PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 57 (71), 1995, 81–90
Đuro Kurepa memorial volume

# EXPANSIONS OF THE KUREPA FUNCTION

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*Dedicated to the memory of Professor Đ. Kurepa*

**Abstract**. The Taylor series expansions of the Kurepa function $K(a+z)$, $a \geq 0$, and numerical determination of their coefficients $b_\nu(a)$ for $a=0$ and $a=1$ are given. An asymptotic behaviour of $b_\nu(a)$ as well as that $|b_\nu(a)/b_{\nu+1}(a)| \sim a+1$, when $\nu \to \infty$, are shown. Using this fact, a transformation of series with much faster convergence is done. Numerical values of coefficients in such a transformed series for $a=0$ and $a=1$ are given with 30 decimal digits. Also, the Chebyshev expansions of $K(1+z)$ and $1/K(1+z)$ are obtained.

## 1. Introduction

In 1971 Professor Đ. Kurepa (see [**8**–**9**]) defined so-called left factorial as

$$!n = 0! + 1! + \cdots + (n-1)!$$

and extended it to the complex plane

$$K(z) = \int_0^\infty \frac{t^z - 1}{t-1}\, e^{-t}\, dt \qquad (\operatorname{Re} z > 0). \tag{1.1}$$

This function can be extended analytically to the hole complex plane by

$$K(z) = K(z+1) - \Gamma(z+1), \tag{1.2}$$

where $\Gamma(z)$ is the gamma function defined by

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\, dt \quad (\operatorname{Re} z > 0) \quad \text{and} \quad z\Gamma(z) = \Gamma(z+1).$$

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*AMS Subject Classification* (1991): Primary 33B15, 65D20

Research partly supported by Science Fund of Serbia under grant 0401F.

Kurepa [9] proved that $K(z)$ is a meromorphic function with simple poles at the points $z_k = -k$ ($k \in \mathbb{N} \setminus \{2\}$). Slavić [17] found the representation

$$K(z) = -\frac{\pi}{e}\cot \pi z + \frac{1}{e}\left(\sum_{n=1}^{\infty}\frac{1}{n!n} + \gamma\right) + \sum_{n=0}^{\infty}\Gamma(z-n),$$

where $\gamma$ is Euler's constant. These formulas were mentioned also in the book [11]. A number of problems and hypotheses, especially in number theory, were posed by Kurepa and then considered by several mathematicians. For details and a complete list of references see a recent survey written by Ivić and Mijajlović [6].

Evaluation of $K(z)$ for some specific $z$ in $(0,1)$, using quadrature formulas with relatively small accuracy, was done by Slavić and the author of this paper (see [9]).

In this paper we give power series expansions of the Kurepa function $K(a+z)$, $a \geq 0$, and determine numerical values of their coefficients $b_\nu(a)$ for $a = 0$ and $a = 1$. Using an asymptotic behaviour of $b_\nu(a)$, when $\nu \to \infty$, we give a transformation of series with much faster convergence. Also, we obtain the Chebyshev expansions for $K(1+z)$ and $1/K(1+z)$. For similar expansions of the gamma function see e.g. Davis [3], Luke [10], Fransén and Wrigge [4], and Bohman and Fröberg [1].

## 2. Power series expansions

There are many investigations about the gamma function $\Gamma(z)$, including numerical calculations. Some old references can be found in Luke [10]. In 1980 Fransén and Wrigge [4] determined numerical values of the coefficients in the Taylor series expansion

$$\Gamma(a+x)^m = \sum_{k=0}^{\infty} g_k(m,a)x^k$$

for certain values of $m$ and $a$ and used these values to calculate $\Gamma(p/q)$ ($p, q = 1(1)10$; $p < q$) with a high precision. Recently, Bohman and Fröberg [1] have given power series expansions of the form

$$\Gamma(n+1+z) = n!(1 + d_1 z + d_2 z^2 + \cdots) \qquad (n = 2, 3, 4, 10)$$

and

$$(-1)^n n!\Gamma(-n+z) = \frac{n}{1-z} - \frac{1}{(n+1)(1+z)} + \frac{1}{z}\left(1 + f_1 z + f_2 z^2 + \cdots\right)$$

for $n = 0$ 1, 2, 10.

