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ON THE ERDŐS–LAX INEQUALITY CONCERNING POLYNOMIALS

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Abstract. If $P(z)$ is a polynomial of degree n which does not vanish in $|z| < k$, where $k \leqslant 1$, then N. K. Govil [On a theorem of S. Bernstein, *Proc. Nat. Acad. Sci.*, **50** (1980), 50–52] proved that

$$\max_{|z|=1} |P'(z)| \leqslant \frac{n}{1+k^n} \max_{|z|=1} |P(z)|,$$

provided $|P'(z)|$ and $|Q'(z)|$ attain maximum at the same point on $|z| = 1$, where $Q(z) = z^n\overline{P(1/\overline{z})}$. In this paper, we obtain certain refinements and generalizations of this inequality and related results.

1. Introduction

Let $P(z) := \sum_{j=0}^{n} c_j z^j$ be a polynomial of degree n in the complex plane and $P'(z)$ its derivative. The study of Bernstein type inequalities that relate the norm of a polynomial to that of its derivative and their various versions are a classical topic in analysis. One basic result is that: for $P(z)$ to be a polynomial of degree n, it is true that

$$\max_{|z|=1} |P'(z)| \leqslant n \max_{|z|=1} |P(z)|. \tag{1}$$

Inequality (1) is a famous result due to Bernstein [2] who proved it in 1912. Over the last forty years many different authors produced a large number of different versions and generalizations of (1) by introducing restrictions on the zeros of $P(z)$, the modulus of large root of $P(z)$, restrictions on coefficients etc. For more information on these inequalities and related results, one can consult the books of Milovanović et al. [10], Marden [9] and Rahman and Schmeisser [11]. Since equality in (1) holds if and only if $P(z)$ has all its zeros at the origin, one would expect a relationship between the bound n and the distance of the zeros of the polynomial $P(z)$ from the origin. This fact was observed by Erdős and later verified by Lax [7] by proving that if $P(z)$ does not vanish in $|z| < 1$, then (1) can be replaced by

$$\max_{|z|=1} |P'(z)| \leqslant \frac{n}{2} \max_{|z|=1} |P(z)|. \tag{2}$$

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It was proposed by R. P. Boas Jr. to study the class of polynomials having no zeros in $|z| < k$, where $k > 0$, and obtain an inequality analogous to (2). This proposed problem has been studied extensively by many people, for example see Malik [8], Govil and Rahman [4], Govil, Qazi and Rahman [5] and others. In 1969, Malik [8] proved a partial extension of (2) for polynomials $P(z)$ not vanishing in $|z| < k$, where $k \geqslant 1$, by obtaining

$$\max_{|z|=1} |P'(z)| \leqslant \frac{n}{1+k} \max_{|z|=1} |P(z)|.$$

For the class of polynomials not vanishing in $|z| < k$, where $k \leqslant 1$, the precise estimate of maximum $|P'(z)|$ on $|z| = 1$ does not seem to the easily obtainable in general. For quite some time it was believed that if $P(z) \neq 0$ in $|z| < k$, $k \leqslant 1$, then the inequality analogous to (2) should be

$$\max_{|z|=1} |P'(z)| \leqslant \frac{n}{1+k^n} \max_{|z|=1} |P(z)|, \tag{3}$$

till E. B. Saff gave the example $P(z) = (z-\frac{1}{2})(z+\frac{1}{3})$ to counter this belief. Finally, in 1980, it was shown by N. K. Govil [3] that (3) holds with some additional hypothesis and proved the following result.

THEOREM A. *Let $P(z)$ be a polynomial of degree n having no zeros in $|z| < k$, where $k \leqslant 1$, and let $Q(z) = z^n \overline{P(1/\overline{z})}$. If $|P'(z)|$ and $|Q'(z)|$ attain maximum at the same point on $|z| = 1$, then*

$$\max_{|z|=1} |P'(z)| \leqslant \frac{n}{1+k^n} \max_{|z|=1} |P(z)|. \tag{4}$$

The result is best possible and equality in (4) holds for $P(z) = z^n + k^n$.

In 1997, Aziz and Ahmad [1] improved the bound in (4) and proved the following result.

THEOREM B. *Let $P(z)$ be a polynomial of degree n having no zeros in $|z| < k$, where $k \leqslant 1$, and let $Q(z) = z^n \overline{P(1/\overline{z})}$. If $|P'(z)|$ and $|Q'(z)|$ attain maximum at the same point on $|z| = 1$, then*

$$\max_{|z|=1} |P'(z)| \leqslant \frac{n}{1+k^n} \left\{ \max_{|z|=1} |P(z)| - \min_{|z|=k} |P(z)| \right\}. \tag{5}$$

The result is best possible and equality in (5) holds for $P(z) = z^n + k^n$.

