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# CERTAIN ESTIMATES OF TURÁN'S-TYPE FOR THE MAXIMUM MODULUS OF THE POLAR DERIVATIVE OF A POLYNOMIAL

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*Dedicated to Professor Hari Mohan Srivastava on the occasion of his 80$^{th}$ birthday*

Abstract. We establish some lower bound estimates for the maximum modulus of the polar derivative of a polynomial on the unit disk under the assumption that the polynomial has all zeros in another disk. The obtained results sharpen as well as generalize some estimates of Turán's-type that relate the uniform-norm of the polar derivative and the polynomial.

## 1. Introduction

Let $P(z) := \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree $n$ in the complex plane and $P'(z)$ its derivative. A classical inequality that provides an estimate to the size of the derivative of a given polynomial on the unit disk, relative to size of the polynomial itself on the same disk is the famous Bernstein inequality [**4**]. It states that: if $P(z)$ is a polynomial of degree $n$, then on $|z| = 1$,

$$|P'(z)| \leqslant n \max_{|z|=1} |P(z)|. \tag{1.1}$$

Over the years, this Bernstein inequality has been generalized and extended in several directions. In 1930, (see [**5**]) Bernstein himself revisited (1.1) and proved that; for two polynomials $P(z)$ and $Q(z)$ with degree of $P(z)$ not exceeding that of $Q(z)$ and $Q(z) \neq 0$ for $|z| > 1$, the inequality $|P(z)| \leqslant |Q(z)|$ on the unit disk $|z| = 1$ implies the inequality of their derivatives $|P'(z)| \leqslant |Q'(z)|$ on $|z| = 1$. In fact, this inequality gives (1.1) in particular by taking $Q(z) = z^n \max_{|z|=1} |P(z)|$.

On the other hand, Turán's classical inequality [**16**] provides a lower bound estimate to the size of the derivative of a polynomial on the unit circle in the

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complex plane, relative to the size of the polynomial itself when there is a restriction on its zeros. It states that, if $P(z)$ is a polynomial of degree $n$ having all its zeros in $|z| \leqslant 1$, then

$$\max_{|z|=1} |P'(z)| \geqslant \frac{n}{2} \max_{|z|=1} |P(z)|. \tag{1.2}$$

Inequality (1.2) was refined by Aziz and Dawood [**1**] in the form

$$\max_{|z|=1} |P'(z)| \geqslant \frac{n}{2} \Big\{ \max_{|z|=1} |P(z)| + \min_{|z|=1} |P(z)| \Big\}. \tag{1.3}$$

Equality in (1.2) and (1.3) holds for any polynomial which has all its zeros on $|z| = 1$.

Inequalities (1.2) and (1.3) have been generalized and extended in many ways. For a polynomial $P(z)$ of degree $n$ having all its zeros in $|z| \leqslant k$, $k \geqslant 1$, Govil [**9**], proved that

$$\max_{|z|=1} |P'(z)| \geqslant \frac{n}{1+k^n} \max_{|z|=1} |P(z)|. \tag{1.4}$$

As is easy to see that (1.4) becomes equality if $P(z) = z^n + k^n$, one would expect that if we exclude the class of polynomials having all zeros on $|z| = k$, then it may be possible to improve the bound in (1.4). In this direction, it was shown by Govil [**11**] that if $P(z) = \sum_{v=0}^{n} a_v z^v$ is a polynomial of degree $n$ having all its zeros in $|z| \leqslant k$, $k \geqslant 1$, then

$$\max_{|z|=1} |P'(z)| \geqslant \frac{n}{1+k^n} \Big\{ \max_{|z|=1} |P(z)| + \min_{|z|=k} |P(z)| \Big\}. \tag{1.5}$$

Recently, Govil and Kumar [**12**] obtained a sharpening of (1.4) by involving some coefficients of $P(z)$. They proved that if $P(z) = \sum_{v=0}^{n} a_v z^v$ is a polynomial of degree $n$ having all its zeros in $|z| \leqslant k$, $k \geqslant 1$, then

$$\max_{|z|=1} |P'(z)| \geqslant \Big\{ \frac{n}{1+k^n} + \frac{(k^n|a_n| - |a_0|)}{(1+k^n)(k^n|a_n| + |a_0|)} \Big\} \max_{|z|=1} |P(z)|. \tag{1.6}$$

Different versions of the these Turán-type inequalities have appeared in the literature in more generalized forms in which the underlying polynomial is replaced by more general classes of functions. Such a generalization is moving from the domain of ordinary derivative of polynomials to their polar derivative.

