ON THE INEQUALITIES OF ZYGMUND AND DE BRUIJN

ROMAN R. AKOPYAN[1,2], PRASANNA KUMAR[3] AND GRADIMIR V. MILOVANOVIĆ[4,5]

Abstract. For the polar derivative $D_\alpha P(z) = nP(z) + (\alpha - z)P'(z)$ of a polynomial $P(z)$ of degree n, most of the L^p inequalities available in the literature are for restricted values of α, and in this paper we extend few such fundamental results to all of α in the complex plane.

1. Introduction and Statement of Results

Let us denote by $\mathcal{P}_n$, the space of polynomials of degree not more than n, and let $\mathcal{P}_n(K)$ be the collection of polynomials from $\mathcal{P}_n$ having no zeros in the disc $|z| < K$ with $K > 0$. *It is customary to assume that for a polynomial $P(z) \in \mathcal{P}_n$ of exact degree $n-m$, the point at $z = \infty$ is a zero of $P(z)$ of multiplicity m.* We know by a classical result due to Bernstein (see [18]), which states that if $P(z) \in \mathcal{P}_n$ is of degree n, then

$$\max_{|z|=1} |P'(z)| \le n \max_{|z|=1} |P(z)|. \tag{1.1}$$

The inequality (1.1) is sharp and equality holds, if $P(z)$ has all its zeros at the origin. If $P(z) \in \mathcal{P}_n(1)$ is of degree n, then Erdős conjectured and later Lax [12] proved that

$$\max_{|z|=1} |P'(z)| \le \frac{n}{2} \max_{|z|=1} |P(z)|. \tag{1.2}$$

The inequality (1.2) is best possible for $P(z) = (z+1)^n$. Some generalizations of the inequality (1.2) to the class of Hurwitz polynomials may be seen in the recent paper of Kumar [11].

For $p > 0$ and any $P(z) \in \mathcal{P}_n$, we now consider the L^p-mean value on the circle which is defined by

$$M_p(P) := \left(\frac{1}{2\pi} \int_0^{2\pi} \left| P(e^{i\theta}) \right|^p \, d\theta \right)^{1/p}.$$

It is known [9] that $M_p(P)$ is nondecreasing for $p \in (0, \infty)$, and is a norm of $P(z)$ for $p \ge 1$. Moreover, the limits $p \to 0$ and $p \to \infty$ exist and

$$\lim_{p \to 0+} M_p(P) = \exp\left(\frac{1}{2\pi} \int_0^{2\pi} \log \left| P(e^{i\theta}) \right| \, d\theta \right) =: M_0(P),$$

$$\lim_{p \to \infty} M_p(P) = \max_{|z|=1} |P(z)| =: M_\infty(P).$$

With reference to [13], the mean value $M_0(P)$ has often been called the *Mahler measure* of P.

Date: October 11, 2023.

2010 *Mathematics Subject Classification.* 30A10.

Key words and phrases. L^p inequalities, polynomials, polar derivatives, zeros.

Zygmund [20] extended the Bernstein's inequality (1.1) to L^p norm as

$$M_p(P') \leq nM_p(P) \tag{1.3}$$

for any $P(z) \in \mathcal{P}_n$ of degree n and for any $p \geq 1$. The result (1.3) is sharp and equality holds if $P(z)$ has all its zeros at the origin.

The above inequality of Zygmund was extended by Arestov [1] for $0 < p < 1$.

De Bruijn [5] improved Zygmund's result for the class of polynomials having no zeros in the disc $|z| < 1$. He proved that, if $P(z) \in \mathcal{P}_n(1)$ is of degree n, then for any $p \geq 1$,

$$M_p(P') \leq n\frac{M_p(P)}{M_p(E)}, \tag{1.4}$$

where $E(z) := 1 + z$. Equality holds in (1.4) for $P(z) = 1 + z^n$.

Rahman and Schmeisser [17] showed that de Bruijn's inequality (1.4) is true for all positive values of p.

If $P(z) \in \mathcal{P}_n$ is of degree n, then the polar derivative of $P(z)$ with respect to a complex number α is defined as

$$D_\alpha P(z) = nP(z) + (\alpha - z)P'(z).$$

Note that $D_\alpha P(z)$ is a polynomial of degree at most $n-1$, and

$$\lim_{\alpha\to\infty} \frac{D_\alpha P(z)}{\alpha} = P'(z)$$

uniformly with respect to z for $|z| \leq R$, $R > 0$. The polar derivative of a polynomial appeared apparently in the works of E. Laguere and G. Szegö as a generalization of the classical derivative of a polynomial in connection with the issues related to the location of the zeros of polynomials, and is a classical object of research. In particular, the operator D_α for $|\alpha| \geq 1$ is a B_n-operator [16, §14.5].

The Bernstein-type inequalities and their generalizations have been extended to the polar derivatives of complex polynomials significantly, for which we refer to the monographs [15] and [16]. Before going into the details, let us introduce some notations. For any subset π_n of $\mathcal{P}_n$, let us denote by $\mathcal{K}_p(D_\alpha,\ \pi_n)$, the smallest exact possible π_n-*constant* in the inequality

$$M_p(D_\alpha P) \leq \mathcal{K}_p(D_\alpha,\ \pi_n)M_p(P), \qquad P \in \pi_n. \tag{1.5}$$

The exact values of $\mathcal{K}_p(D_\alpha,\ \pi_n)$ are known only for some parameter values. They are most interesting in the cases $\pi_n = \mathcal{P}_n$ and $\pi_n = \mathcal{P}_n(K)$.

