\frac{C_{k-m,2\ell}}{(k-m)!}\\
&=\sum_{m=0}^{k}\frac{(-1)^m}{m!(k-m)!}(-1)^\ell 2^{2\ell}\frac{Q(k-m,2\ell;2)}{\binom{k-m+2\ell}{k-m}}\\
&=(-1)^\ell 2^{2\ell}\sum_{m=0}^{k}\frac{(-1)^{k-m}}{m!(k-m)!}\frac{Q(m,2\ell;2)}{\binom{m+2\ell}{m}}\\
&=(-1)^{\ell+k}2^{2\ell}\frac{(2\ell)!}{(2\ell+k)!}\sum_{m=0}^{k}(-1)^m\binom{2\ell+k}{k-m}Q(m,2\ell;2)\\
&=(-1)^{\ell+k}\frac{(4\ell)!!}{(2\ell+k)!}\sum_{m=0}^{k}(-1)^m\binom{2\ell+k}{k-m}Q(m,2\ell;2)
\end{aligned}$$

for $k,\ell\in\mathbb{N}$ *with* $2\ell\ge k$. *These results coincide with* (2.7) *and* (2.8).

Example 2.2. *For* $z\in\mathbb{C}$, *the function*

$$\operatorname{sinc} z=\begin{cases}\dfrac{\sin z}{z}, & z\ne 0\\ 1, & z=0\end{cases}$$

is called the sinc function [23]. In [19, Theorem 2.1], it was obtained that

$$\operatorname{sinc}^\ell z=1+\sum_{j=1}^{\infty}(-1)^j\frac{T(\ell+2j,\ell)}{\binom{\ell+2j}{\ell}}\frac{(2z)^{2j}}{(2j)!} \tag{2.11}$$

for $\ell\in\mathbb{N}_0$ *and* $z\in\mathbb{C}$, *where* $T(n,\ell)$ *for* $n\ge\ell\in\mathbb{N}_0$ *denotes central factorial numbers of the second kind, which can be explicitly computed [10] by*

$$T(n,\ell)=\frac{1}{\ell!}\sum_{j=0}^{\ell}(-1)^j\binom{\ell}{j}\biggl(\frac{\ell}{2}-j\biggr)^n \tag{2.12}$$

with $T(0,0)=1$ *and* $T(n,0)=0$ *for* $n\in\mathbb{N}$. *Applying* $f(z)=\operatorname{sinc} z$ *to* (1.2) *and considering* (2.11) *acquire that*

$$C_{\ell,n}=\begin{cases}0, & n=2j-1\\ (-1)^j2^{2j}\dfrac{T(\ell+2j,\ell)}{\binom{\ell+2j}{\ell}}, & n=2j\end{cases} \tag{2.13}$$

for $j\in\mathbb{N}$. *From*

$$\operatorname{sinc} z=\sum_{j=0}^{\infty}\frac{(-1)^j}{2j+1}\frac{z^{2j}}{(2j)!},\quad z\in\mathbb{C},$$

it follows that

$$(\operatorname{sinc} z)^{(2j)}\big|_{z=0}=\frac{(-1)^j}{2j+1}\quad\text{and}\quad(\operatorname{sinc} z)^{(2j-1)}\big|_{z=0}=0,\quad j\in\mathbb{N}. \tag{2.14}$$

Employing (2.13) *in* (2.2) *and simplifying lead to*

$$\sum_{m=0}^{k}(-1)^m\frac{T(2n+k-m,k-m)}{(2n+k-m)!m!}=0,\quad 0\le 2n<k\in\mathbb{N}.$$

Letting $f(z)=\operatorname{sinc} z$ *in* (2.1) *and utilizing the values in* (2.13) *and* (2.14) *give*

$$B_{2n-1,k}\left(0,-\frac{1}{3},0,\frac{1}{5},\dots,\frac{(-1)^{2n-k-1}}{2n-k+1}\sin\frac{(2n-k-1)\pi}{2}\right)=0,\quad 2n-1\ge k\ge 0$$

and

$$B_{2n,k}\left(0,-\frac{1}{3},0,\frac{1}{5},\dots,\frac{(-1)^{2n-k}}{2n-k+2}\sin\frac{(2n-k)\pi}{2}\right)=(-1)^{n+k}(4n)!!\sum_{j=0}^{k}(-1)^j\frac{T(2n+j,j)}{(2n+j)!(k-j)!},\quad 2n\ge k\ge 0.$$

These two conclusions recover [19, Theorem 3.1].

3. Maclaurin's series expansions of real powers of functions

As the second step to reach our aim, by virtue of the explicit formula (2.1) in Theorem 2.1, we will present a general formula for $C_{\alpha,n}$ in terms of the sequence $C_{j,n}$. In other words, we give an explicit expression of the series expansion (1.4) in terms of the sequence $C_{j,n}$.

Theorem 3.1. *For* $j\in\mathbb{N}_0$ *and* $\alpha\in\mathbb{R}$, *if the series expansion* (1.2) *is valid, then*

$$C_{\alpha,n}=\sum_{k=0}^{n}\frac{(-\alpha)_k}{k!}\sum_{q=0}^{k}(-1)^q\binom{k}{q}f^{\alpha-q}(0)C_{q,n},\tag{3.1}$$

that is,

$$f^{\alpha}(z)=\sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}\frac{(-\alpha)_k}{k!}\sum_{q=0}^{k}(-1)^q\binom{k}{q}f^{\alpha-q}(0)C_{q,n}\right]\frac{z^n}{n!},\tag{3.2}$$

where

$$(\lambda)_m=\prod_{j=0}^{m-1}(\lambda+j)=\begin{cases}1, & m=0\\ \lambda(\lambda+1)\cdots(\lambda+m-1), & m\in\mathbb{N}\end{cases}\tag{3.3}$$

denotes the rising factorial of a complex number $\lambda\in\mathbb{C}$.