The authors are curious to know how the inequalities in Theorems A and B can be sharpened by using some of the coefficients of $P(z)$. Indeed, this paper is mainly motivated by the desire to establish some more refined bounds than given by (4) and (5).

2. Main results

Here, we further sharpen the bounds in (4) and (5) by involving some of the coefficients of $P(z)$. We begin, by presenting the following strengthening of (4).

THEOREM 1. *Let* $P(z)=\sum_{j=0}^{n}c_jz^j$ *be a polynomial of degree* n *having no zeros in* $|z|<k$*, where* $k\leqslant 1$*, and let* $Q(z)=z^n\overline{P(1/\overline{z})}$*. If* $|P'(z)|$ *and* $|Q'(z)|$ *attain maximum at the same point on* $|z|=1$*, then*

$$\max_{|z|=1}|P'(z)|\leqslant n\left(\frac{k^{n-1}|c_n|+|c_0|}{(k^{n-1}+k^{2n})|c_n|+(k^{n-1}+1)|c_0|}\right)\max_{|z|=1}|P(z)|. \tag{6}$$

The result is best possible and equality in (6) *holds for* $P(z)=z^n+k^n$.

REMARK 1. In fact excepting the case when $P(z)$ has all its zeros on $|z|=k$, where $k\leqslant 1$, the bound obtained in Theorem 1 is always sharper than the bound obtained from Theorem A and for this it needs to show that

$$\frac{k^{n-1}|c_n|+|c_0|}{(k^{n-1}+k^{2n})|c_n|+(k^{n-1}+1)|c_0|}\leqslant\frac{1}{1+k^n},$$

which is equivalent to showing

$$|c_n|(k^{2n-1}-k^{2n})\leqslant|c_0|(k^{n-1}-k^n),$$

that is

$$k^n|c_n|\leqslant|c_0|,$$

which clearly holds because all the zeros of $P(z)$ lie in $|z|\geqslant k$ and $k\leqslant 1$.

REMARK 2. Since $P(z)=\sum_{j=0}^{n}c_jz^j\neq 0$ in $|z|<k$, $k\leqslant 1$, and if $z_1,z_2,\ldots,z_n$, are the zeros of $P(z)$ then

$$\left|\frac{c_0}{c_n}\right|=|z_1z_2\cdots z_n|\geqslant k^n. \tag{7}$$

Here, we show that for $0\leqslant\lambda\leqslant 1$,

$$k^n|c_n|\leqslant|c_0|-\lambda m, \tag{8}$$

where $m=\min_{|z|=k}|P(z)|$.

We can assume, without loss of generality, that $P(z)$ has no zeros on $|z|=k$, for otherwise (8) holds trivially by (7). Now, $P(z)$ is analytic in $|z|\leqslant k$ and has no zeros in $|z|\leqslant k$, by the Minimum Modulus Principle,

$$|P(z)|\geqslant m\quad\text{for}\quad|z|\leqslant k.$$

This implies $|P(z)|>m$ for $|z|<k$, which in particular implies

$$|c_0|=|P(0)|>m. \tag{9}$$

By Rouché's theorem, the polynomial $P(z)-\alpha m=(c_0-\alpha m)+\sum_{j=1}^{n}c_jz^j$, with $|\alpha|\leqslant 1$ has no zeros in $|z|<k$, $k\leqslant 1$, hence

$$\left|\frac{c_0-\alpha m}{c_n}\right|\geqslant k^n. \tag{10}$$

Choosing the argument of α suitably in (10), so that $|c_0 - \alpha m| = |c_0| - |\alpha| m$, which is possible by (9), we get

$$k^n |c_n| \leqslant |c_0| - |\alpha| m. \tag{11}$$

If in (11), we take $|\alpha| = \lambda$, so that $0 \leqslant \lambda \leqslant 1$, we get (8).