In this section we investigate the Kurepa function in a similar way.

Let $a \geq 0$. Differentiating (1.1) we find

$$K^{(\nu)}(a) = \int_0^\infty \frac{t^a \log^\nu t}{t-1}\, e^{-t}\, dt \qquad (\nu \geq 1), \tag{2.1}$$

so that the Taylor series expansion is given by

$$K(a+z)=b_0(a)+b_1(a)z+b_2(a)z^2+b_3(a)z^3+\cdots\,, \tag{2.2}$$

where

$$b_0(a)=K(a), \qquad b_\nu(a)=\frac{1}{\nu!}\,K^{(\nu)}(a) \qquad (\nu\ge 1). \tag{2.3}$$

Taking $a$ to be an integer, we have that $b_0(a)=!a$. For example, $b_0(0)=0$, $b_0(1)=1$, $b_0(2)=2$, $b_0(3)=4$, etc.

The following theorem gives an asymptotic behaviour of $b_\nu(a)$.

THEOREM 2.1. *For every $a>0$, $b_\nu(a)\to 0$, when $\nu\to\infty$. If $a=0$ we have*

$$\lim_{\nu\to\infty} b_{2\nu}(0)=-1, \qquad \lim_{\nu\to\infty} b_{2\nu+1}(0)=1.$$

*Precisely,*

$$b_\nu(a)\sim\frac{(-1)^{\nu+1}}{(a+1)^{\nu+1}}, \qquad \textit{when}\ \ \nu\to\infty.$$

*Proof.* We start with the identity

$$K(a)=R_1(a)+R_2(a), \tag{2.4}$$

where

$$R_1(a)=\int_0^1\frac{t^a-1}{t-1}\,e^{-t}\,dt \quad\text{and}\quad \int_1^\infty\frac{t^a-1}{t-1}\,e^{-t}\,dt.$$

Differentiating (2.4) $\nu$ times with respect to $a$, we obtain

$$b_\nu(a)=\frac{K^{(\nu)}(a)}{\nu!}=-\frac{1}{\nu!}\int_0^1\frac{t^a\log^\nu t}{1-t}\sum_{k=0}^\infty\frac{(-1)^k t^k}{k!}\,dt+\frac{1}{\nu!}\,R_2^{(\nu)}(a), \tag{2.5}$$

i.e.,

$$b_\nu(a)=-\frac{1}{\nu!}\sum_{k=0}^\infty\frac{(-1)^k}{k!}\int_0^1\frac{t^{k+a}\log^\nu t}{1-t}\,dt+\frac{1}{\nu!}\,R_2^{(\nu)}(a). \tag{2.6}$$

Using the Karamata inequality [**7**] (cf. [**14**, p. 272]),

$$\frac{\log t}{t-1}\le\frac{1}{\sqrt{t}} \qquad (t\ge 1),$$

for the last term on the right in (2.6), we have

$$\begin{aligned}0\le\frac{1}{\nu!}\,R_2^{(\nu)}(a)&=\frac{1}{\nu!}\int_1^\infty\frac{t^a\log^\nu t}{t-1}\,e^{-t}\,dt\\&\le\frac{1}{\nu!}\int_1^\infty\frac{(t-1)^{\nu-1}}{t^{\nu/2-a}}\,e^{-t}\,dt\\&\le\frac{1}{\nu!e}\int_0^\infty\frac{z^{\nu-1}}{(z+1)^{\nu/2-a}}\,e^{-z}\,dz,\end{aligned}$$

i.e.,

$$\frac{1}{\nu!}\,R_2^{(\nu)}(a) < \frac{1}{\nu!e}\int_0^{\infty} z^{\nu/2+a-1}e^{-z}\,dz = \frac{1}{\nu!e}\,\Gamma\Big(\frac{\nu}{2}+a\Big) \to 0,$$

when $\nu \to \infty$.