Proof. For $n\in\mathbb{N}_0$, the Faà di Bruno formula, see [2, Theorem 11.4] and [3, p. 139, Theorem C], can be described in terms of $B_{n,k}(x_1,x_2,\dots,x_{n-k+1})$ by

$$\frac{\mathrm{d}^n}{\mathrm{d}\,z^n}F\circ f(z)=\sum_{k=0}^{n}F^{(k)}(f(z))B_{n,k}\left(f'(z),f''(z),\dots,f^{(n-k+1)}(z)\right).\tag{3.4}$$

Applying (3.4) to $F(u)=u^{\alpha}$ and $u=f(z)$ yields

$$\frac{\mathrm{d}^n}{\mathrm{d}\,z^n}f^{\alpha}(z)=\sum_{k=0}^{n}F^{(k)}(u)B_{n,k}\left(f'(z),f''(z),\dots,f^{(n-k+1)}(z)\right)$$

$$
\begin{aligned}
&= \sum_{k=0}^{n} \langle\alpha\rangle_k u^{\alpha-k} B_{n,k}\Bigl(f'(z), f''(z), \dots, f^{(n-k+1)}(z)\Bigr)\\
&= \sum_{k=0}^{n} \langle\alpha\rangle_k f^{\alpha-k}(z) B_{n,k}\Bigl(f'(z), f''(z), \dots, f^{(n-k+1)}(z)\Bigr)\\
&\to \sum_{k=0}^{n} \langle\alpha\rangle_k f^{\alpha-k}(0) B_{n,k}\Bigl(f'(0), f''(0), \dots, f^{(n-k+1)}(0)\Bigr)
\end{aligned}
$$

as $z \to 0$ for $n \in \mathbb{N}$. Employing the formula (2.1) in Theorem 2.1 results in

$$
\begin{aligned}
C_{\alpha,n} &= \lim_{z\to0} \frac{\mathrm{d}^n}{\mathrm{d}\, z^n} f^{\alpha}(z)\\
&= \sum_{k=0}^{n} \langle\alpha\rangle_k f^{\alpha-k}(0) B_{n,k}\Bigl(f'(0), f''(0), \dots, f^{(n-k+1)}(0)\Bigr)\\
&= \sum_{k=0}^{n} \langle\alpha\rangle_k f^{\alpha-k}(0) \sum_{m=0}^{k} (-1)^m \frac{f^m(0)}{m!} \frac{C_{k-m,n}}{(k-m)!}\\
&= \sum_{k=0}^{n} \langle\alpha\rangle_k \sum_{m=0}^{k} (-1)^m \frac{f^{\alpha-k+m}(0)}{m!} \frac{C_{k-m,n}}{(k-m)!}\\
&= \sum_{k=0}^{n} \frac{(-\alpha)_k}{k!} \sum_{q=0}^{k} (-1)^q \binom{k}{q} f^{\alpha-q}(0) C_{q,n}.
\end{aligned}
$$

The proof of Theorem 3.1 is complete. □

Example 3.1. *Maclaurin's series expansion* (2.4) *was generalized in [13, Section 4] as*

$$
\biggl(\frac{\arcsin t}{t}\biggr)^{\alpha} = 1 + \sum_{n=1}^{\infty} (-1)^n \Biggl[\sum_{k=1}^{2n} \frac{(-\alpha)_k}{(2n+k)!} \sum_{q=1}^{k} (-1)^q \binom{2n+k}{k-q} Q(q,2n;2)\Biggr] (2t)^{2n} \tag{3.5}
$$

for $\alpha \in \mathbb{R}$ and $|t| < 1$ by rediscovering a special case of (2.7) *and the closed-form formula* (2.8)*, where $Q(k,2n;2)$ is given by* (2.5)*.*

In the formula (3.1)*, taking $f(z)$ as the one in* (2.9) *and using the expression* (2.10) *arrive at*

$$
C_{\alpha,2n-1} = \sum_{k=0}^{2n-1} \langle\alpha\rangle_k \sum_{m=0}^{k} \frac{(-1)^m}{m!(k-m)!} C_{k-m,2n-1} = 0
$$

and

$$
\begin{aligned}
C_{\alpha,2n} &= \sum_{k=0}^{2n} \langle\alpha\rangle_k \sum_{m=0}^{k} \frac{(-1)^m}{m!(k-m)!} C_{k-m,2n}\\
&= \sum_{k=1}^{2n} \langle\alpha\rangle_k \sum_{m=0}^{k-1} \frac{(-1)^m}{m!(k-m)!} C_{k-m,2n}\\
&= \sum_{k=1}^{2n} \langle\alpha\rangle_k \sum_{m=0}^{k-1} \frac{(-1)^m}{m!(k-m)!} (-1)^n 2^{2n} \frac{Q(k-m,2n;2)}{\binom{k-m+2n}{k-m}}\\
&= \sum_{k=1}^{2n} \langle\alpha\rangle_k \sum_{\ell=1}^{k} \frac{(-1)^{k-\ell}}{(k-\ell)!\ell!} (-1)^n 2^{2n} \frac{Q(\ell,2n;2)}{\binom{\ell+2n}{\ell}}
\end{aligned}
$$

$$= (-1)^n \sum_{k=1}^{2n} \frac{(-\alpha)_k}{(2n+k)!} \sum_{\ell=1}^{k} (-1)^\ell \binom{2n+k}{k-\ell} \frac{2^{2n}(2n)!}{(\ell+2n)!} Q(\ell, 2n; 2)$$

for $n \in \mathbb{N}$, *where we used the relation* $(-1)^k \langle \alpha \rangle_k = (\alpha)_k$. *These results coincide with Maclaurin's series expansion* (3.5).