Next, we will prove the following strengthening of Theorem 1 which in turn refines the bounds in Theorems A and B. In fact, these two theorems can be represented in a unified form even for each λ, $0 \leqslant \lambda \leqslant 1$,

$$\max_{|z|=1} |P'(z)| \leqslant \frac{n}{1+k^n} \left\{ \max_{|z|=1} |P(z)| - \lambda \min_{|z|=k} |P(z)| \right\}, \tag{12}$$

i.e.,

$$M' \leqslant \frac{n}{1+k^n} (M - \lambda m), \tag{13}$$

where

$$M' = \max_{|z|=1} |P'(z)|, \quad M = \max_{|z|=1} |P(z)|, \quad m = \min_{|z|=k} |P(z)|. \tag{14}$$

For $\lambda = 0$ and $\lambda = 1$, the inequality (12) reduces to (4) and (5), respectively.

THEOREM 2. *Let $P(z) = \sum_{j=0}^{n} c_j z^j$ be a polynomial of degree n having no zeros in $|z| < k$, where $k \leqslant 1$, and let $Q(z) = z^n \overline{P(1/\overline{z})}$. If $|P'(z)|$ and $|Q'(z)|$ attain maximum at the same point on $|z| = 1$, then for $0 \leqslant \lambda \leqslant 1$,*

$$\begin{aligned} \max_{|z|=1} |P'(z)| \leqslant n &\left(\frac{k^n |c_n| + \lambda m + k|c_0|}{(k^n|c_n| + \lambda m)(k^{n+1}+1) + k|c_0|(k^{n-1}+1)} \right) \max_{|z|=1} |P(z)| \\ &- \lambda n \left(\frac{k^{n+1}|c_n| + k\lambda m + |c_0|}{(k^n|c_n| + \lambda m)(k^{n+1}+1) + k|c_0|(k^{n-1}+1)} \right) \min_{|z|=k} |P(z)|, \end{aligned} \tag{15}$$

where $m = \min_{|z|=k} |P(z)|$. The result is best possible and equality in (15) holds for $P(z) = z^n + k^n$.

REMARK 3. Using the notation (14), the inequality (15) can be written in a form similar to (13), i.e.,

$$M' \leqslant n \frac{(k^n|c_n| + \lambda m + k|c_0|) M - \lambda \left(k^{n+1}|c_n| + k\lambda m + |c_0| \right) m}{(k^n|c_n| + \lambda m)(k^{n+1}+1) + k|c_0|(k^{n-1}+1)}, \tag{16}$$

where $0 \leqslant \lambda \leqslant 1$.

REMARK 4. As shown in Remark 2 for $0 \leqslant \lambda \leqslant 1$, that $k^n|c_n| + \lambda m \leqslant |c_0|$, i.e.,

$$X = \frac{k^n|c_n| + \lambda m}{|c_0|} \leqslant 1. \tag{17}$$

Then the inequality (16) becomes

$$M' \leqslant \frac{n}{X(k^{n+1}+1)+k^n+k}\left[(X+k)M-\lambda(kX+1)m\right].$$

In order to prove that the bound in (16) is better than one in (13) we should check the inequality

$$\frac{(X+k)M-\lambda(kX+1)m}{X(k^{n+1}+1)+k^n+k} \leqslant \frac{M-\lambda m}{1+k^n},$$

which clearly holds because the function

$$X \mapsto f(X) = \frac{(X+k)M-\lambda(kX+1)m}{X(k^{n+1}+1)+k^n+k}$$

is increasing in $[0,1]$, hence

$$f(X) \leqslant f(1) = \frac{M-\lambda m}{1+k^n} \quad \text{as } X \leqslant 1.$$

Thus, Theorem 2 improves (12).

REMARK 5. It is important to mention that by virtue of Remark 2, the bound obtained from Theorem 2 is optimal when $\lambda = 1$ and the same is true for the inequality (12) which gives the most desirable bound for $\lambda = 1$ in the form of the inequality (5). However, the parameter λ plays a vital role for making Theorem 2 more general and to get different bounds from it by simply giving different values to it from 0 to 1 and without changing the hypothesis of the theorem. For example, for $\lambda = 0$ (without assuming that $P(z)$ has a zero on $|z| = k$) it gives the inequality (6). Thus, Theorem 1 is a corollary of Theorem 2.

Now we illustrate the obtained results by means of the following example.