Before proceeding to our main results, let us introduce the concept of the polar derivative involved. For a polynomial $P(z)$ of degree $n$, we define

$$D_\alpha P(z) := nP(z) + (\alpha - z)P'(z),$$

the polar derivative of $P(z)$ with respect to the point $\alpha$. The polynomial $D_\alpha P(z)$ is of degree at most $n-1$ and it generalizes the ordinary derivative in the sense that

$$\lim_{\alpha\to\infty} \Big\{ \frac{D_\alpha P(z)}{\alpha} \Big\} = P'(z),$$

uniformly with respect to $z$ for $|z| \leqslant R$, $R > 0$.

As mentioned before different authors produced a large number of different versions and generalizations of the above inequalities by introducing restrictions on the multiplicity of zero at $z = 0$, the modulus of the largest root of $P(z)$, restrictions on coefficients etc. Many of these generalizations involve the comparison of polar derivative $D_\alpha P(z)$ with various choices of $P(z)$, $\alpha$ and other parameters. For more information on the polar derivative of polynomials, one can consult the books of Marden [**13**], Milovanović et al. [**14**] or Rahman and Schmeisser [**15**].

In 1998, Aziz and Rather [**2**] established the polar derivative analogue of (1.4) by proving that if $P(z)$ is a polynomial of degree $n$ having all its zeros in $|z| \leqslant k$, $k \geqslant 1$, then for every $\alpha \in \mathbb{C}$ with $|\alpha| \geqslant k$,

$$\max_{|z|=1} |D_\alpha P(z)| \geqslant n\Big(\frac{|\alpha| - k}{1 + k^n}\Big) \max_{|z|=1} |P(z)|. \tag{1.7}$$

Aziz and Rather [**3**] further generalized (1.7) by using a parameter $\lambda$ and established that if $P(z)$ is a polynomial of degree $n$ having all its zeros in $|z| \leqslant k$, $k \geqslant 1$, and $m = \min_{|z|=k} |P(z)|$, then for every complex numbers $\alpha$, $\lambda$ with $|\alpha| \geqslant k$ and $|\lambda| \leqslant 1$,

$$\max_{|z|=1} |D_\alpha P(z) + \lambda nm| \geqslant n\Big(\frac{|\alpha| - k}{1 + k^n}\Big)\Big\{ \max_{|z|=1} |P(z)| + |\lambda| m\Big\}. \tag{1.8}$$

Clearly, for $\lambda = 0$, (1.8) reduces to (1.7). Although the inequality (1.8) sharpens inequality (1.7) but it has a drawback that if there is a zero on $|z| = k$, then $\min_{|z|=k} |P(z)| = 0$, and so the inequality (1.8) fails to give any improvement over (1.7). Therefore, it is quite natural to ask now; is it possible to obtain a better bound for polynomials $P(z) = \sum_{v=0}^n a_v z^v$, where not all the coefficients $a_0, a_1, \dots, a_{n-1}, a_n$ are zero and is more informative than the one given in (1.7)? In this paper, we consider this problem by using various coefficients of the underlying polynomial, and our results thus obtained sharpen the inequalities (1.4)–(1.8) and other related inequalities.

## 2. Auxiliary results

For the proofs of the theorems, we shall make use of the following lemmas.

The first lemma is due to Dubinin [**7**].