Example 3.2. *In Maclaurin's series expansion* (3.2), *taking* $f(z) = \operatorname{sinc} z$ *and applying the formula* (2.13) *yield*

$$\operatorname{sinc}^\alpha z = \sum_{n=0}^{\infty} (-1)^n 2^{2n} \left[\sum_{k=0}^{2n} (-\alpha)_k \sum_{j=0}^{k} (-1)^j \frac{T(2n+j, j)}{(2n+j)!(k-j)!} \right] z^{2n},$$

where $(-\alpha)_k$ *is defined by* (3.3). *This conclusion recovers the first Maclaurin's series expansion in [19, Theorem 4.1].*

Example 3.3. *Taking* $f(z) = \mathrm{e}^z$ *gives*

$$\mathrm{e}^{kz} = (\mathrm{e}^z)^k = \sum_{n=0}^{\infty} k^n \frac{z^n}{n!}, \quad k \in \mathbb{N}.$$

This means that

$$C_{k,n} = k^n, \quad k \in \mathbb{N}, \quad n \in \mathbb{N}_0. \tag{3.6}$$

Substituting this into (3.2) *in Theorem 3.1 arrives at*

$$\mathrm{e}^{\alpha z} = \sum_{n=0}^{\infty} \left[\sum_{k=0}^{n} \langle \alpha \rangle_k \sum_{m=0}^{k} \frac{(-1)^m}{m!} \frac{(k-m)^n}{(k-m)!} \right] \frac{z^n}{n!}$$

for $\alpha \in \mathbb{R}$. *Comparing this with*

$$\mathrm{e}^{\alpha z} = \sum_{n=0}^{\infty} \alpha^n \frac{z^n}{n!}$$

results in an identity

$$\sum_{k=0}^{n} \frac{\langle \alpha \rangle_k}{k!} \sum_{m=0}^{k} (-1)^m \binom{k}{m} (k-m)^n = \sum_{k=0}^{n} \langle \alpha \rangle_k \frac{(-1)^k}{k!} \sum_{\ell=0}^{k} (-1)^\ell \binom{k}{\ell} \ell^n = \alpha^n, \tag{3.7}$$

where $\alpha \in \mathbb{R}$ *and* $n \in \mathbb{N}_0$. *Since the Stirling numbers of the second kind* $S(n,k)$ *can be analytically computed by*

$$S(n,k) = \begin{cases} \dfrac{(-1)^k}{k!} \displaystyle\sum_{\ell=0}^{k} (-1)^\ell \binom{k}{\ell} \ell^n, & n > k \in \mathbb{N}_0; \\ 1, & n = k \in \mathbb{N}_0, \end{cases} \tag{3.8}$$

the identity (3.7) *can be written as*

$$\sum_{k=0}^{n} S(n,k) \langle \alpha \rangle_k = \alpha^n, \quad \alpha \in \mathbb{R}, \quad n \in \mathbb{N}_0.$$

This is a recovery of the equation (1.27) on page 19 in the monograph [24]. See also Remark 3.2 in the paper [9].

Example 3.4. *The equation* (2.6) *can be rearranged as Maclaurin's series expansions of the power function*

$$\left[\frac{\ln(1+x)}{x}\right]^k = \sum_{n=0}^{\infty} \frac{s(k+n,k)}{\binom{k+n}{k}} \frac{x^n}{n!}$$

for $|x| < 1$ and $k \ge 0$. Meanwhile, the Stirling numbers of the second kind $S(n,k)$ for $n \ge k \ge 0$ can be generated [22, pp. 131–132] by

$$\frac{(\mathrm{e}^x - 1)^k}{k!} = \sum_{n=k}^{\infty} S(n,k) \frac{x^n}{n!}. \tag{3.9}$$

The equation (3.9) *can be rearranged as Maclaurin's series expansions of the power function*

$$\left(\frac{\mathrm{e}^x - 1}{x}\right)^k = \sum_{n=0}^{\infty} \frac{S(k+n,k)}{\binom{k+n}{k}} \frac{x^n}{n!}, \quad k \ge 0.$$

See also [16, Section 2]. Applying the result (3.2) *in Theorem 3.1 yields*

$$\left[\frac{\ln(1+x)}{x}\right]^\alpha = \sum_{n=0}^{\infty} \left[\sum_{k=0}^{n} \frac{(-\alpha)_k}{k!} \sum_{\ell=0}^{k} (-1)^\ell \binom{k}{\ell} \frac{s(n+\ell,\ell)}{\binom{n+\ell}{\ell}}\right] \frac{z^n}{n!} \tag{3.10}$$

and

$$\left(\frac{\mathrm{e}^x - 1}{x}\right)^\alpha = \sum_{n=0}^{\infty} \left[\sum_{k=0}^{n} \frac{(-\alpha)_k}{k!} \sum_{\ell=0}^{k} (-1)^\ell \binom{k}{\ell} \frac{S(n+\ell,\ell)}{\binom{n+\ell}{\ell}}\right] \frac{z^n}{n!}, \quad \alpha \in \mathbb{R}. \tag{3.11}$$