EXAMPLE 1. Consider the polynomial $P(z) = z^3 - z^2 + z - 1$, then clearly $P(z)$ has all its zeros $\{1, \mathrm{i}, -\mathrm{i}\}$ on $|z| = 1$. Further,

$$Q(z) = z^n\overline{P(1/\overline{z})} = -P(z),$$

so that $|P'(z)|$ and $|Q'(z)|$ attain maximum at the same point on $|z| = 1$. We take $k = 1/2$, so that $P(z) \neq 0$ in $|z| < k = 1/2$. By Theorem A and Theorem B we obtain the following estimates

$$\max_{|z|=1} |P'(z)| \leqslant 10.7 \quad \text{and} \quad \max_{|z|=1} |P'(z)| \leqslant 9.0,$$

respectively, but by Theorem 2, with $\lambda = 1$, we get

$$\max_{|z|=1} |P'(z)| \leqslant 8.7.$$

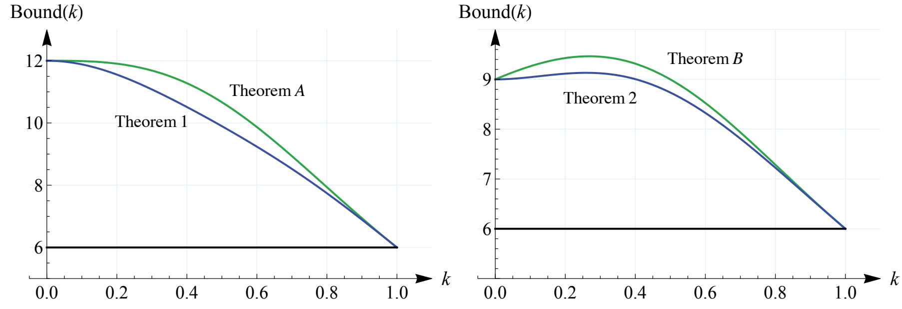

Figure 1: Bounds obtained by Theorem A and Theorem 1 (left) and by Theorem B and Theorem 2 (for $\lambda = 1$) (right) when $0 \leqslant k \leqslant 1$

In Figure 1 we displayed the bounds obtained by the previous theorems when k runs over $[0,1]$.

Finally, we compare bounds obtained by (12) and by (15) from Theorem 2, for different values of $\lambda = 0, 0.25, 0.5, 0.75, 1$ and $0 \leqslant k \leqslant 1$. Namely, in Figure 2 we give the graphics of difference between these bounds, i.e.,

$$d(k,\lambda) = \text{right side of (13)} - \text{right side of (16)},$$

confirming the theoretical result presented in Remark 4.

Otherwise, in this example we have that

$$M' = \max_{|z|=1} |P'(z)| = 6, \quad M = \max_{|z|=1} |P(z)| = 4, \quad m = \min_{|z|=k} |P(z)| = (1-k)(1+k^2).$$

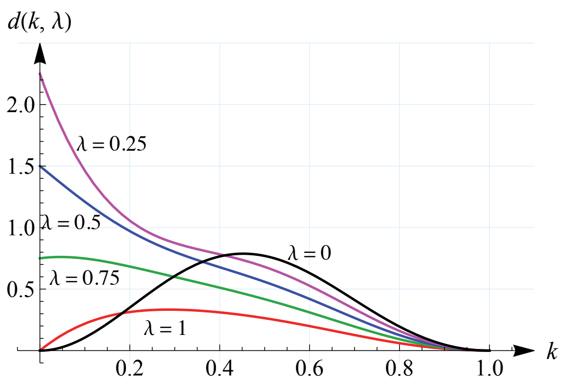

Figure 2: The function $k \mapsto d(k,\lambda)$, when $0 \leqslant k \leqslant 1$ for $\lambda = 0, 1/4, 1/2, 3/4, 1$

3. Auxiliary results

In order to prove our main results, we need the following lemmas.

LEMMA 1. *If $P(z)$ is a polynomial of degree n and $Q(z) = z^n\overline{P(1/\overline{z})}$, then on $|z| = 1$,*

$$|P'(z)| + |Q'(z)| \leqslant n \max_{|z|=1} |P(z)|.$$

This lemma is a special case of a result due to Govil and Rahman [4].

LEMMA 2. *Let $P(z) = \sum_{j=0}^{n} c_j z^j$ be a polynomial of degree n having all its zeros in $|z| \leqslant k$, $k \geqslant 1$, then*

$$\max_{|z|=1} |P'(z)| \geqslant n\left(\frac{|c_0| + |c_n|k^{n+1}}{|c_0|(1+k^{n+1}) + |c_n|(k^{n+1}+k^{2n})}\right) \max_{|z|=1} |P(z)|.$$

The above lemma is due to Jain [6].

