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ON CERTAIN GAUSS-TYPE QUADRATURE RULES

By

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ABSTRACT

In this short note we give some comments to recent paper by M.A. Bokhari, A. Qadir, and H. Al-Attas, On Gauss-type quadrature rules. Numer. Funct. Anal. Optime. 31 (2010), 1120-1134. Their polynomials are a special case of the Jacobi polynomials on (0,1). In addition we construct orthogonal polynomials $\pi_n(x), n=0,1,...,$ and the corresponding Gaussian quadrature rules with respect to the linear B-spline (as a weight function) and give some numerical examples in order to illustrate an application of such quadratures.

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1. Introduction. Recently Bokhari, Qadir, and Al-Attas [1] considered Gauss-type quadrature rules based on polynomials $p_n(t)$ orthogonal on (0,1) with respect to the linear weight function $\omega(t) := 1-t$. They discussed a development of $p_n(t)$ via Gaussian hypergeometric differential equation, narrated some of its properties, derived the three-term recurrence relation for the monic polynomials

$$p_{n+1}(t) = \left(t - \frac{2(n+1)^2-1}{4(n+1)^2-1}\right)p_n(t) - \frac{n(n+1)}{4(2n+1)^2}p_{n-1}(t), n=0,1,... \tag{1.1}$$

where $p_0(t)=1$ and $p_{-1}(t)=0$, and considered several numerical examples of such kind of quadratures.

In this short note we show that these polynomials $p_n(t)$ are a special case of the well-known Jacobi polynomials on (0,1). In Section 3 we construct orthogonal polynomials $\pi_n(x), n=0,1,...,$ and the corresponding Gaussian quadrature rules with respect to the linear B-spline (as a weight function). Finally, in Section 4 we give some numerical examples to illustrate and application of such quadratures.

2. Jacobi Polynomials on (0,1). Let $w(x)=(1-x)^{\alpha}(1+x)^{\beta},\alpha,\beta>-1,$ and $\{\hat{P}_n^{(\alpha,\beta)}(x)\}$ be a sequence of the corresponding monic Jacobi polynomials, which satisfy the three-term recurrence relation (cf. [8, pp. 131-140])

$$\hat{P}_{n+1}^{(\alpha,\beta)}(x)=(x-\hat{\alpha}_n)\hat{P}_n^{(\alpha,\beta)}(x)-\hat{\beta}_n\hat{P}_{n-1}^{(\alpha,\beta)}(x),n=0,1,...,$$

where $\hat{P}_0^{(\alpha,\beta)}(x)=1,\hat{P}_{-1}^{(\alpha,\beta)}(x)=0$, and

$$\hat{\alpha}_n=\frac{\beta^2-\alpha^2}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)}\qquad(n\geq 0),$$

$$\hat{\beta}_n=\frac{4n(n+\alpha)(n+\beta)(n+\alpha+\beta)}{(2n+\alpha+\beta)^2\left[(2n+\alpha+\beta)^2-1\right]}\qquad(n\geq 1),$$

except when ; then .

By a change of variables $x=2t-1$ we get the monic orthogonal polynomials $p_n^{(\alpha,\beta)}(t)\left(=2^{-n}\hat{P}_n^{(\alpha,\beta)}(2t-1)\right)$ orthogonal on (0,1) with respect to the weight function $\omega(t):=(1-t)^{\alpha}t^{\beta},\alpha,\beta>-1$. The coefficients in their three-term recurrence relation

$$p_{n+1}^{(\alpha,\beta)}(t)=(x-\alpha_n)p_n^{(\alpha,\beta)}(t)-\beta_n p_{n-1}^{(\alpha,\beta)}(t),\quad n=0,1,..., \tag{2.1}$$

are

$$\alpha_n=\frac{1}{2}(1+\hat{\alpha}_n)=\frac{(2n+\alpha+\beta+1)^2-(1+\alpha^2-\beta^2)}{2\left[(2n+\alpha+\beta+1)^2-1\right]}(n\geq 0),$$

$$\beta_n=\frac{1}{4}\hat{\beta}_n=\frac{n(n+\alpha)(n+\beta)(n+\alpha+\beta)}{(2n+\alpha+\beta)^2\left[(2n+\alpha+\beta)^2-1\right]}\quad(n\geq 1).$$

For $\alpha=1$ and $\beta=0$, this relation (2.1) gives orthogonal polynomials $p_n(t)$ discussed in [1], which satisfy the relation (1.1).