Taking $\alpha = -1$ in (3.10) *gives*

$$\frac{x}{\ln(1+x)} = \sum_{n=0}^{\infty} \left[\sum_{k=0}^{n} \sum_{\ell=0}^{k} (-1)^\ell \binom{k}{\ell} \frac{s(n+\ell,\ell)}{\binom{n+\ell}{\ell}}\right] \frac{z^n}{n!}.$$

Comparing this equation with the generating function

$$\frac{x}{\ln(1+x)} = \sum_{n=0}^{\infty} b_n x^n$$

of the Bernoulli numbers of the second kind b_n results in

$$b_n = \frac{1}{n!} \sum_{k=0}^{n} \sum_{\ell=0}^{k} (-1)^\ell \binom{k}{\ell} \frac{s(n+\ell,\ell)}{\binom{n+\ell}{\ell}} \tag{3.12}$$

for $n \ge 0$. The formula (3.12) *recovers the first result in [21, Theorem 3].*

Interchanging the order of double sums in (3.12) *shows that*

$$b_n = \frac{1}{n!} \sum_{\ell=0}^{n} (-1)^\ell \left[\sum_{k=\ell}^{n} \binom{k}{\ell}\right] \frac{s(n+\ell,\ell)}{\binom{n+\ell}{\ell}} = \frac{1}{n!} \sum_{\ell=0}^{n} (-1)^\ell \binom{n+1}{\ell+1} \frac{s(n+\ell,\ell)}{\binom{n+\ell}{\ell}}. \tag{3.13}$$

The last formula in (3.13) *is not appeared in the papers [11, 12, 14, 17, 20, 21].*

Setting $\alpha = -1$ in (3.11) *and interchanging the order of double sums lead to*

$$\frac{x}{\mathrm{e}^x - 1} = \sum_{n=0}^{\infty} \left[\sum_{k=0}^{n} \sum_{\ell=0}^{k} (-1)^\ell \binom{k}{\ell} \frac{S(n+\ell,\ell)}{\binom{n+\ell}{\ell}}\right] \frac{z^n}{n!}$$

$$
\begin{aligned}
&= \sum_{n=0}^{\infty}\left\{\sum_{\ell=0}^{n}(-1)^{\ell}\left[\sum_{k=\ell}^{n}\binom{k}{\ell}\right]\frac{S(n+\ell,\ell)}{\binom{n+\ell}{\ell}}\right\}\frac{z^n}{n!}\\
&= \sum_{n=0}^{\infty}\left[\sum_{\ell=0}^{n}(-1)^{\ell}\binom{n+1}{\ell+1}\frac{S(n+\ell,\ell)}{\binom{n+\ell}{\ell}}\right]\frac{z^n}{n!}.
\end{aligned}
$$

Comparing this equation with the generating function

$$
\frac{z}{\mathrm{e}^z-1}=\sum_{n=0}^{\infty}B_n\frac{z^n}{n!}=1-\frac{z}{2}+\sum_{n=1}^{\infty}B_{2n}\frac{z^{2n}}{(2n)!},\quad |z|<2\pi
$$

of the classical Bernoulli numbers B_n *reveals*

$$
B_{2n}=\sum_{\ell=0}^{2n}(-1)^{\ell}\binom{2n+1}{\ell+1}\frac{S(2n+\ell,\ell)}{\binom{2n+\ell}{\ell}} \tag{3.14}
$$

and

$$
\sum_{\ell=0}^{2n+1}(-1)^{\ell}\binom{2n+2}{\ell+1}\frac{S(2n+\ell+1,\ell)}{\binom{2n+\ell+1}{\ell}}=0 \tag{3.15}
$$

for $n\ge 1$*. The formulas* (3.14) *and* (3.15) *recover the last formula in [7, Theorem 1].*

4. Maclaurin's series expansions of composite functions

Under the assumption that the series expansion (1.2) is valid, we now derive a general expression for Maclaurin's series expansion of a composite function $F\circ f(z)$ around $z=0$. As an example, from this general expression, we will present some new results of the Bell numbers B_n which are generated by

$$
\mathrm{e}^{\mathrm{e}^{\pm z}}=\mathrm{e}\sum_{n=0}^{\infty}(\pm1)^nB_n\frac{z^n}{n!}. \tag{4.1}
$$

For more knowledge about the Bell numbers B_n, see [1, Section 2.1] and related references therein.

Theorem 4.1. *For* $j\in\mathbb{N}_0$ *and* $\alpha\in\mathbb{R}$*, if the series expansion* (1.2) *is valid and the composite function* $F\circ f$ *is defined, then*

$$
F\circ f(z)=\sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}(-1)^k\frac{F^{(k)}(f(0))}{k!}\sum_{q=0}^{k}(-1)^q\binom{k}{q}f^{k-q}(0)C_{q,n}\right]\frac{z^n}{n!}, \tag{4.2}
$$

where, if $f(0)=0$*, we regard* 0^0 *as* 1.