On the other side, we have that (cf. [**16**, p. 491])

$$\int_0^1 \frac{t^{k+a}\log^{\nu} t}{1-t}\,dt = (-1)^{\nu}\nu!\zeta(\nu+1,k+a+1),$$

where $\zeta(z,\alpha) = \sum\limits_{m=0}^{\infty}(\alpha+m)^{-z}$ is the generalized Riemman function (the Hurwitz Zeta function).

Therefore,

$$b_r(a) = (-1)^{\nu+1}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\,\zeta(\nu+1,k+a+1) + \frac{1}{\nu!}\,R_2^{(\nu)}(a). \tag{2.7}$$

When $\nu \to \infty$, we see that

$$\zeta(\nu+1,q+1) = \frac{1}{(q+1)^{\nu+1}} + \sum_{m=1}^{\infty}\frac{1}{(q+1+m)^{\nu+1}} \to \begin{cases} 0, & \text{if } \ q>0,\\ 1, & \text{if } \ q=0. \end{cases}$$

Thus, when $\nu \to \infty$ all terms in the right-hand side of (2.7) approaches zero except the term $(-1)^{\nu+1}$ in the case when $a = 0$ ($k = 0,\ q = k+a = 0$). The last statement of the theorem follows immediately from (2.7). □

From this theorem we obtain:

Corollary 2.2. *For the coefficients* $b_\nu(0)$ *in the series* (2.2) *we have that* $b_\nu(0) = (-1)^{\nu+1}(1+\varepsilon_\nu)$, *where* $\lim\limits_{\nu\to\infty}\varepsilon_\nu = 0$.

*Remark* 2.1. We note that the radius of convergence of the series (2.2) increases as $a$ increases, because of $|b_\nu(a)/b_{\nu+1}(a)| \sim a+1$, when $\nu \to \infty$.

The following theorem gives an expansion with a faster convergence.

Theorem 2.3. *For* $|z| < 1$ *we have*

$$K(z) = \frac{1}{1+z}\sum_{\nu=1}^{\infty}\alpha_\nu z^\nu, \tag{2.8}$$

*where* $\alpha_\nu = b_\nu(0) + b_{\nu-1}(0) = (-1)^{\nu-1}\Delta\varepsilon_{\nu-1}$, $\nu \ge 1$. *The coefficients* $b_\nu(0)$ *and* $\varepsilon_\nu$ *are given in Corollary* 2.2.

*Proof.* Applying the Euler-Abel transformation (cf. Milovanović [**12**, pp. 49–50]) to (2.2), with $a = 0$, we obtain (2.8). □

*Remark* 2.2. The series (2.8) converges also for $z = 1$. Notice that $\sum\limits_{\nu=1}^{\infty}\alpha_\nu = 2$.

Similar to Theorem 2.3 we can state and prove the following more general result:

THEOREM 2.4. *For* $|z| < a + 1$ *we have*

$$K(a+z) = \frac{1}{a+1+z}\sum_{\nu=0}^{\infty}\beta_\nu(a)z^\nu, \tag{2.9}$$

*where*

$$\beta_0(a) = (a+1)b_0(a), \quad \beta_\nu(a) = (a+1)b_{\nu+1}(a) + b_\nu(a), \quad \nu \geq 1. \tag{2.10}$$

*Proof.* Put $K(a+z) = b_0(a) + \big(z/(a+1)\big)g(z)$, where

$$g(z) = \sum_{\nu=1}^{\infty}(a+1)^\nu b_\nu(a)\Big(\frac{z}{a+1}\Big)^{\nu-1}.$$

Since

$$\Big(1+\frac{z}{a+1}\Big)g(z) = \sum_{\nu=0}^{\infty}\big\{(a+1)b_{\nu+1}(a)+b_\nu(a)\big\}z^\nu - b_0(a),$$

we obtain (2.9), where the coefficients $\beta_\nu(a)$ are given by (2.10). This is, in fact, the Euler-Abel transformation of the series (2.2). □

*Remark* 2.3. Suppose that $a \geq 1$. For $z = 1$ and $z = -1$, (2.9) gives

$$\sum_{\nu=0}^{\infty}\beta_\nu(a) = (a+2)K(a+1) \quad \text{and} \quad \sum_{\nu=0}^{\infty}(-1)^\nu\beta_\nu(a) = aK(a-1),$$

respectively. If $a = 1$, we have that

$$\sum_{\nu=0}^{\infty}\beta_\nu(1) = 6 \quad \text{and} \quad \sum_{\nu=0}^{\infty}(-1)^\nu\beta_\nu(1) = 0.$$

