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ON A CONNECTION BETWEEN SOME TRIGONOMETRIC QUADRATURE RULES AND GAUSS–RADAU FORMULAS WITH RESPECT TO THE CHEBYSHEV WEIGHT

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A b s t r a c t. In this short note we prove that two trigonometric quadrature formulae which are very often in applications, are equivalent to the trigonometric version of the Gauss-Radau formulas with respect to the Chebyshev weight of the first kind on $(-1,1)$*. Also, we give a short account on the classical results of Gauss-Radau quadrature rules which are related to the Chebyshev weight functions.*

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1. *Introduction*

In this short note we consider the $(2n+1)$-point trigonometric quadrature formula

$$\int_0^{2\pi} f(x)\,\mathrm{d}x = \frac{2\pi}{2n+1}\sum_{k=0}^{2n} f(x_k) + R_{2n+1}[f], \tag{1.1}$$

with the nodes

$$x_k = x_0 + \frac{2k\pi}{2n+1}, \qquad k = 0, 1, \ldots, 2n,$$

where $0 \le x_0 < 2\pi/(2n+1)$. Formula (1.1) is exact for every trigonometric polynomial of degree at most $2n$ (cf. [14]). Such kind of quadratures are known as quadrature formulas of Gaussian type and they have applications in numerical integration of 2π-periodic functions. A brief historical survey of available approaches for the construction of quadrature formulas with maximal trigonometric degree of exactness can be found in [9].

Two special cases of the quadrature formula (1.1) for which $x_0 = 0$ and $x_0 = \pi/(2n+1)$ are very interesting in applications. The corresponding quadrature sums on the right hand side of (1.1) in these cases, we will denote by

$$Q_{2n+1}^{T}(f) = \frac{2\pi}{2n+1} \sum_{k=0}^{2n} f\left(\frac{2k\pi}{2n+1}\right) \tag{1.2}$$

and

$$Q_{2n+1}^{M}(f) = \frac{2\pi}{2n+1} \sum_{k=0}^{2n} f\left(\frac{(2k+1)\pi}{2n+1}\right), \tag{1.3}$$

respectively. Some details on $Q_{2n+1}^{T}(f)$ and its applications in the trigonometric approximation can be found in [8, Chap. 3]. The second formula $Q_{2n+1}^{M}(f)$ has been recently analyzed in [9].

If we put $h = 2\pi/(2n+1)$ and $f_\alpha \equiv f(\alpha h)$, we can write these formulas (1.1) and (1.3) in the forms

$$Q_{2n+1}^{T}(f) = h\left\{\frac{1}{2}f_0 + f_1 + \cdots + f_{2n} + \frac{1}{2}f_{2n+1}\right\}$$

and

$$Q_{2n+1}^{M}(f) = h\left\{f_{1/2} + f_{3/2} + \cdots + f_{2n} + f_{2n+1/2}\right\},$$

where, because of periodicity, we introduced $f_{2n+1} = f(2\pi) = f(0) = f_0$. As we can see, quadratures (1.2) and (1.3) are symmetric with respect to the point $x = \pi$, and they are, in fact, the composite *trapezoidal* and *midpoint* rules, respectively.

In this short note we prove that these two trigonometric quadrature rules are equlvalent to the trigonometric version of the Gauss-Radau formulas with respect to the Chebyshev weight of the first kind on $(-1, 1)$. The paper is organized as follows. In Section 2 we mention classical results of Stieltjes [13] and Markov [7] on algebraic Gauss-Radau type quadratures with respect to Chebyshev weights. The main result is given in Section 3.

2. *Gaussian algebraic quadratures with respect to the Chebyshev weight*

Let

$$w_1(t) = \frac{1}{\sqrt{1-t^2}}, \quad w_2(t) = \sqrt{1-t^2}, \quad w_3(t) = \sqrt{\frac{1+t}{1-t}}, \quad w_4(t) = \sqrt{\frac{1-t}{1+t}}$$

be Chebyshev weights of the first, second, third, and fourth kind, respectively (cf. [1], [4], [8, p. 122]).