Lemma 2.1. *If $P(z) = \sum_{v=0}^n a_\nu z^v$ is a polynomial of degree $n$ having all its zeros in $|z| \leqslant 1$, then for each point $z$ on $|z| = 1$ at which $P(z) \neq 0$, we have*

$$\operatorname{Re}\Big(\frac{zP'(z)}{P(z)}\Big) \geqslant \frac{n-1}{2} + \frac{|a_n|}{|a_n| + |a_0|}.$$

Lemma 2.2. *If $P(z) = \sum_{v=0}^n a_v z^v$ is a polynomial of degree $n$ having no zeros in $|z| \leqslant 1$, then for $R \geqslant 1$,*

$$\max_{|z|=R\geqslant 1} |P(z)| \leqslant \Big(\frac{R^n + 1}{2}\Big) M(P, 1) - |a_1| \Big(\frac{R^n - 1}{n} - \frac{R^{n-2} - 1}{n - 2}\Big), \tag{2.1}$$

*if $n > 2$, and*

$$\max_{|z|=R\geqslant 1} |P(z)| \leqslant \Big(\frac{R^n + 1}{2}\Big) M(P, 1) - |a_1| \frac{(R - 1)^n}{2}, \tag{2.2}$$

*if $n = 2$.*

The above lemma is due to Govil [**10**].

LEMMA 2.3. *If* $P(z)=\sum_{v=0}^{n}a_vz^v$ *is a polynomial of degree* $n\geqslant 1$*, then for* $R\geqslant 1$*,*

$$\max_{|z|=R}|P(z)|\leqslant R^n\max_{|z|=1}|P(z)|-(R^n-R^{n-2})|P(0)|,\quad \textit{if}\ \ n\geqslant 2. \tag{2.3}$$

*and*

$$\max_{|z|=R}|P(z)|\leqslant R\max_{|z|=1}|P(z)|-(R-1)|P(0)|,\quad \textit{if}\ \ n=1. \tag{2.4}$$

*The coefficient of* $|P(0)|$ *is best possible for each* $n$*.*

The above lemma is due to Frappier, Rahman and Ruscheweyh [**8**].

## 3. Main results and proofs

We begin by proving the following result involving the polar derivative of a polynomial having all zeros in $|z|\leqslant k$, $k\geqslant 1$. As it will be shown, this result generalize inequalities (1.4), (1.6) and (1.7) and will be used to obtain sharpening of the inequalities (1.5) and (1.8).

THEOREM 3.1. *If* $P(z)=\sum_{v=0}^{n}a_vz^v$ *is a polynomial of degree* $n\geqslant 2$ *having all its zeros in* $|z|\leqslant k$*,* $k\geqslant 1$*, then for every complex number* $\alpha$ *with* $|\alpha|\geqslant k$*,*

$$\begin{aligned}\max_{|z|=1}|D_\alpha P(z)|\geqslant (|\alpha|-k)\Big\{\frac{n}{1+k^n}+\frac{|a_n|k^n-|a_0|}{(1+k^n)(|a_n|k^n+|a_0|)}\Big\}\max_{|z|=1}|P(z)|\\ +\frac{|\alpha|-k}{k}|a_{n-1}|\Big\{\frac{n}{1+k^n}+\frac{|a_n|k^n-|a_0|}{(1+k^n)(|a_n|k^n+|a_0|)}\Big\}\phi(k)\\ +|na_0+\alpha a_1|\psi(k),\end{aligned} \tag{3.1}$$

*where*

$$\phi(k)=\begin{cases}\dfrac{1}{n}(k^n-1)-\dfrac{1}{n-2}(k^{n-2}-1), & \textit{if}\ \ n>2,\\ \dfrac{1}{2}(k-1)^2, & \textit{if}\ \ n=2,\end{cases}$$

*and*

$$\psi(k)=\begin{cases}1-\dfrac{1}{k^2}, & \textit{if}\ \ n>2,\\ 1-\dfrac{1}{k}, & \textit{if}\ \ n=2.\end{cases}$$