Numerical calculation of $b_\nu(a)$ for $\nu \geq 1$ is not so easy, because of singular integrand in (2.1). Some Gaussian type of quadrature formulas, e.g. Gauss-Laguerre formulas or, after changing variable, Gauss-Einstein formulas on $(0,\infty)$ (cf. Gautschi and Milovanović [**5**]) show a slowly convergence. Using the fact that for an integral of an analytic function over $(-\infty,\infty)$ the trapezoidal rule with an equal mesh size is asymptotically optimal among formulas with the same density of sampling points (see Takahasi and Mori [**18**]), we will transform our integrals to ones over $(-\infty,\infty)$, taking $t = \exp(\pi\sinh x)$, and then apply the trapezoidal quadrature formula

$$I = \int_{-\infty}^{\infty} g(x)\,dx \approx I_h = h\sum_{n=-\infty}^{\infty} g(nh) \tag{2.11}$$

For a such kind of quadratures obtained by variable transformation, especially for the DE-rule, see a survey written by Mori [**15**].

TABLE 2.1: The coefficients in the series (2.8) and (2.9) when $a = 1$

| $\nu$ | $\beta_\nu(0)$ | $\beta_\nu(1)$ |
|---|---|---|
| 0 | 0.00000 00000 00000 00000 00000 00000 | 2.00000 00000 00000 00000 00000 00000 |
| 1 | 1.43220 57346 53224 41481 10310 06215 | 2.70998 01395 03383 10840 90378 32265 |
| 2 | 0.46888 45450 62632 23498 89857 40662 | 0.90645 96812 26452 30535 12196 88028 |
| 3 | 0.06386 42753 51364 60958 84436 58189 | 0.26514 75834 59521 37636 12078 98719 |
| 4 | 0.02916 62163 10236 51587 58769 27935 | 0.08712 40652 66430 32799 74653 52816 |
| 5 | 0.00384 29068 21880 78393 78013 20684 | 0.02344 30113 03591 62453 71921 03613 |
| 6 | 0.00183 20459 86444 14916 92261 41329 | 0.00610 50968 58694 42506 90720 59033 |
| 7 | 0.00009 73017 96964 77704 35972 03248 | 0.00137 03067 97499 58760 40919 95658 |
| 8 | 0.00010 88168 64034 75461 36273 53524 | 0.00029 98495 87774 13386 75427 15149 |
| 9 | -0.00000 83124 80989 14474 80756 83316 | 0.00005 71279 77327 82762 62249 42045 |
| 10 | 0.00000 76093 36747 52906 90149 05470 | 0.00001 09812 77997 28649 23721 64909 |
| 11 | -0.00000 16730 50119 56080 52965 01901 | 0.00000 17875 37686 00232 08394 11611 |
| 12 | 0.00000 06860 95605 33952 08525 32143 | 0.00000 03177 80645 24735 74359 66125 |
| 13 | -0.00000 02068 17935 17460 02890 90873 | 0.00000 00427 02999 64944 38064 00163 |
| 14 | 0.00000 00715 53267 36586 97321 77786 | 0.00000 00078 08364 44191 12066 23204 |
| 15 | -0.00000 00233 56503 79198 66091 37138 | 0.00000 00007 17143 23915 78700 00764 |
| 16 | 0.00000 00078 18902 04675 37681 86066 | 0.00000 00001 87654 45234 13299 33159 |
| 17 | -0.00000 00025 94278 60137 72193 76381 | 0.00000 00000 01586 36642 25452 36344 |
| 18 | 0.00000 00008 64332 38788 78858 80196 | 0.00000 00000 05714 87184 82397 51814 |
| 19 | -0.00000 00002 87697 66083 26943 17995 | -0.00000 00000 00680 75693 40826 49802 |
| 20 | 0.00000 00000 95832 46989 44643 89904 | 0.00000 00000 00256 93394 02012 50700 |
| 21 | -0.00000 00000 31923 54355 14763 58582 | -0.00000 00000 00054 25128 23079 80583 |
| 22 | 0.00000 00000 10636 48853 89104 25473 | 0.00000 00000 00014 61950 44184 99559 |
| 23 | -0.00000 00000 03544 28133 05353 85378 | -0.00000 00000 00003 53643 92780 66646 |
| 24 | 0.00000 00000 01181 12896 81039 46306 | 0.00000 00000 00000 89493 51613 20739 |
| 25 | -0.00000 00000 00393 63482 92231 22875 | -0.00000 00000 00000 22239 88173 56536 |
| 26 | 0.00000 00000 00131 19298 21915 64240 | 0.00000 00000 00000 05567 76999 51411 |
| 27 | -0.00000 00000 00043 72633 98654 72079 | -0.00000 00000 00000 01390 20407 68723 |
| 28 | 0.00000 00000 00014 57428 48201 29110 | 0.00000 00000 00000 00347 51114 79725 |
| 29 | -0.00000 00000 00004 85780 47285 87821 | -0.00000 00000 00000 00086 84201 06978 |
| 30 | 0.00000 00000 00001 61919 57473 07582 | 0.00000 00000 00000 00021 70582 21355 |
| 31 | -0.00000 00000 00000 53971 38024 88922 | -0.00000 00000 00000 00005 42530 90710 |
| 32 | 0.00000 00000 00000 17990 00746 96841 | 0.00000 00000 00000 00001 35611 61567 |
| 33 | -0.00000 00000 00000 05996 55604 63516 | -0.00000 00000 00000 00000 33898 54027 |
| 34 | 0.00000 00000 00000 01998 82374 65548 | 0.00000 00000 00000 00000 08473 77501 |
| 35 | -0.00000 00000 00000 00666 26751 66870 | -0.00000 00000 00000 00000 02118 27096 |
| 36 | 0.00000 00000 00000 00222 08740 61997 | 0.00000 00000 00000 00000 00529 53328 |
| 37 | -0.00000 00000 00000 00074 02869 39615 | -0.00000 00000 00000 00000 00132 37643 |
| 38 | 0.00000 00000 00000 00024 67612 09747 | 0.00000 00000 00000 00000 00033 09273 |
| 39 | -0.00000 00000 00000 00008 22534 60745 | -0.00000 00000 00000 00000 00008 27291 |