LEMMA 3. *Let $P(z) = \sum_{j=0}^{n} c_j z^j$ be a polynomial of degree n having all its zeros in $|z| \leqslant k$, $k \geqslant 1$, then for $0 \leqslant \lambda \leqslant 1$, we have*

$$\max_{|z|=1} |P(z)| \geqslant n\left(\frac{|c_0| + \lambda m + |c_n|k^{n+1}}{(|c_0| + \lambda m)(1+k^{n+1}) + |c_n|(k^{n+1}+k^{2n})}\right) \left\{\max_{|z|=1} |P(z)| + \lambda m\right\}, \tag{18}$$

where $m = \min_{|z|=k} |P(z)|$.

Proof. If $P(z) = \sum_{j=0}^{n} c_j z^j$ has a zero on $|z| = k$, then $m = \min_{|z|=k} |P(z)| = 0$ and the result follows from Lemma 2 in this case. Henceforth, we suppose that $P(z)$ has all its zeros in $|z| < k$, where $k \geqslant 1$.

Let $H(z) = P(kz)$ and $G(z) = z^n\overline{H(1/\overline{z})} = z^n\overline{P(k/\overline{z})}$. Then all the zeros of $G(z)$ lie in $|z| > 1$ and $|H(z)| = |G(z)|$ for $|z| = 1$.

This gives

$$\left|z^n\overline{P\left(\frac{k}{\overline{z}}\right)}\right| = |P(kz)| \geqslant m \quad \text{for} \quad |z| = 1.$$

It follows by the Minimum Modulus Principle, that

$$\left|z^n\overline{P\left(\frac{k}{\overline{z}}\right)}\right| \geqslant m \ \text{ for } \ |z| \leqslant 1.$$

Replacing z by $1/\overline{z}$, it implies that

$$|P(kz)| \geqslant m|z|^n \ \text{ for } \ |z| \geqslant 1,$$

or

$$|P(z)| \geqslant m\left|\frac{z}{k}\right|^n \ \text{ for } \ |z| \geqslant k. \tag{19}$$

Now, consider the polynomial $F(z) = P(z) + \alpha m$, where α is a complex number with $|\alpha| \leqslant 1$, then all the zeros of $F(z)$ lie in $|z| \leqslant k$. Because, if for some $z = z_1$ with $|z_1| > k$, we have $F(z_1) = P(z_1) + \alpha m = 0$, then

$$|P(z_1)| = |\alpha m| \leqslant m < m\left|\frac{z_1}{k}\right|^n,$$

which contradicts (19). Hence, for every complex number α with $|\alpha| \leqslant 1$, the polynomial

$$F(z) = P(z) + \alpha m = (c_0 + \alpha m) + \sum_{j=1}^{n} c_j z^j,$$

has all its zeros in $|z| \leqslant k$, where $k \geqslant 1$. Applying Lemma 2 to the polynomial $F(z)$, we get for every complex α with $|\alpha| \leqslant 1$ and $|z| = 1$,

$$\max_{|z|=1} |P'(z)| \geqslant n \left(\frac{|c_0 + \alpha m| + |c_n| k^{n+1}}{|c_0 + \alpha m|(1 + k^{n+1}) + |c_n|(k^{n+1} + k^{2n})} \right) |P(z) + \alpha m|. \tag{20}$$

For every $\alpha \in \mathbb{C}$, we have

$$|c_0 + \alpha m| \leqslant |c_0| + |\alpha| m,$$

and since the function

$$x \mapsto \frac{x + |c_n| k^{n+1}}{x(1 + k^{n+1}) + |c_n|(k^{n+1} + k^{2n})} \qquad (x \geqslant 0)$$

is decreasing for $k \geqslant 1$, it follows from (20) that for every α with $|\alpha| \leqslant 1$ and $|z| = 1$,

$$\max_{|z|=1} |P'(z)| \geqslant n \left(\frac{|c_0| + |\alpha| m + |c_n| k^{n+1}}{(|c_0| + |\alpha| m)(1 + k^{n+1}) + |c_n|(k^{n+1} + k^{2n})} \right) |P(z) + \alpha m|. \tag{21}$$

Choosing the argument of α on the right hand side of (21) such that

$$|P(z) + \alpha m| = |P(z)| + |\alpha| m,$$

we obtain from (21) that

$$\max_{|z|=1} |P'(z)| \geqslant n \left\{ \frac{|c_0| + |\alpha| m + |c_n| k^{n+1}}{(|c_0| + |\alpha| m)(1 + k^{n+1}) + |c_n|(k^{n+1} + k^{2n})} \right\} \big(|P(z)| + |\alpha| m\big),$$

for every α with $|\alpha| \leqslant 1$ and $|z| = 1$, thereby leading to (18). This completes the proof of Lemma 3.