PROOF. First we suppose that $P(z)=\sum_{v=0}^{n}a_vz^v$ is a polynomial of degree $n>2$. Recall that $P(z)$ has all its zeros in $|z|\leqslant k$, $k\geqslant 1$, therefore, all the zeros of $F(z)=P(kz)$ lie in $|z|\leqslant 1$. Let $G(z)=z^n\overline{F(1/\bar z)}$, then it is clear that

$$|nF(z)-zF'(z)|=|G'(z)|\quad \text{for}\ \ |z|=1. \tag{3.2}$$

By a result of de-Brujin (see [**6**, Theorem 1, p. 1265]), we have

$$|G'(z)|\leqslant |F'(z)|\quad \text{for}\ \ |z|=1. \tag{3.3}$$

Using (3.2) in (3.3), we get

$$|nF(z) - zF'(z)| \leqslant |F'(z)| \quad \text{for} \quad |z| = 1. \tag{3.4}$$

Since $F(z) = P(kz)$ has all its zeros in $|z| \leqslant 1$, it follows by applying Lemma 2.1 to $F(z)$ that for points $e^{i\theta}$, $0 \leqslant \theta < 2\pi$, other than the zeros of $F(z)$,

$$\begin{aligned}\Big|\frac{zF'(z)}{F(z)}\Big|_{z=e^{i\theta}} \geqslant \operatorname{Re}\Big(\frac{zF'(z)}{F(z)}\Big)\Big|_{z=e^{i\theta}} &\geqslant \frac{n-1}{2} + \frac{k^n|a_n|}{k^n|a_n| + |a_0|}\\ &= \frac{1}{2}\Big\{n + \frac{k^n|a_n| - |a_0|}{k^n|a_n| + |a_0|}\Big\}.\end{aligned}$$

Hence, we have

$$|F'(e^{i\theta})| \geqslant \frac{1}{2}\Big\{n + \frac{k^n|a_n| - |a_0|}{k^n|a_n| + |a_0|}\Big\}|F(e^{i\theta})|, \tag{3.5}$$

for the points $e^{i\theta}$, $0 \leqslant \theta < 2\pi$, other than the zeros of $F(z)$. Since (3.5) is true for the points $e^{i\theta}$, $0 \leqslant \theta < 2\pi$, which are the zeros of $F(z)$ also, it follows that

$$|F'(z)| \geqslant \frac{1}{2}\Big\{n + \frac{k^n|a_n| - |a_0|}{k^n|a_n| + |a_0|}\Big\}|F(z)| \quad \text{for} \quad |z| = 1. \tag{3.6}$$

Now, noting by hypothesis, we have $|\alpha|/k \geqslant 1$, hence on using the definition of polar derivative, we get

$$|D_{\alpha/k}F(z)| = \Big|nF(z) - \Big(z - \frac{\alpha}{k}\Big)F'(z)\Big| \geqslant \frac{|\alpha|}{k}|F'(z)| - |nF(z) - zF'(z)|.$$

This gives with the help of (3.4) and (3.6), that for $|z| = 1$,

$$|D_{\alpha/k}F(z)| \geqslant \frac{|\alpha| - k}{2k}\Big\{n + \frac{k^n|a_n| - |a_0|}{k^n|a_n| + |a_0|}\Big\}|F(z)|,$$

which gives by replacing $F(z)$ by $P(kz)$,

$$|D_{\alpha/k}P(kz)| \geqslant \frac{|\alpha| - k}{2k}\Big\{n + \frac{k^n|a_n| - |a_0|}{k^n|a_n| + |a_0|}\Big\}|P(kz)| \quad \text{for} \quad |z| = 1.$$

This further implies

$$\max_{|z|=1}\Big|nP(kz) + \Big(\frac{\alpha}{k} - z\Big)kP'(kz)\Big| \geqslant \frac{|\alpha| - k}{2k}\Big\{n + \frac{k^n|a_n| - |a_0|}{k^n|a_n| + |a_0|}\Big\}\max_{|z|=1}|P(kz)|,$$

which on using the fact that

$$\max_{|z|=1}\Big|nP(kz) + \Big(\frac{\alpha}{k} - z\Big)kP'(kz)\Big| = \max_{|z|=k}|D_\alpha P(z)|,$$

gives

$$\max_{|z|=k}|D_\alpha P(z)| \geqslant \frac{|\alpha| - k}{2k}\Big\{n + \frac{k^n|a_n| - |a_0|}{k^n|a_n| + |a_0|}\Big\}\max_{|z|=k}|P(z)|. \tag{3.7}$$