| | | |
|---|---|---|
| 40 | 0.00000 00000 00000 00002 74177 51295 | 0.00000 00000 00000 00000 00002 06817 |
| 41 | -0.00000 00000 00000 00000 91392 33194 | -0.00000 00000 00000 00000 00000 51703 |
| 42 | 0.00000 00000 00000 00000 30464 06756 | 0.00000 00000 00000 00000 00000 12926 |
| 43 | -0.00000 00000 00000 00000 10154 67841 | -0.00000 00000 00000 00000 00000 03231 |
| 44 | 0.00000 00000 00000 00000 03384 89011 | 0.00000 00000 00000 00000 00000 00808 |
| 45 | -0.00000 00000 00000 00000 01128 29603 | -0.00000 00000 00000 00000 00000 00202 |

Thus, coefficients given by (2.3) and (2.1) reduce to

$$b_\nu(a) = \frac{\pi^{\nu+1}}{2\nu!}\int_{-\infty}^{\infty} \frac{\sinh^\nu x\cosh x\exp\left((a+\frac{1}{2})\pi\sinh x - e^{\pi\sinh x}\right)}{\sinh\left(\frac{\pi}{2}\sinh x\right)}\,dx \qquad (\nu \ge 1).$$

Similarly,

$$\alpha_\nu = \frac{\pi^{\nu+1}}{2(\nu+1)!}\int_{-\infty}^{\infty} \frac{\sinh^\nu x\cosh x\exp\left(\frac{1}{2}\pi\sinh x - e^{\pi\sinh x}\right)}{\sinh\left(\frac{\pi}{2}\sinh x\right)}\left[\pi\sinh x+\nu+1\right]dx$$

for $\nu \ge 1$.