4. Proofs of main results

According to Remark 5 we need only to prove Theorem 2.

Proof of Theorem 2. Recall that $P(z) = \sum_{j=0}^{n} c_j z^j \neq 0$ in $|z| < k$, $k \leqslant 1$, it follows that all the zeros of $Q(z) = z^n \overline{P(1/\overline{z})}$ lie in $|z| \leqslant 1/k$, $1/k \geqslant 1$. Applying Lemma 3 to the polynomial $Q(z)$, we get for $0 \leqslant \lambda \leqslant 1$,

$$\max_{|z|=1} |Q'(z)| \geqslant n \frac{|c_n| + \lambda m' + |c_0| \frac{1}{k^{n+1}}}{\left(1 + \frac{1}{k^{n+1}}\right)(|c_n| + \lambda m') + \left(\frac{1}{k^{n+1}} + \frac{1}{k^{2n}}\right)|c_0|} \left\{ \max_{|z|=1} |Q(z)| + \lambda m' \right\}, \tag{22}$$

where

$$m' = \min_{|z|=\frac{1}{k}} |Q(z)| = \frac{1}{k^n} \min_{|z|=k} |P(z)| = \frac{m}{k^n}.$$

Since

$$\max_{|z|=1} |P(z)| = \max_{|z|=1} |Q(z)|,$$

we have inequality (22) is equivalent to

$$\max_{|z|=1} |Q'(z)| \geqslant n \frac{(k^{n+1}|c_n| + k\lambda m + |c_0|)k^n}{(k^{n+1}+1)(k^n|c_n| + \lambda m) + k|c_0|(k^{n-1}+1)} \left\{ \max_{|z|=1} |P(z)| + \frac{\lambda m}{k^n} \right\}. \tag{23}$$

By the given hypothesis, $|P'(z)|$ and $|Q'(z)|$ attain maximum at the same point on $|z| = 1$. Let

$$\max_{|z|=1} |P'(z)| = |P'(\mathrm{e}^{\mathrm{i}\alpha})|, \quad 0 \leqslant \alpha < 2\pi, \tag{24}$$

then

$$\max_{|z|=1} |Q'(z)| = |Q'(\mathrm{e}^{\mathrm{i}\alpha})|. \tag{25}$$

Also by Lemma 1, we have

$$|P'(\mathrm{e}^{\mathrm{i}\alpha})| + |Q'(\mathrm{e}^{\mathrm{i}\alpha})| \leqslant n \max_{|z|=1} |P(z)|,$$

which gives with the help of (23), (24) and (25), that

$$\begin{aligned} n \max_{|z|=1} |P(z)| \geqslant& \max_{|z|=1} |P'(z)| \\ &+ n \frac{(k^{n+1}|c_n| + k\lambda m + |c_0|)k^n}{(k^n|c_n| + \lambda m)(k^{n+1}+1) + (k^{n-1}+1)k|c_0|} \left\{ \max_{|z|=1} |P(z)| + \frac{\lambda m}{k^n} \right\}, \end{aligned}$$

which implies

$$\begin{aligned} \max_{|z|=1} |P'(z)| \leqslant n& \left\{ 1 - \frac{k^n(k^{n+1}|c_n| + k\lambda m + |c_0|)}{(k^n|c_n| + \lambda m)(k^{n+1}+1) + k|c_0|(k^{n-1}+1)} \right\} \max_{|z|=1} |P(z)| \\ &- \lambda n \left(\frac{k^{n+1}|c_n| + k\lambda m + |c_0|}{(k^n|c_n| + \lambda m)(k^{n+1}+1) + k|c_0|(k^{n-1}+1)} \right) m. \end{aligned}$$

From this, we get

$$\begin{aligned} \max_{|z|=1} |P'(z)| \leqslant n& \left\{ \frac{k^n|c_n| + \lambda m + k|c_0|}{(k^n|c_n| + \lambda m)(k^{n+1}+1) + k|c_0|(k^{n-1}+1)} \right\} \max_{|z|=1} |P(z)| \\ &- \lambda n \left(\frac{k^{n+1}|c_n| + k\lambda m + |c_0|}{(k^n|c_n| + \lambda m)(k^{n+1}+1) + k|c_0|(k^{n-1}+1)} \right) m, \end{aligned}$$

which is (15) and this completes the proof of Theorem 2.

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