Again since $P(z)$ has all its zeros in $0 < |z| \leqslant k$, $k \geqslant 1$, therefore, the polynomial $Q(z) = z^n P(1/z)$ has all its zeros in $|z| \geqslant 1/k$ and hence the polynomial $Q(z/k)$ is of degree $n > 2$ and has all its zeros in $|z| \geqslant 1$. Applying Lemma 2.2 for the case $n > 2$ to the polynomial $Q(z/k)$ for $k \geqslant 1$, we get

$$\max_{|z|=k}\Big|Q\Big(\frac{z}{k}\Big)\Big| \leqslant \frac{k^n+1}{2}\max_{|z|=1}\Big|Q\Big(\frac{z}{k}\Big)\Big| - \frac{|a_{n-1}|}{k}\Big(\frac{k^n-1}{n} - \frac{k^{n-2}-1}{n-2}\Big),$$

which is equivalent to

$$\max_{|z|=k}|P(z)| \geqslant \frac{2k^n}{1+k^n}\max_{|z|=1}|P(z)| + \frac{2|a_{n-1}|k^{n-1}}{1+k^n}\Big(\frac{k^n-1}{n} - \frac{k^{n-2}-1}{n-2}\Big). \tag{3.8}$$

Further, for $|\alpha| = k$, the coefficient of $z^{n-1}$ in $D_\alpha P(z)$ vanishes if $\alpha = z_1 = z_2 = \cdots = z_n = ke^{i\theta}$, that is, when $P(z)$ is of the form $(z-\alpha)^n$, for which inequality (2.1) is obvious. Therefore, let $|\alpha| > k$, and since $P(z)$ is of degree $n > 2$ and so the polynomial $D_\alpha P(z)$ is of degree $n-1$, where $n-1 \geqslant 2$, and hence on applying inequality (2.3) of Lemma 2.3 to the polynomial $D_\alpha P(z)$, we get for $k \geqslant 1$,

$$\max_{|z|=k}|D_\alpha P(z)| \leqslant k^{n-1}\max_{|z|=1}|D_\alpha P(z)| - (k^{n-1}-k^{n-3})|D_\alpha P(0)|,$$

or

$$\max_{|z|=k}|D_\alpha P(z)| \leqslant k^{n-1}\max_{|z|=1}|D_\alpha P(z)| - (k^{n-1}-k^{n-3})|na_0+\alpha a_1|. \tag{3.9}$$

On using (3.8) and (3.9) in (3.7), we get

$$\begin{aligned} k^{n-1}\Big\{&\max_{|z|=1}|D_\alpha P(z)| - \Big(1-\frac{1}{k^2}\Big)|na_0+\alpha a_1|\Big\} \\ &\geqslant \max_{|z|=k}|D_\alpha P(z)| \\ &\geqslant \Big(\frac{|\alpha|-k}{2k}\Big)\Big(n + \frac{k^n|a_n|-|a_0|}{k^n|a_n|+|a_0|}\Big) \\ &\quad\times\Big\{\frac{2k^n}{1+k^n}\max_{|z|=1}|P(z)| + \frac{2|a_{n-1}|k^{n-1}}{1+k^n}\Big(\frac{k^n-1}{n} - \frac{k^{n-2}-1}{n-2}\Big)\Big\}, \end{aligned}$$

which gives

$$\begin{aligned} \max_{|z|=1}|D_\alpha P(z)| \geqslant& \frac{|\alpha|-k}{1+k^n}\Big(n + \frac{k^n|a_n|-|a_0|}{k^n|a_n|+|a_0|}\Big) \\ &\times\Big\{\max_{|z|=1}|P(z)| + \frac{|a_{n-1}|}{k}\Big(\frac{k^n-1}{n} - \frac{k^{n-2}-1}{n-2}\Big)\Big\} \\ &+ \Big(1-\frac{1}{k^2}\Big)|na_0+\alpha a_1|, \end{aligned}$$

which is equivalent to (3.1) for $n > 2$.