Proof. From the formulas (2.1) and (3.4), it follows that

$$
\begin{aligned}
\lim_{z\to0}\frac{\mathrm{d}^n}{\mathrm{d}\,z^n}F\circ f(z)&=\sum_{k=0}^{n}\lim_{z\to0}F^{(k)}(f(z))\lim_{z\to0}B_{n,k}\Big(f'(z),f''(z),\dots,f^{(n-k+1)}(z)\Big)\\
&=\sum_{k=0}^{n}F^{(k)}(f(0))B_{n,k}\Big(f'(0),f''(0),\dots,f^{(n-k+1)}(0)\Big)\\
&=\sum_{k=0}^{n}F^{(k)}(f(0))\sum_{m=0}^{k}(-1)^m\frac{f^m(0)}{m!}\frac{C_{k-m,n}}{(k-m)!}
\end{aligned}
$$

$$= \sum_{k=0}^{n}(-1)^k \frac{F^{(k)}(f(0))}{k!} \sum_{q=0}^{k}(-1)^q \binom{k}{q} f^{k-q}(0) C_{q,n}.$$

The proof of Theorem 4.1 is complete. □

Example 4.1. *We can regard the generating function* e^{e^z} *of the Bell numbers* B_n *as a composite of the functions* $f(z) = F(z) = e^z$. *Combining this with the formula* (3.6) *in Example 3.3 and applying the series* (4.2) *in Theorem 4.1 yield*

$$e^{e^z} = e + \sum_{n=1}^{\infty}\left[\sum_{k=0}^{n} e^{e^0} \sum_{m=0}^{k} \frac{(-1)^m}{m!} \frac{(k-m)^n}{(k-m)!}\right]\frac{z^n}{n!} = e + e\sum_{n=1}^{\infty}\left[\sum_{k=0}^{n} \sum_{m=0}^{k} \frac{(-1)^m}{m!} \frac{(k-m)^n}{(k-m)!}\right]\frac{z^n}{n!}.$$

Comparing this series with (4.1) *gives* $B_0 = 1$ *and*

$$B_n = \sum_{k=0}^{n} \sum_{m=0}^{k} \frac{(-1)^m}{m!} \frac{(k-m)^n}{(k-m)!} = \sum_{k=0}^{n} \frac{1}{k!} \sum_{m=0}^{k} (-1)^m \binom{k}{m} (k-m)^n = \sum_{k=0}^{n} S(n,k)$$

for $n \in \mathbb{N}$, *where we used the formula* (3.8). *Consequently, we recover an explicit formula for the Bell numbers* B_n *in terms of the Stirling numbers of the second kind* $S(n,k)$. *For new properties of the Bell numbers* B_n, *please refer to [1, Section 2.1] and the paper [15].*

Example 4.2. *We now consider Maclaurin's series expansion of the function* $\sin(\sin z)$. *For this, we write the series expansion* (2.11) *as*

$$(\sin z)^\ell = \ell! \sum_{j=0}^{\infty} (-1)^j 2^{2j} T(\ell + 2j, \ell) \frac{z^{2j+\ell}}{(2j+\ell)!}.$$

This means that

$$C_{\ell,n} = \begin{cases} 0, & n < \ell \\ 0, & n = \ell + 2j - 1 \\ (-1)^j \ell! 2^{2j} T(\ell + 2j, \ell), & n = \ell + 2j \end{cases}$$

for $j \in \mathbb{N}_0$. *Substituting this result into the series* (4.2) *in Theorem 4.1 and simplifying arrive at*

$$\sin(\sin z) = \sum_{n=0}^{\infty} (-1)^n 2^{2n} \left[\sum_{k=0}^{n} \frac{T(2n+1, 2k+1)}{2^{2k}}\right] \frac{z^{2n+1}}{(2n+1)!}. \tag{4.3}$$

Alternatively, by virtue of the formula (3.4), *we have*

$$\begin{aligned} \frac{d^n \sin(\sin z)}{d z^n} &= \sum_{k=0}^{n} (\sin u)^{(k)} B_{n,k}\Big((\sin z)', (\sin z)'', \dots, (\sin z)^{(n-k+1)}\Big) \\ &= \sum_{k=0}^{n} \sin\Big(u + \frac{k\pi}{2}\Big) B_{n,k}\Big(\sin\Big(z + \frac{\pi}{2}\Big), \sin\Big(z + \frac{2\pi}{2}\Big), \dots, \sin\Big[z + \frac{(n-k+1)\pi}{2}\Big]\Big) \\ &\to \sum_{k=0}^{n} \sin\frac{k\pi}{2} B_{n,k}\Big(\sin\frac{\pi}{2}, \sin\frac{2\pi}{2}, \dots, \sin\frac{(n-k+1)\pi}{2}\Big), \quad z \to 0, \end{aligned}$$

where $u = \sin z \to 0$ *as* $z \to 0$. *Further making use of the formula*

$$B_{n,k}\Big(1, 0, -1, 0, \dots, \sin\frac{(n-k)\pi}{2}, \sin\frac{(n-k+1)\pi}{2}\Big) = (-1)^{n-k} 2^{n-k} \Big[\cos\frac{(n-k)\pi}{2}\Big] T(n,k), \tag{4.4}$$