Now, we list the parameters $\alpha_n(n\geq 0)$ and $\beta_n(n\geq 1)$ for some other special cases:

(a) $\alpha=0,\beta=1$.

$$\alpha_n=\frac{2(n+1)^2}{(2n+1)(2n+3)},\beta_n=\frac{n(n+1)}{4(2n+1)^2};$$

(b) $\alpha = 1, \beta = 1.$

$$\alpha_n = \frac{1}{2}, \beta_n = \frac{n(n+2)}{4(2n+1)(2n+3)};$$

(c) $\alpha = 2, \beta = 0.$

$$\alpha_n = \frac{n^2+3n+1}{2(n+1)(n+2)}, \beta_n = \frac{n^2(n+2)^2}{4(n+1)^2(2n+1)(2n+3)};$$

(d) $\alpha = 2, \beta = 1.$

$$\alpha_n = \frac{2(n+1)(n+3)}{(2n+3)(2n+5)}, \beta_n = \frac{n(n+3)}{4(2n+3)^2};$$

(e) $\alpha = 2, \beta = 2.$

$$\alpha_n = \frac{1}{2}, \beta_n = \frac{n(n+4)}{4(2n+3)(2n+5)};$$

(f) $\alpha = 3, \beta = 0.$

$$\alpha_n = \frac{2n^2+8n+3}{(2n+3)(2n+5)}, \beta_n = \frac{n^2(n+3)^2}{4(n+1)(n+2)(2n+3)^2};$$

(g) $\alpha = 3, \beta = 1.$

$$\alpha_n = \frac{(n+1)(n+4)}{2(n+2)(n+3)}, \beta_n = \frac{n(n+1)(n+3)(n+4)}{4(n+2)^2(2n+3)(2n+5)};$$

(h) $\alpha = \beta = -1/2.$

$$\alpha_n = \frac{1}{2}, \beta_1 = \frac{1}{8}, \beta_n = \frac{1}{16}(n \geq 2), \text{ etc.}$$

Remark 2.1. The relation (2.1) can be obtained taking $w(x) = |x|^\gamma (1-x^2)^\alpha$, with and the corresponding generalized Gegenbauer polynomials , where , which were introduced by Lasčĕnov [7] (see, also, [2, pp. 155-156] and [8, pp. 147-148]). Their three-term recurrence relation is

$$W_{n+1}^{(\alpha,\beta)}(x) = xW_n^{(\alpha,\beta)}(x) - B_n W_{n-1}^{(\alpha,\beta)}(x), W_0^{(\alpha,\beta)}(x) = 1, W_{-1}^{(\alpha,\beta)}(x) = 0,$$

with recursion coefficients

$$B_{2n} = \frac{n(n+\alpha)}{(2n+\alpha+\beta)(2n+\alpha+\beta+1)}, B_{2n-1} = \frac{(n+\beta)(n+\alpha+\beta)}{(2n+\alpha+\beta-1)(2n+\alpha+\beta)},$$

except $\alpha + \beta = -1$; then $B_1 = \beta + 1$.

Since the weight function is even on (–1,1), using Theorems 2.2.11 and 2.2.12 from [8, pp. 102-103], we get (2.1) for polynomials orthogonal with respect to the weight $\omega(t)=w(\sqrt{t})/\sqrt{t}=t^{\beta}(1-t)^{\alpha}$, with $\alpha_0=B_1=(\beta+1)/(\alpha+\beta+2)$, $\alpha_n=B_{2n}+B_{2n+1}$, $\beta_n=B_{2n-1}B_{2n}, n\geq 1$.

3. A Gaussian quadrature formula. Sometimes in applications it could be of some interest to construct orthogonal polynomials $\pi_n(x), n=0,1,...,$ and the corresponding Gaussian quadrature rules with respect to the linear B-spline (as a weight function)

$$w(x)=B_1(x)=\begin{cases}1+x, & -1\leq x\leq 0\\ 1-x & 0<x\leq 1\\ 0, & \text{otherwise},\end{cases}$$

i.e.,

$$\int_R f(x)B_1(x)dx=\int_{-1}^{1}f(x)(1+|x|)dx=\sum_{k=1}^{n}A_k f(x_k)+R_n(f), \tag{3.1}$$

where $R_n(p)=0$ for all polynomials p of degree at most $2n$–1.