For the case $n = 2$, the proof follows along the same lines as that of $n > 2$, but instead of (2.3) of Lemma 2.3 we use (2.4) of the same lemma and for Lemma 2.2 using the inequality (2.2) true for polynomials of degree 2. This proves Theorem 3.1 completely. □

If we divide both sides of (3.1) by $|\alpha|$ and let $|\alpha| \to \infty$, we get the following result.

Corollary 3.1. *If $P(z)=\sum_{v=0}^{n} a_v z^v$ is a polynomial of degree $n \geqslant 2$ having all its zeros in $|z| \leqslant k$, $k \geqslant 1$, then*

$$\begin{aligned}\max_{|z|=1}|P'(z)| \geqslant \Big\{\frac{n}{1+k^n}+\frac{|a_n|k^n-|a_0|}{(1+k^n)(|a_n|k^n+|a_0|)}\Big\}\max_{|z|=1}|P(z)| \\ +\frac{|a_{n-1}|}{k}\Big\{\frac{n}{1+k^n}+\frac{|a_n|k^n-|a_0|}{(1+k^n)(|a_n|k^n+|a_0|)}\Big\}\phi(k)+|a_1|\psi(k),\end{aligned} \tag{3.10}$$

*where $\phi(k)$ and $\psi(k)$ are as defined in Theorem* 3.1. *Equality in* (3.10) *holds for $P(z)=z^n+k^n$.*

Remark 3.1. Since $k \geqslant 1$ and $n \geqslant 2$, we have $\phi(k)$ and $\psi(k)$ both non-negative, hence, it is clear that, in general, for any polynomial having all zeros in $|z| \leqslant k$, $k \geqslant 1$, inequalities (3.1) and (3.10) would give improvements over the bounds obtained from inequalities (1.7) and (1.6) respectively. Also for the class of polynomials having a zero on $|z|=k$, inequalities (3.1) and (3.10) will give bounds better than obtained from inequalities (1.8) and (1.5) respectively.

Remark 3.2. It is easy to see that Corollary 3.1 sharpens (1.6) in all cases except when $a_1=a_{n-1}=0$. Also for $k=1$, Corollary 3.1 reduces to a result of Govil and Kumar [**12**, Corollary 1.6].

Finally, we prove the following generalization of Theorem 3.1, which also provides strengthening of (1.8).

Theorem 3.2. *If $P(z)=\sum_{v=0}^{n} a_v z^v$ is a polynomial of degree $n \geqslant 2$ having all its zeros in $|z| \leqslant k$, $k \geqslant 1$, then for every complex number $\alpha$ with $|\alpha| \geqslant k$ and $|\lambda| \leqslant 1$,*

$$\begin{aligned}\max_{|z|=1}|D_\alpha P(z)+\lambda m n| \geqslant \frac{|\alpha|-k}{1+k^n}\Big(n+\frac{k^n|a_n|-|a_0|-|\lambda|m}{k^n|a_n|+|a_0|+|\lambda|m}\Big) \\ \times\Big\{\max_{|z|=1}|P(z)|+|\lambda|m+\frac{|a_{n-1}|}{k}\phi(k)\Big\}+(|na_0+\alpha a_1|-|\lambda|m)\psi(k),\end{aligned} \tag{3.11}$$

*where here and throughout $m=\min_{|z|=k}|P(z)|$ and $\phi(k)$ and $\psi(k)$ are defined in Theorem* 3.1.