which is a variant of the formula (1.15) in [18, Section 1.6], we arrive at

$$\begin{aligned}
\lim_{z\to0}\frac{\mathrm{d}^n\sin(\sin z)}{\mathrm{d}\,z^n}&=\sum_{k=0}^{n}\biggl(\sin\frac{k\pi}{2}\biggr)(-1)^{n-k}2^{n-k}\biggl[\cos\frac{(n-k)\pi}{2}\biggr]T(n,k)\\
&=\begin{cases}\displaystyle\sum_{k=0}^{m-1}(-1)^k2^{2m-2k-1}\biggl[\cos\frac{(2m-2k-1)\pi}{2}\biggr]T(2m,2k+1), & n=2m\\
\displaystyle\sum_{k=0}^{m}(-1)^k2^{2m-2k}\biggl[\cos\frac{(2m-2k)\pi}{2}\biggr]T(2m+1,2k+1), & n=2m+1\end{cases}\\
&=\begin{cases}0, & n=2m\\
\displaystyle(-1)^m2^{2m}\sum_{k=0}^{m}\frac{T(2m+1,2k+1)}{2^{2k}}, & n=2m+1\end{cases}
\end{aligned}$$

for $m\in\mathbb{N}_0$. As a result, the series expansion (4.3) *is derived once again.*

Similar to the formula (4.4), *in terms of central factorial numbers of the second kind $T(n,k)$ defined by* (2.12), *another two formulas in [18, Section 1.6] can be rewritten as*

$$B_{n,k}\biggl(1,0,1,0,\dots,\frac{1-(-1)^{n-k}}{2},\frac{1-(-1)^{n-k+1}}{2}\biggr)=2^{n-k}T(n,k)$$

and

$$B_{n,k}\biggl(0,1,0,1,\dots,\frac{1+(-1)^{n-k}}{2},\frac{1+(-1)^{n-k+1}}{2}\biggr)=\frac{(2k)!}{(2k)!!}T(n,2k).$$

Finally, we notice that the formula

$$B_{n,k}\biggl(0,-1,0,1,\dots,\cos\frac{(n-k)\pi}{2},\cos\frac{(n-k+1)\pi}{2}\biggr)=\biggl(\cos\frac{n\pi}{2}\biggr)\frac{(-1)^k}{k!}\sum_{\ell=0}^{k}\frac{(-1)^\ell}{2^\ell}\binom{k}{\ell}\sum_{q=0}^{\ell}\binom{\ell}{q}(2q-\ell)^n$$

deduced in [18, Section 1.6] seemingly cannot be represented in terms of central factorial numbers of the second kind $T(n,k)$.

Example 4.3. *At the website* `https://math.stackexchange.com/a/4249446`, *the author Feng Qi gave an answer to a problem as follows.*

The hyperbolic secant function is defined by

$$\operatorname{sech} z=\frac{2}{\mathrm{e}^z+\mathrm{e}^{-z}}=\frac{2\,\mathrm{e}^z}{1+\mathrm{e}^{2z}}.$$

Then, when setting $u = u(z) = 1 + e^{2z}$, by virtue of the Leibniz rule and the Faà di Bruno formula (3.4), *we obtain*

$$\begin{aligned}
\frac{d^n \operatorname{sech}^\alpha z}{d z^n} &= 2^\alpha \frac{d^n}{d z^n} \frac{e^{\alpha z}}{(1+e^{2z})^\alpha} \\
&= 2^\alpha \sum_{k=0}^{n} \binom{n}{k} (e^{\alpha z})^{(n-k)} \Big[\big(1+e^{2z}\big)^{-\alpha}\Big]^{(k)} \\
&= 2^\alpha \sum_{k=0}^{n} \binom{n}{k} \alpha^{n-k} e^{\alpha z} \sum_{\ell=0}^{k} (u^{-\alpha})^{(\ell)} B_{k,\ell}\big(u', u'', \dots, u^{(k-\ell+1)}\big) \\
&= 2^\alpha \sum_{k=0}^{n} \binom{n}{k} \alpha^{n-k} e^{\alpha z} \sum_{\ell=0}^{k} \frac{\langle -\alpha\rangle_\ell}{u^{\alpha+\ell}} B_{k,\ell}\big(2 e^{2z}, 2^2 e^{2z}, \dots, 2^{(k-\ell+1)} e^{2z}\big) \\
&= 2^\alpha \sum_{k=0}^{n} \binom{n}{k} \alpha^{n-k} e^{\alpha z} \sum_{\ell=0}^{k} \langle -\alpha\rangle_\ell \big(1+e^{2z}\big)^{-\alpha-\ell} 2^k e^{2\ell z} B_{k,\ell}(1,1,\dots,1) \\
&= \sum_{k=0}^{n} \binom{n}{k} \alpha^{n-k} \sum_{\ell=0}^{k} S(k,\ell) \langle -\alpha\rangle_\ell \frac{2^{\alpha+k} e^{(\alpha+2\ell)z}}{(1+e^{2z})^{\alpha+\ell}} \\
&= \operatorname{sech}^\alpha z \sum_{k=0}^{n} \binom{n}{k} \alpha^{n-k} \sum_{\ell=0}^{k} S(k,\ell) \langle -\alpha\rangle_\ell 2^{k-\ell} (1+\tanh z)^\ell
\end{aligned} \tag{4.5}$$

for any real number $\alpha \ne 0$ and any integer $n \in \mathbb{N}_0$. In particular, for $n \in \mathbb{N}_0$, taking $\alpha = 2$ in the above formula (4.5) *yields*

$$\frac{d^n}{d z^n} \operatorname{sech}^2 z = 2^n \operatorname{sech}^2 z \sum_{k=0}^{n} \binom{n}{k} \sum_{\ell=0}^{k} (-1)^\ell \frac{(\ell+1)! S(k,\ell)}{2^\ell} (1+\tanh z)^\ell.$$