This weight function is an even extension of $\omega(t)=1-t$ from (0,1) to (-1,1). The coefficients in the three-term recurrence relation for the corresponding monic orthogonal polynomials $\pi_n(x)$,

$$\pi_{n+1}(x)=x\pi_n(x)-\beta_n\pi_{n-1}(x), n=1,2,...$$

are

$\beta_0=1, \beta_1=1/6, \beta_2=7/30, \beta_3=57/245, \beta_4=683/2793, \beta_5=207725/856482,$

$\beta_6=286749501/1159331030, \beta_7=286268318986/1164429355245,$

$\beta_8=272609711230510/1097298927604497,$ etc.

For example,

$$\pi_0(x)=1, \pi_1(x)=x, \pi_2(x)=x^2-\frac{1}{6}, \pi_3(x)=x^3-\frac{2x}{5}, \pi_4(x)=x^4-\frac{31x^2}{49}+\frac{19}{490},$$

$$\pi_5(x)=x^5-\frac{50x^3}{57}+\frac{109x}{798}, \pi_6(x)=x^6-\frac{16825x^4}{15026}+\frac{2179x^2}{7513}-\frac{5935}{631092}, \text{ etc.}$$

In numerical construction we use our Mathematica Package "Orthogonal Polynomials" [3]. In order to construct quadrature rules up to n points we need the moment $\mu_k = \int_R x^k B_1(x)\,dx, k = 0,1,\ldots,2n-1,$ which are in our case given by

$$\mu_k = \frac{1+(-1)^k}{(k+1)(k+2)}, k \geq 0.$$

By the mentioned Package, in this case, we can obtain the recursion coefficients, α_k (=0 in this symmetric case) and , in a symbolic form for a reasonable n:

{alpha, beta}=aChebyshevAlgorithm[moments, Algorithm->Symbolic];
taking, for example, the first 800 moments,

moments= Table [(1+(-1)^k)/((1+k)(2+k)), {k,0,800}];

This enables us to construct parameters in Gaussian quadrature

$$Q_n(f) = \sum_{k=1}^{n} A_k^{(n)} f\left(x_k^{(n)}\right)$$

up to $n \leq 400$ nodes. For example, for n=100, with Precision ->40, the statement is the following

{n100,w100}=aGaussianNodesWeigths[100,alpha,beta,
WorkingPrecision->50, Precision->40];

where n100 and w100 are sequences of nodes $x_k^{(n)}, k = 1,\ldots,n,$ and Christoffel numbers $A_k^{(n)}, k = 1,\ldots,n,$ respectively. The last command implements the wellknown Golub-Welsch algorithm [6].

The corresponding software in Matlab was given by Gautschi [4], The first FORTRAN package of routines ORTHPOL was also developed by Walter Gautschi [5] in 1994.

4. Numerical results. In this section we consider three integrals

$$I_k = \int_R f_k(x) B_1(x)\,dx, k = 1,2,3,$$

for the functions $f_1(x) = \cos(\pi x)$,

$$f_2(x) = \frac{1}{\left(x+\frac{3}{10}\right)^2 + \frac{1}{100}} + \frac{1}{\left(x-\frac{9}{10}\right)^2 + \frac{1}{25}} \text{ and } f_3(x) = \frac{2}{2+\sin(10\pi x)},$$

and then apply the quadrature formula (3.1) for their calculation. The first function is smooth, the second is quasi-singular, and the last is an oscillatory function. The graphics of integrands $f_k(x)B_1(x), k = 1,2,$ are displayed in Fig. 4.1, and the graphic of the oscillatory function $f_3(x)B_1(x)$ is presented in Fig. 4.2

Using our Mathematica Package "Orthogonal Polynomials" [5] we constructed n-point Gaussian rules for $n=5$; 10(10)50; 100(50)400. Then, we applied these rules to integrals $I_k, k=1,2,3,$ and compared the obtained results with the exact values