Proof. If $P(z)=\sum_{v=0}^{n} a_v z^v$ has a zero on $|z|=k$, then

$$m=\min_{|z|=k}|P(z)|=0$$

and the result follows from Theorem 3.1 in this case. Henceforth, we suppose that $P(z)$ has all its zeros in $|z|<k$, $k \geqslant 1$. Let $H(z)=P(kz)$ and

$$G(z)=z^n\overline{H\Big(\frac{1}{\bar z}\Big)}=z^n\overline{P\Big(\frac{k}{\bar z}\Big)},$$

then all the zeros of $G(z)$ lie in $|z|>1$ and hence $|H(z)|=|G(z)|$ for $|z|=1$. This gives $m \leqslant |P(kz)|$ for $|z|=1$, and since $m/P(kz)$ is not a constant, it follows by the Minimum Modulus Principle that

$$\Big|z^n\overline{P\Big(\frac{k}{z}\Big)}\Big| = |P(kz)| \geqslant m \quad \text{for} \quad |z| = 1.$$

Replacing $z$ by $1/\bar{z}$, it implies that

$$|P(kz)| \geqslant m|z|^n \quad \text{for} \quad |z| \geqslant 1,$$

or

$$|P(z)| \geqslant m\Big|\frac{z}{k}\Big|^n \quad \text{for} \quad |z| \geqslant k. \tag{3.12}$$

Now, consider the polynomial $F(z) = P(z)+\lambda m$, where $\lambda$ is a complex number with $|\lambda| \leqslant 1$, then all the zeros of $F(z)$ lie in $|z| \leqslant k$. Because, if for some $z = z_0$ with $|z_0| > k$, we have $F(z_0) = P(z_0) + \lambda m = 0$, then $|P(z_0)| = |\lambda m| \leqslant m < m|z_0/k|^n$, which contradicts (3.12).

Hence for every complex number $\lambda$ with $|\lambda| \leqslant 1$, the polynomial

$$F(z) = P(z) + \lambda m = (a_0 + \lambda m) + \sum_{v=1}^{n} a_v z^v,$$

has all its zeros in $|z| \leqslant k$, where $k \geqslant 1$. Applying Theorem 3.1 to the polynomial $F(z)$, we get for any complex number $\alpha$ with $|\alpha| \geqslant k$, $n \geqslant 2$ and $|z| = 1$,

$$\begin{aligned}\max_{|z|=1}|D_\alpha(P(z) + \lambda m)| &\geqslant \frac{|\alpha| - k}{1 + k^n}\Big(n + \frac{k^n|a_n| - |a_0 + \lambda m|}{k^n|a_n| + |a_0 + \lambda m|}\Big)\\ &\quad\times\Big\{|P(z) + \lambda m| + \frac{|a_{n-1}|}{k}\phi(k)\Big\} + |n(a_0 + \lambda m) + \alpha a_1|\psi(k).\end{aligned} \tag{3.13}$$

For every $\lambda \in \mathbb{C}$, we have

$$|a_0 + \lambda m| \leqslant |a_0| + |\lambda|m,$$

and since the function $x \mapsto (k^n|a_n| - x)/(k^n|a_n| + x)$ is non increasing on

$$\{x : x > -k^n|a_n|\} \cup \{x : x < -k^n|a_n|\},$$

for every $k$, it follows from (3.13) that for every $\lambda$ with $|\lambda| \leqslant 1$ and $|z| = 1$,

$$\begin{aligned}\max_{|z|=1}|D_\alpha P(z) + \lambda mn| &\geqslant \frac{|\alpha| - k}{1 + k^n}\Big(n + \frac{k^n|a_n| - |a_0| - |\lambda|m}{k^n|a_n| + |a_0| + |\lambda|m}\Big)\\ &\quad\times\Big\{|P(z) + \lambda m| + \frac{|a_{n-1}|}{k}\phi(k)\Big\} + |na_0 + \alpha a_1 + \lambda mn|\psi(k).\end{aligned} \tag{3.14}$$

Choosing the argument of $\lambda$ on the right-hand side of (3.14) such that

$$|P(z) + \lambda m| = |P(z)| + |\lambda|m,$$

we obtain from (3.14) for $|z| = 1$, that

$$\begin{aligned}\max_{|z|=1}|D_\alpha P(z) + \lambda mn| &\geqslant \frac{|\alpha| - k}{1 + k^n}\Big(n + \frac{k^n|a_n| - |a_0| - |\lambda|m}{k^n|a_n| + |a_0| + |\lambda|m}\Big)\\ &\quad\times\Big\{|P(z)| + |\lambda|m + \frac{|a_{n-1}|}{k}\phi(k)\Big\} + (|na_0 + \alpha a_1| - |\lambda|m)\psi(k),\end{aligned}$$

which is equivalent to (3.11) and hence Theorem 3.2 is completely proved. □

Dividing both sides of (3.11) by $|\alpha|$ and let $|\alpha| \to \infty$, we get the following sharpening of (1.5).