Now we continue to cultivate the above derivatives. From the above derivatives, we arrive at

$$\lim_{z\to 0} \frac{d^n}{d z^n} \operatorname{sech}^\alpha z = \sum_{j=0}^{n} \binom{n}{j} \alpha^{n-j} \sum_{\ell=0}^{j} \langle -\alpha\rangle_\ell 2^{j-\ell} S(j,\ell)$$

for $n \in \mathbb{N}_0$ and $\alpha \ne 0$. In other words, the series expansion

$$\operatorname{sech}^\alpha z = \sum_{n=0}^{\infty} \alpha^n \Bigg[\sum_{j=0}^{n} \binom{n}{j} \Big(\frac{2}{\alpha}\Big)^j \sum_{\ell=0}^{j} \frac{\langle -\alpha\rangle_\ell}{2^\ell} S(j,\ell)\Bigg] \frac{z^n}{n!} \tag{4.6}$$

is valid for $|z| < \frac{\pi}{2}$ and $\alpha \ne 0$. In particular, letting $\alpha = 1, 2$ in (4.6), *we deduce*

$$\operatorname{sech} z = \sum_{n=0}^{\infty} \Bigg[\sum_{j=0}^{n} \binom{n}{j} 2^j \sum_{\ell=0}^{j} (-1)^\ell \frac{\ell!}{2^\ell} S(j,\ell)\Bigg] \frac{z^n}{n!}, \quad |z| < \frac{\pi}{2} \tag{4.7}$$

and

$$\operatorname{sech}^2 z = \sum_{n=0}^{\infty} 2^n \Bigg[\sum_{j=0}^{n} \binom{n}{j} \sum_{\ell=0}^{j} (-1)^\ell \frac{(\ell+1)!}{2^\ell} S(j,\ell)\Bigg] \frac{z^n}{n!}, \quad |z| < \frac{\pi}{2}.$$

In [4, p. 42], it was listed that

$$\operatorname{sech} x = 1 - \frac{x^2}{2} + \frac{5x^4}{24} - \frac{61x^6}{720} + \dots = 1 + \sum_{k=1}^{\infty} \frac{E_{2k}}{(2k)!} x^{2k}, \quad |x| < \frac{\pi}{2}, \tag{4.8}$$

where E_n denotes the classical Euler numbers with $E_{2n+1}=0$ for $n\in\mathbb{N}_0$. When comparing (4.7) with (4.8), we find two identities

$$\sum_{j=0}^{2n-1}\binom{2n-1}{j}2^j\sum_{\ell=0}^{j}(-1)^\ell\frac{\ell!}{2^\ell}S(j,\ell)=0,\quad n\in\mathbb{N}$$

and

$$E_{2n}=\sum_{j=0}^{2n}\binom{2n}{j}2^j\sum_{\ell=0}^{j}(-1)^\ell\frac{\ell!}{2^\ell}S(j,\ell),\quad n\in\mathbb{N}_0. \tag{4.9}$$

The identity (4.9) gives an explicit formula for the Euler numbers E_{2n} in terms of the Stirling numbers of the second kind $S(n,k)$.

Setting $\alpha=m\in\mathbb{N}$ in (4.6) gives

$$\operatorname{sech}^m z=\sum_{n=0}^{\infty}\left[\sum_{j=0}^{n}\binom{n}{j}m^{n-j}\sum_{\ell=0}^{j}(-1)^\ell(m)_\ell 2^{j-\ell}S(j,\ell)\right]\frac{z^n}{n!},\quad |z|<\frac{\pi}{2}.$$

This means that

$$C_{m,n}=\sum_{j=0}^{n}\binom{n}{j}m^{n-j}\sum_{\ell=0}^{j}(-1)^\ell(m)_\ell 2^{j-\ell}S(j,\ell) \tag{4.10}$$

for $m\in\mathbb{N}$ and $n\in\mathbb{N}_0$. Considering the formula (2.1) in Theorem 2.1 reveals

$$\begin{aligned}B_{2n-1,k}(0,E_2,0,E_4,\dots,E_{2n-k-1},E_{2n-k})&=\frac{(-1)^k}{k!}\sum_{q=1}^{k}(-1)^q\binom{k}{q}C_{q,2n-1}\\&=\frac{1}{k!}\sum_{q=1}^{k}(-1)^{k-q}\binom{k}{q}\sum_{j=0}^{2n-1}\binom{2n-1}{j}q^{2n-j-1}\sum_{\ell=0}^{j}\langle -q\rangle_\ell 2^{j-\ell}S(j,\ell)\end{aligned} \tag{4.11}$$

for $2n-1\ge k\ge 0$ and

$$\begin{aligned}B_{2n,k}\big(0,E_2,0,E_4,\dots,E_{2n-k},E_{2n-k+1})&=\frac{(-1)^k}{k!}\sum_{q=1}^{k}(-1)^q\binom{k}{q}C_{q,2n}\\&=\frac{1}{k!}\sum_{q=1}^{k}(-1)^{k-q}\binom{k}{q}\sum_{j=0}^{2n}\binom{2n}{j}q^{2n-j}\sum_{\ell=0}^{j}\langle -q\rangle_\ell 2^{j-\ell}S(j,\ell)\end{aligned} \tag{4.12}$$

for $2n\ge k\ge 0$ and $n\in\mathbb{N}$.