In a short note in 1884 Stieltjes [13] gave the explicit expressions for algebraic Gaussian quadrature formulas for the Chebyshev weights w_1, w_2, and w_4,

$$\int_{-1}^{1} w_\nu(t) g(t)\,\mathrm{d}t = Q_{n,\nu}(g) + R_{n,\nu}[g], \tag{2.1}$$

where

$$\begin{aligned} Q_{n,1}(g) &= \frac{\pi}{n}\sum_{k=1}^{n} g\left(\cos\frac{(2k-1)\pi}{2n}\right), \\ Q_{n,2}(g) &= \frac{\pi}{n+1}\sum_{k=1}^{n} \sin^2\frac{k\pi}{n+1}\, g\left(\cos\frac{k\pi}{n+1}\right), \\ Q_{n,4}(g) &= \frac{4\pi}{2n+1}\sum_{k=1}^{n} \sin^2\frac{k\pi}{2n+1}\, g\left(\cos\frac{2k\pi}{2n+1}\right) \end{aligned} \tag{2.2}$$

and $R_{n,\nu}(g) = 0$ for all algebraic polynomials of degree at most $2n-1$.

The corresponding formula (2.1) for w_3 can be obtained by changing $t := -t$ and using (2.2), so that

$$\int_{-1}^{1} w_3(t) g(t)\,\mathrm{d}t = \int_{-1}^{1} w_4(t) g(-t)\,\mathrm{d}t = Q_{n,4}(g(-\,\cdot\,)) + R_{n,3}[g],$$

where

$$\begin{aligned} Q_{n,3}(g) &= \frac{4\pi}{2n+1}\sum_{k=1}^{n} \sin^2\frac{k\pi}{2n+1}\, g\left(-\cos\frac{2k\pi}{2n+1}\right) \\ &= \frac{4\pi}{2n+1}\sum_{k=1}^{n} \cos^2\left(\frac{\pi}{2}-\frac{k\pi}{2n+1}\right) g\left(\cos\left(\pi-\frac{2k\pi}{2n+1}\right)\right) \end{aligned}$$

and $R_{n,3}(g(\cdot)) = R_{n,4}(g(-\cdot))$.

After changing $k := n-k+1$ in the previous quadrature sum, we get

$$Q_{n,3}(g) = \frac{4\pi}{2n+1}\sum_{k=1}^{n}\cos^2\frac{(2k-1)\pi}{2(2n+1)}\,g\left(\cos\frac{(2k-1)\pi}{2n+1}\right). \tag{2.3}$$

Shortly after Stieltjes' results, Markov [7] obtained the explicit expressions for Gauss-Radau formulas with respect to the Chebyshev weight of the first kind (for each of the end points),

$$\int_{-1}^{1}\frac{g(t)\,\mathrm{d}t}{\sqrt{1-t^2}} = \frac{2\pi}{2n+1}\left[\frac{1}{2}g(-1)+\sum_{k=1}^{n} g\left(\cos\frac{(2k-1)\pi}{2n+1}\right)\right] + R_{n+1}^{(-1)}[g] \tag{2.4}$$

and

$$\int_{-1}^{1}\frac{g(t)\,\mathrm{d}t}{\sqrt{1-t^2}} = \frac{2\pi}{2n+1}\left[\frac{1}{2}g(1)+\sum_{k=1}^{n} g\left(\cos\frac{2k\pi}{2n+1}\right)\right] + R_{n+1}^{(+1)}[g], \tag{2.5}$$

as well as the corresponding Gauss-Lobatto formula

$$\int_{-1}^{1}\frac{g(t)\,\mathrm{d}t}{\sqrt{1-t^2}} = \frac{\pi}{n+1}\left[\frac{1}{2}g(-1)+\sum_{k=1}^{n} g\left(\cos\frac{k\pi}{n+1}\right)+\frac{1}{2}g(1)\right] + R_{n+2}^{L}[g].$$

Supposing $g \in C^{2n+1}[-1,1]$, Markov [7] expressed the corresponding error terms in the Gauss-Radau formulas as

$$R_{n+1}^{(-\varepsilon)}[g] = \varepsilon\frac{\pi g^{(2n+1)}(\xi)}{(2n+1)!2^{2n}}, \quad -1<\xi<1,$$

where $\varepsilon = \pm 1$. Also, if $g \in C^{2n+2}[-1,1]$ he found the expression for the Gauss-Lobatto formula in the form

$$R_{n+2}^{L}[g] = -\frac{\pi g^{(2n+2)}(\xi)}{(2n+2)!2^{2n+1}}, \quad -1<\xi<1.$$

These formulas for remainder terms are of little practical use, because of the higher-order derivative that contains.