$$I_1 = 4/\pi^2 = 0.405284734569351085775517852838910555617435...$$

$$I_2 = 22.385907490835113164082274654298571250346 4...$$

$$I_3 = 1.154700538379251529018297561003914911295 20...$$

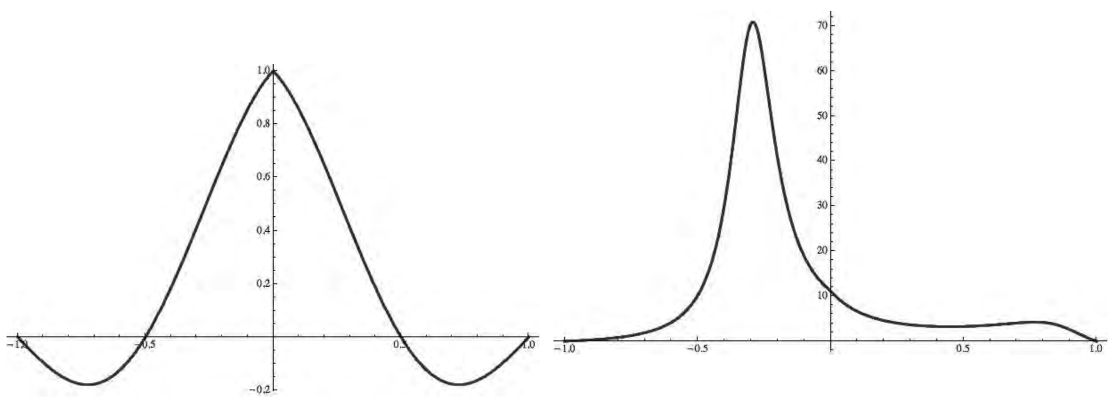

Figure 4.1. *Graphics of function* $x \to B_1(x)f_1(x)$ *(left) and* $x \to B_1(x)f_2(x)$ *(right)*

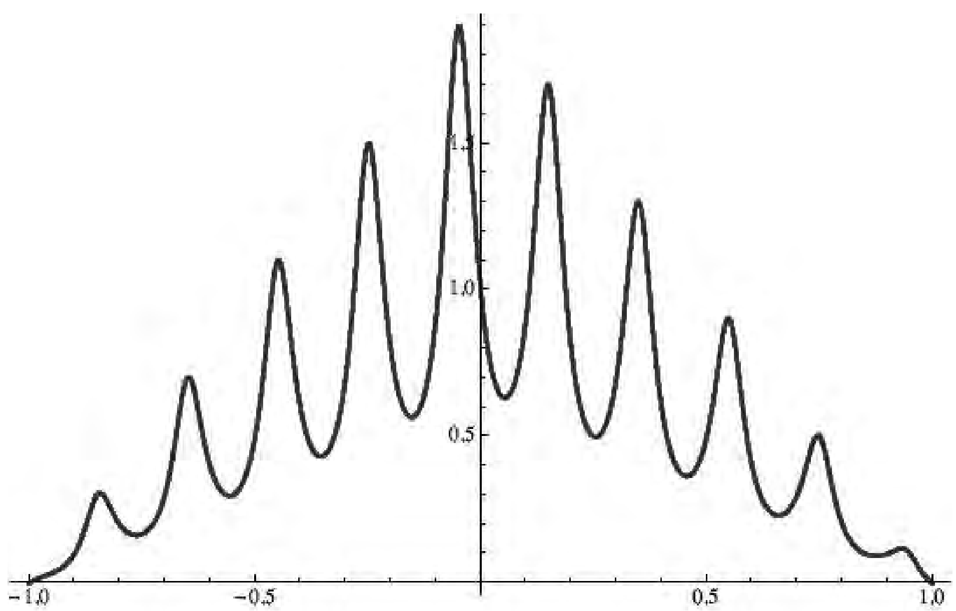

Figure 4.2. Graphics of the function $x \to B_1(x)f_3(x)$

In Table 4.1 we give the relative error in Gaussian approximations

$$REL_k(n) = \left| \frac{Q_n(f_k) - I_k}{I_k} \right|, k = 1,2,3.$$

Numbers in parentheses indicate decimal exponents.

Table 4.1. Relative errors $REL_k(n)$ in Gaussian approximations $Q_n(f_k)$ for k=1,2,3

n	$REL_1(n)$	$REL_2(n)$	$REL_3(n)$
5	2.89(–5)	3.33(–1)	1.39(–2)
10	4.01(–15)	1.25(–1)	2.97(–2)
20	1.01(–40)	1.81(–2)	7.79(–2)
30		2.50(–3)	1.39(–2)
40		3.26(–4)	3.69(–4)
50		4.05(–5)	3.32(–4)
100		4.22(–10)	2.68(–6)
150		2.28(–14)	1.05(–6)
200		9.20(–19)	3.63(–9)
250		4.38(–24)	3.79(–10)
300		6.49(–28)	1.49(–12)
350		1.99(–32)	1.81(–14)
400		3.40(–38)	7.64(–16)

As we can see in the case of a smooth function f_1 the convergence is very fast. The relative error is about 10^{-40} for n=20 points. As we expected the convergence in the last case is very slow. We must use 400 points in the quadrature rule in order to obtain the so-called double precision result (with about 16 decimal digits).

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