Comparing (4.11) with (2.7) results in

$$\sum_{q=1}^{k}(-1)^q\binom{k}{q}\sum_{j=0}^{2n-1}\binom{2n-1}{j}q^{2n-j-1}\sum_{\ell=0}^{j}\langle -q\rangle_\ell 2^{j-\ell}S(j,\ell)=0,\quad 2n-1\ge k\ge 0.$$

Substituting (4.10) into the formula (3.2) in Theorem 3.1 shows

$$\operatorname{sech}^\alpha z=1+\sum_{n=1}^{\infty}\left[\sum_{k=1}^{n}\frac{(-\alpha)_k}{k!}\sum_{q=1}^{k}(-1)^q\binom{k}{q}\sum_{j=0}^{n}\binom{n}{j}q^{n-j}\sum_{\ell=0}^{j}\langle -q\rangle_\ell 2^{j-\ell}S(j,\ell)\right]\frac{z^n}{n!}$$

for $|z| < \frac{\pi}{2}$ *and* $\alpha \neq 0$. *Comparing this series expansion with* (4.6) *leads to*

$$\sum_{k=1}^{n} \frac{(-\alpha)_k}{k!} \sum_{q=1}^{k} (-1)^q \binom{k}{q} \sum_{j=0}^{n} \binom{n}{j} q^{n-j} \sum_{\ell=0}^{j} \langle -q \rangle_\ell 2^{j-\ell} S(j,\ell) = \sum_{j=0}^{n} \binom{n}{j} \alpha^{n-j} \sum_{\ell=0}^{j} \langle -\alpha \rangle_\ell 2^{j-\ell} S(j,\ell)$$

for $n \in \mathbb{N}$ *and* $\alpha \in \mathbb{R}$.

Generally, the formula (4.10) *and the series expansion* (4.2) *in Theorem 4.1, or the formula* (4.12) *together with* (2.7)*, can be applied to derive Maclaurin's series expansions of the composite function* $F(\operatorname{sech} z)$ *around* $z = 0$*, only if the derivatives* $F^{(n)}(1)$ *for* $n \in \mathbb{N}_0$ *are computable. Now we discuss Maclaurin's series expansion of the function* $\operatorname{sech}(\operatorname{sech} z)$ *around* $z = 0$ *as follows.*

Letting $\alpha = 1$ *in* (4.5) *and taking the limit* $z \to 1$ *lead to*

$$\lim_{z\to 1} \frac{\mathrm{d}^k \operatorname{sech} z}{\mathrm{d} z^k} = \sum_{j=0}^{k} \binom{k}{j} \sum_{\ell=0}^{j} (-1)^\ell \ell! 2^{j-\ell} (1+\tanh 1)^\ell (\operatorname{sech} 1) S(j,\ell), \quad k \in \mathbb{N}_0.$$

Combining this with (4.10) *and* (4.2)*, we obtain*

$$\begin{aligned} \operatorname{sech}(\operatorname{sech} z) = \operatorname{sech} 1 + (\operatorname{sech} 1) \sum_{n=1}^{\infty} \Biggl(\sum_{k=1}^{n} \frac{(-1)^k}{k!} \Biggl[\sum_{j=0}^{k} \binom{k}{j} \sum_{\ell=0}^{j} (-1)^\ell \ell! 2^{j-\ell} (1+\tanh 1)^\ell S(j,\ell) \Biggr] \\ \times \Biggl[\sum_{q=1}^{k} (-1)^q \binom{k}{q} \sum_{j=0}^{n} \binom{n}{j} q^{n-j} \sum_{\ell=0}^{j} (-1)^\ell (q)_\ell 2^{j-\ell} S(j,\ell) \Biggr] \Biggr) \frac{z^n}{n!} \end{aligned}$$

for $|\operatorname{sech} z| < \frac{\pi}{2}$. *Since* $\operatorname{sech}(\operatorname{sech} z)$ *is an even function, we acquire*

$$\begin{aligned} \operatorname{sech}(\operatorname{sech} z) = \operatorname{sech} 1 + (\operatorname{sech} 1) \sum_{n=1}^{\infty} \Biggl(\sum_{k=1}^{2n} \frac{(-1)^k}{k!} \Biggl[\sum_{j=0}^{k} \binom{k}{j} \sum_{\ell=0}^{j} (-1)^\ell \ell! 2^{j-\ell} (1+\tanh 1)^\ell S(j,\ell) \Biggr] \\ \times \Biggl[\sum_{q=1}^{k} (-1)^q \binom{k}{q} \sum_{j=0}^{2n} \binom{2n}{j} q^{2n-j} \sum_{\ell=0}^{j} (-1)^\ell (q)_\ell 2^{j-\ell} S(j,\ell) \Biggr] \Biggr) \frac{z^{2n}}{(2n)!} \end{aligned}$$

for $|\operatorname{sech} z| < \frac{\pi}{2}$ *and*

$$\begin{aligned} \sum_{k=1}^{2n-1} \frac{(-1)^k}{k!} \Biggl[\sum_{j=0}^{k} \binom{k}{j} \sum_{\ell=0}^{j} (-1)^\ell \ell! 2^{j-\ell} (1+\tanh 1)^\ell S(j,\ell) \Biggr] \\ \times \Biggl[\sum_{q=1}^{k} (-1)^q \binom{k}{q} \sum_{j=0}^{2n-1} \binom{2n-1}{j} q^{2n-j-1} \sum_{\ell=0}^{j} (-1)^\ell (q)_\ell 2^{j-\ell} S(j,\ell) \Biggr] = 0 \end{aligned}$$

for $n \in \mathbb{N}$.

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