Using (2.11) in Q-arthmetic on the MICROVAX 3400 computer (machine precision $\approx 1.93\times 10^{-34}$), we computed the coefficients $b_\nu(a)$ and $\beta_\nu(a)$ ($\nu \le 60$) for $a=0$ and $a=1$. Here, we give in Table 2.1 the coefficients $\beta_\nu(0)$ and $\beta_\nu(1)$ only for $\nu = 0(1)45$ with 30D.

## 3. Chebyshev expansions

Let $T_k^*(z)$ be the shifted Chebyshev polynomial of the first kind of degree $k$ given by $T_k^*(z) = T_k(2z-1)$, $T_k(x) = \cos k\theta$ ($x=\cos\theta$). Numerical values of the coefficients in the Chebyshev expansions for the gamma function

$$\Gamma(z+1) = \sum_{k=0}^{\infty}\gamma_k T_k^*(z) \quad\text{and}\quad \frac{1}{\Gamma(z+1)} = \sum_{k=0}^{\infty}\delta_k T_k^*(z) \qquad (0\le z\le 1),$$

were given by Clenshaw [**2**] to 20D (see also Luke [**10**, p. 13]).

In this section we obtain similar expansions for the Kurepa function. Namely, we determine the coefficients $c_k$ and $d_k$ in the Chebyshev expansions

$$K(z+1) = \sum_{k=0}^{\infty}c_k T_k^*(z) \quad\text{and}\quad \frac{1}{K(z+1)} = \sum_{k=0}^{\infty}d_k T_k^*(z) \qquad (0\le z\le 1). \quad (3.1)$$

Taking $x = 2z-1$, we can reduce our problem to the interval $(-1,1)$ and then use the least square approximation, with respect to the Chebyshev weight $w(x) = (1-x^2)^{-1/2}$, in order to find the coefficients $c_k$ and $d_k$. Thus,

$$c_k = \frac{\left(K((x+3)/2), T_k\right)}{\|T_k\|^2}, \qquad d_k = \frac{\left(1/K((x+3)/2), T_k\right)}{\|T_k\|^2} \qquad (k\ge 0),$$

where the inner product and the norm are given by

$$(f,g)=\int_{-1}^{1} f(x)\overline{g(x)}w(x)\,dx, \qquad \|f\|^2=(f,f),$$

respectively.

TABLE 3.1: The coefficients $c_k$ and $d_k$ in series (3.1)