The remainder terms of Gauss-Lobatto and Gauss-Radau quadratures for analytic functions were estimated by Gautschi [2]. For analytic functions in $|z|<r$ and continuous on

$$C_r = \left\{z\in\mathbb{C} : |z| = r\right\}, \qquad r>1,$$

for the Gauss-Radau formulae (2.4) and (2.5), Gauutschi [2] proved that

$$|R^{\varepsilon}_{n+1}[g]| \le r\left(\max_{z\in C_r}|K_{n+1}(z)|\right)\left(\max_{z\in C_r}|f(z)|\right), \tag{2.6}$$

where the maximum of the kernel K_{n+1} can be expressed in the form

$$\max_{z\in C_r}|K_{n+1}(z)| = |K_{n+1}(-r)| = \frac{4\pi}{R-R^{-1}}\cdot\frac{1}{R^{2n+1}-1},$$

where $R = r+\sqrt{r^2-1}$. The first approach of this type for Gaussian quadratures was developed by Gautschi and Varga [5] (see also [6]).

Recently Notaris [10] has computed or, if that is not possible, estimated the norm of the error term in the Gauss-Radau formulas for all Chebyshev weights w_ν, $\nu = 1,2,3,4$. His equivalent result for the Gauss-Radau quadrature rules (2.4) and (2.5)is given by

$$||R^{(\pm 1)}_{n+1}[g]|| = \frac{2\pi r\tau^{2n+1}}{(1-\tau^{2n+1})\sqrt{r^2-1}}, \quad n \ge 1,$$

where $\tau = r-\sqrt{r^2-1}$. Notice that $R\tau = 1$.

Remark 2.1. In [3] Gautschi obtained explicit expressions for the weights of the Gauss-Radau quadrature formula for integration over the interval $[-1,1]$ relative to the Jacobi weight function $w^{\alpha,\beta}(t) = (1-t)^\alpha(1+t)^\beta$, $\alpha,\beta > -1$, i.e.,

$$\int_{-1}^{1} g(t)w^{\alpha,\beta}(t)\,\mathrm{d}t = \lambda_0^{\alpha,\beta}g(-1) + \sum_{k=1}^{n}\lambda_k^{\alpha,\beta}f(\tau_k^{\alpha,\beta}) + \widetilde{R}_{n+1}[g],$$

in the form

$$\lambda_0^{\alpha,\beta} = \frac{2^{\alpha+\beta+1}\Gamma(\beta+1)}{\binom{n+\beta+1}{n}}\frac{\Gamma(n+\alpha+1)}{\Gamma(n+\alpha+\beta+2)}$$

and

$$\lambda_k^{\alpha,\beta} = \frac{2^{\alpha+\beta}}{n+\beta+1}\frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(n+1)\Gamma(n+\alpha+\beta+2)}\frac{1-\tau_k}{[P_n^{(\alpha,\beta)}(\tau_k)]^2}, \quad k = 1,\dots,n.$$

Here $\tau_k = \tau_k^{\alpha,\beta}$ are the zeros of the Jacobi polynomial$P_n^{(\alpha,\beta+1)}(t)$ (cf. [8, p. 329]).

3. *Main result*

In this section we prove that the trigonometric quadrature rules (1.3) and (1.2) can be obtained directly by applying the (algebraic) Gauss-Radau quadrature formulas with respect to the Chebyshev weight function of the first kind on $(-1,1)$.

Proposition 3.1. *Trigonometric quadrature formula* (1.3) *is equlvalent to the trigonometric version of the Gauss-Radau formula* (2.4)*, relative to the Chebyshev weight of the first kind and with a fixed node at the end-point* -1.

PROOF. First, we transform the integral of a 2π-periodic function over $[0,2\pi]$ to an integral with respect to the Chebyshev weight function of the first kind on $(-1,1)$,

$$\begin{aligned} \int_0^{2\pi} f(x)\,\mathrm{d}x &= \int_0^{\pi} [f(x)+f(2\pi-x)]\,\mathrm{d}x \\ &= \int_{-1}^{1} [f(\arccos t)+f(2\pi-\arccos t)]\frac{\mathrm{d}t}{\sqrt{1-t^2}}. \end{aligned} \tag{3.1}$$