COROLLARY 3.2. *If $P(z) = \sum_{v=0}^{n} a_v z^v$ is a polynomial of degree $n \geqslant 2$ having all its zeros in $|z| \leqslant k$, $k \geqslant 1$, then for $|\lambda| \leqslant 1$, we have*

$$\max_{|z|=1} |P'(z)| \geqslant \frac{1}{1+k^n}\Big(n + \frac{k^n|a_n| - |a_0| - |\lambda|m}{k^n|a_n| + |a_0| + |\lambda|m}\Big) \times \Big\{ \max_{|z|=1} |P(z)| + |\lambda|m + \frac{|a_{n-1}|}{k}\phi(k)\Big\} + |a_1|\psi(k), \tag{3.15}$$

*where $\phi(k)$ and $\psi(k)$ are defined in Theorem* 3.1. *Equality in* (3.15) *holds for $P(z) = z^n + k^n$.*

REMARK 3.3. Since $P(z) = \sum_{v=0}^{n} a_v z^v \neq 0$ in $|z| > k$, $k \geqslant 1$ and if $z_1, z_2, \dots, z_n$, are the zeros of $P(z)$, then

$$\Big|\frac{a_0}{a_n}\Big| = |z_1 z_2 \dots z_n| = |z_1||z_2| \dots |z_n| \leqslant k^n.$$

Also, as seen in the proof of Theorem 3.2, we have for every $\lambda$ with $|\lambda| \leqslant 1$, the polynomial $P(z) + \lambda m$ has all its zeros in $|z| \leqslant k$, $k \geqslant 1$, hence

$$\Big|\frac{a_0 + \lambda m}{a_n}\Big| \leqslant k^n. \tag{3.16}$$

If in (3.16), we choose the argument of $\lambda$ suitably, we get

$$|a_0| + |\lambda|m \leqslant k^n|a_n|.$$

Hence in fact, except the case when the polynomial $P(z)$ has all its zeros on $|z| = k$, $a_1 = 0$ and $a_{n-1} = 0$, the bounds obtained in Theorem 3.2 and Corollary 3.2 are always sharper than the bounds obtained in the inequalities (1.8) and (1.5), respectively.

REMARK 3.4. For $\lambda = 0$, Theorem 3.2 reduces to Theorem 3.1 and Corollary 3.2 reduces to Corollary 3.1.

If we take $k = 1$ in Corollary 3.2, we get the following refinement of a result due to Govil and Kumar [**12**, Corollary 1.6]).

COROLLARY 3.3. *If $P(z) = \sum_{v=0}^{n} a_v z^v$, is a polynomial of degree $n$ having all its zeros in $|z| \leqslant 1$, then for $|\lambda| \leqslant 1$ and $m' = \min_{|z|=1} |P(z)|$, we have*

$$\max_{|z|=1} |P'(z)| \geqslant \frac{n}{2}\Big(\max_{|z|=1} |P(z)| + |\lambda|m'\Big) + \frac{1}{2}\,\frac{|a_n| - |a_0| - |\lambda|m'}{|a_n| + |a_0| + |\lambda|m'}\Big(\max_{|z|=1} |P(z)| + |\lambda|m'\Big). \tag{3.17}$$

*Equality in* (3.17) *holds for $P(z) = z^n + 1$.*

Clearly, Corollary 3.3 sharpens inequality (1.3) due to Aziz and Dawood [**1**] in all cases except when $P(z)$ has all its zeros on $|z| = 1$.

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