| $k$ | $c_k$ | $d_k$ |
|---|---|---|
| 0 | 1.47404 93837 23786 91003 52009 68753 | 0.71933 88033 78564 80199 26920 51918 |
| 1 | 0.49590 32661 94158 22700 15501 19905 | -0.24486 53611 27056 06741 80872 17436 |
| 2 | 0.02579 30869 03070 45788 53165 66164 | 0.02976 94905 09647 98622 36085 35374 |
| 3 | 0.00407 08812 31791 61386 34001 49176 | -0.00498 33867 56921 33739 51852 46148 |
| 4 | 0.00015 73577 18751 30746 82134 28808 | 0.00086 57497 58509 74532 38298 56153 |
| 5 | 0.00002 57037 05826 92036 26292 93518 | -0.00014 67971 82657 38722 42886 97797 |
| 6 | 0.00000 01775 04779 26512 29793 59549 | 0.00002 51924 09034 18335 36671 90502 |
| 7 | 0.00000 01478 12089 72264 27761 10704 | -0.00000 43238 54800 40900 43392 52848 |
| 8 | -0.00000 00057 65834 77355 97860 46608 | 0.00000 07414 54647 70219 98837 74381 |
| 9 | 0.00000 00010 46615 98880 68659 86996 | -0.00000 01272 20052 30389 19464 41242 |
| 10 | -0.00000 00000 83632 76388 29507 65862 | 0.00000 00218 27577 03559 46944 44963 |
| 11 | 0.00000 00000 09422 94336 75632 53320 | -0.00000 00037 44928 13250 51616 37824 |
| 12 | -0.00000 00000 00910 90916 80081 86592 | 0.00000 00006 42530 56969 56371 69126 |
| 13 | 0.00000 00000 00093 62274 78249 23554 | -0.00000 00001 10240 75381 57048 29851 |
| 14 | -0.00000 00000 00009 39687 44816 46570 | 0.00000 00000 18914 30308 43243 17659 |
| 15 | 0.00000 00000 00000 95140 34153 77730 | -0.00000 00000 03245 18210 36461 92610 |
| 16 | -0.00000 00000 00000 09603 61612 70033 | 0.00000 00000 00556 78518 95627 89374 |
| 17 | 0.00000 00000 00000 00970 39299 20617 | -0.00000 00000 00095 52923 18136 84543 |
| 18 | -0.00000 00000 00000 00098 02159 05061 | 0.00000 00000 00016 39022 51658 48302 |
| 19 | 0.00000 00000 00000 00009 90238 99413 | -0.00000 00000 00002 81211 80450 47223 |
| 20 | -0.00000 00000 00000 00001 00033 64229 | 0.00000 00000 00000 48248 31777 95406 |
| 21 | 0.00000 00000 00000 00000 10105 45983 | -0.00000 00000 00000 08278 10261 20992 |
| 22 | -0.00000 00000 00000 00000 01020 85778 | 0.00000 00000 00000 01420 29786 64079 |
| 23 | 0.00000 00000 00000 00000 00103 12756 | -0.00000 00000 00000 00243 68458 86448 |
| 24 | -0.00000 00000 00000 00000 00010 41800 | 0.00000 00000 00000 00041 80966 55298 |
| 25 | 0.00000 00000 00000 00000 00001 05243 | -0.00000 00000 00000 00007 17340 45282 |
| 26 | -0.00000 00000 00000 00000 00000 10632 | 0.00000 00000 00000 00001 23076 16403 |
| 27 | 0.00000 00000 00000 00000 00000 01074 | -0.00000 00000 00000 00000 21116 53134 |
| 28 | -0.00000 00000 00000 00000 00000 00108 | 0.00000 00000 00000 00000 03623 02400 |
| 29 | 0.00000 00000 00000 00000 00000 00011 | -0.00000 00000 00000 00000 00621 61264 |
| 30 | -0.00000 00000 00000 00000 00000 00001 | 0.00000 00000 00000 00000 00106 65187 |

Since $\|T_0\|^2=\pi$ and $\|T_k\|^2=\pi/2$ $(k\ge 1)$, we have

$$c_0=\frac{1}{\pi}\int_{-1}^{1}\frac{1}{\sqrt{1-x^2}}K\Big(\frac{x+3}{2}\Big)\,dx, \quad c_k=\frac{2}{\pi}\int_{-1}^{1}\frac{1}{\sqrt{1-x^2}}K\Big(\frac{x+3}{2}\Big)T_k(x)\,dx,$$

where $k \geq 1$. Similarly,

$$d_0 = \frac{1}{\pi} \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} \frac{1}{K\left(\frac{x+3}{2}\right)}\, dx, \quad d_k = \frac{2}{\pi} \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} \frac{T_k(x)}{K\left(\frac{x+3}{2}\right)}\, dx \quad (k \geq 1).$$

For numerical determination of these coefficients, we apply the $N$-point Gauss-Chebyshev quadrature formula (see [**13**])

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} f(x)\, dx = \frac{\pi}{N} \sum_{k=1}^{N} f\left(x_k^{(N)}\right) + R_N(f),$$

where $x_k^{(N)}$, $k = 1, \ldots, N$, are the zeros of $N$-th Chebyshev polynomial $T_N(x)$, i.e., $x_k^{(N)} = \cos((2k-1)\frac{\pi}{2n}$. This formula is exact for all algebraic polynomials of degree at most $2N-1$, i.e., $R_N(\mathcal{P}_{2N-1}) \equiv 0$. To compute values of the Kurepa function we use the power series expansion from Section 2. In order to obtain numerical values to 30D of coefficients $c_k$ and $d_k$ for $k \leq 30$, the Gaussian formula (3.1) needs $N = 40$ and $N = 35$ points, respectively. Numerical values are displayed in Table 3.1.

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(Received 20 09 1995)