Now, if we apply the Gauss-Radau formula (2.4) to the last integral in (3.1), we get

$$\int_0^{2\pi} f(x)\,\mathrm{d}x = \frac{2\pi}{2n+1}\left\{\frac{1}{2}\cdot 2f(\pi)+\sum_{k=1}^{n}\left[f\left(\frac{(2k-1)\pi}{2n+1}\right)+f\left(2\pi-\frac{(2k-1)\pi}{2n+1}\right)\right]\right\}$$

$$+R_{n+1}^{(-1)}[f(\arccos(\cdot))+f(2\pi-\arccos(\cdot))].$$

Define the nodes x_k^M by

$$x_k^M = \frac{(2k+1)\pi}{2n+1}, \quad k=0,1,\ldots,2n.$$

It is easy to see that $x_n^M=\pi$, as well as the following sums

$$\sum_{k=1}^{n} f\left(\frac{(2k-1)\pi}{2n+1}\right) = \sum_{k=0}^{n-1} f(x_k^M),$$

$$\begin{aligned}\sum_{k=1}^{n} f\left(2\pi-\frac{(2k-1)\pi}{2n+1}\right) &= \sum_{k=1}^{n}\left(\frac{(2(2n-k+1)+1)\pi}{2n+1}\right)\\ &= \sum_{k=1}^{n} f(x_{2n-k+1}^M)\\ &= \sum_{k=n+1}^{2n} f(x_k^M),\end{aligned}$$

so that

$$\int_0^{2\pi} f(x)\,\mathrm{d}x = \frac{2\pi}{2n+1}\sum_{k=0}^{2n} f(x_k^M) + R_{2n+1}^M[f],$$

i.e.,

$$\int_0^{2\pi} f(x)\,\mathrm{d}x = Q_{2n+1}^M(f) + R_{2n+1}^M[f],$$

where

$$R_{2n+1}^M[f] = R_{n+1}^{(-1)}[f(\arccos(\cdot)) + f(2\pi-\arccos(\cdot))]$$

is the corresponding error term. □

Proposition 3.2. *Trigonometric quadrature formula* (1.2) *is equlvalent to the trigonometric version of the Gauss-Radau formula* (2.5)*, relative to the Chebyshev weight of the first kind and with a fixed node at the endpoint* 1*.*

Proof. In order to prove this result we apply now the Gauss-Radau formula (2.5) to the last integral in (3.1). Then we obtain

$$\begin{aligned}\int_0^{2\pi} f(x)\,\mathrm{d}x &= \frac{2\pi}{2n+1}\left\{\frac{1}{2}(f(0)+f(2\pi)) + \sum_{k=1}^{n}\left[f\left(\frac{2k\pi}{2n+1}\right) + f\left(2\pi-\frac{2k\pi}{2n+1}\right)\right]\right\}\\ &\quad + R_{n+1}^{(+1)}[f(\arccos(\cdot)) + f(2\pi-\arccos(\cdot))].\end{aligned}$$

Introduce now the nodes x_k^T as

$$x_k^T = \frac{2k\pi}{2n+1}, \quad k=0,1,\dots,2n.$$

We see that $x_0^T = 0$,

$$
\begin{aligned}
\sum_{k=1}^{n} f\left(\frac{2k\pi}{2n+1}\right) &= \sum_{k=1}^{n} f(x_k^T),\\
\sum_{k=1}^{n} f\left(2\pi - \frac{2k\pi}{2n+1}\right) &= \sum_{k=1}^{n} \left(\frac{2(2n-k+1)\pi}{2n+1}\right)\\
&= \sum_{k=1}^{n} f(x_{2n-k+1}^T)\\
&= \sum_{k=n+1}^{2n} f(x_k^T),
\end{aligned}
$$

and then, because of 2π-periodicity of f $(f(0) = f(2\pi))$, we have

$$
\int_0^{2\pi} f(x)\,\mathrm{d}x = \frac{2\pi}{2n+1} \sum_{k=0}^{2n} f(x_k^T) + R_{2n+1}^T[f],
$$

i.e.,

$$
\int_0^{2\pi} f(x)\,\mathrm{d}x = Q_{2n+1}^T(f) + R_{2n+1}^T[f],
$$

where

$$
R_{2n+1}^T[f] = R_{n+1}^{(+1)}[f(\arccos(\cdot)) + f(2\pi - \arccos(\cdot))]
$$

is the corresponding error term. $\square$

An error estimate of $R_{2n+1}^M[f]$ in the form

$$
|R_{2n+1}^M[f]| \le \frac{2\pi}{r^{2n+1} - 1} \left(\max_{z \in C_r} |f(z)|\right),
$$

for analytic functions in a disk $|z| \le r$, $r > 1$, can be found in [12] and [11].

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