In the case of classical Bernstein inequality (1.1), i.e., for the classical derivative whenever $\alpha = \infty$ and for all p, $0 \leq p \leq \infty$, the equality

$$\lim_{\alpha\to\infty} \frac{\mathcal{K}_p(D_\alpha,\ \mathcal{P}_n)}{|\alpha|} = n$$

is well known (see Zygmund [20], Arestov [1]) as mentioned earlier with the inequality (1.3). The inequality (1.5) in this case turns into an equality on the polynomials cz^n.

On the space $\mathcal{P}_n$, consider the operator I, defining a one-to-one mapping of $\mathcal{P}_n$ onto itself according to the formula

$$IP_n(z) = z^n P_n(1/z). \tag{1.6}$$

Similarly we define the operator $\overline{I}P_n(z) = z^n\overline{P_n(1/\overline{z})}$ and use it conveniently.

Now it is easy to see that the following proposition holds.

Proposition 1.1. *For any non-zero complex number α, and any subclass π_n of $\mathcal{P}_n$, the equality*

$$\mathcal{K}_p(D_\alpha,\ \pi_n) = |\alpha|\mathcal{K}_p(D_{1/\alpha},\ I\pi_n)$$

holds for $0 \le p \le \infty$, where $I\pi_n$ is the class of polynomials, which is the image of class π_n when mapped by the operator I. In particular, on the space $\mathcal{P}_n$ the equality

$$\mathcal{K}_p(D_\alpha,\ \mathcal{P}_n) = |\alpha|\mathcal{K}_p(D_{1/\alpha},\ I\mathcal{P}_n)$$

holds.

The exact Bernstein's inequality (1.1) for the polar derivative of polynomials from $\mathcal{P}_n$ for $|\alpha| \ge 1$ in the uniform norm was first obtained by Aziz in 1988 [3]. In fact he proved that, if $P(z) \in \mathcal{P}_n$ is of degree n, then

$$\max_{|z|=1} |D_\alpha P(z)| \le n|\alpha| \max_{|z|=1} |P(z)|, \tag{1.7}$$

whenever $|\alpha| \ge 1$. This result is sharp and equality holds for $P(z) = z^n$ whenever $\alpha \ge 1$.

In the same paper, Aziz [3] established the polar derivative version of the inequality (1.2) by proving that, if $P(z) \in \mathcal{P}_n(1)$ is of degree n, then for any complex number α with $|\alpha| \ge 1$,

$$\max_{|z|=1} |D_\alpha\{P(z)\}| \le \frac{n}{2}(1+|\alpha|) \max_{|z|=1} |P(z)|. \tag{1.8}$$

The inequality (1.8) is best possible for $P(z) = 1 + z^n$ whenever $\alpha \ge 1$.

Taking Proposition 1.1 into account with the input of $p = \infty$, we obtain the following.

Corollary 1.2. *For any complex number α*

$$\mathcal{K}_\infty(D_\alpha,\ \mathcal{P}_n) = n \max\{1, |\alpha|\}.$$

Again the inequality (1.5) on $\mathcal{P}_n$ turns into an equality for $|\alpha| \ge 1$ on the polynomials z^n and for $|\alpha| < 1$ on the constants.

One can see in the literature that the inequalities for the polar derivative $D_\alpha P(z)$ of a polynomial $P(z)$ are available mostly for restricted values of α, and for more information we refer to [7].

To the best of our knowledge, the L^p extension of the inequality (1.7) for any complex values of α is not available and therefore an attempt is made in this paper to settle this problem to some extent.

The inequality (1.8), and its extension in L^p settings are available only for $|\alpha| \ge 1$, and one of the more generalized results in this direction is due to Govil and Kumar [8], which is given by

$$M_p(D_\alpha(P)) \le \frac{n(t_0 + |\alpha|)}{M_p(E_{t_0,1})} M_p(P) \tag{1.9}$$

or equivalently

$$\mathcal{K}_p(D_\alpha,\ \mathcal{P}_n(1)) \le \frac{n(t_0 + |\alpha|)}{M_p(E_{t_0,1})}$$

for any $p \ge 0$, and for every complex number α with $|\alpha| \ge 1$, where all the zeros $z_m,\ 1 \le m \le n$ (counting multiplicity) of any polynomial $P(z) \in \mathcal{P}_n(1)$ of degree n are such that $|z_m| \ge K_m,\ 1 \le m \le n$. Here $E_{t_0,1}(z) := t_0 + z$, and $t_0 =$

$1+n\left(\sum_{m=1}^{n}(K_m-1)^{-1}\right)^{-1}$ if $K_m>1$ $(1\le m\le n)$, and $t_0=1$ if $K_m=1$ for some m, $1\le m\le n$.

When $K_m=K$ for all m, then for any $p\ge 0$ and for every complex number α with $|\alpha|\ge 1$, we have

$$\mathcal{K}_p(D_\alpha,\ \mathcal{P}_n(K))\ \le\ \frac{n(K+|\alpha|)}{M_p(E_{K,1})}\,, \tag{1.10}$$

where $E_{K,1}(z):=K+z$.

As a particular case, when $K=1$ in the above inequality (1.10), we obtain the bound for L^p copy of the inequality (1.8), which states that, for any $p\ge 0$ and for every complex number α with $|\alpha|\ge 1$,

$$\mathcal{K}_p(D_\alpha,\ \mathcal{P}_n(1))\ \le\ \frac{n(1+|\alpha|)}{M_p(E)}, \tag{1.11}$$

where $E(z)=1+z$.

The case $p=\infty$ in (1.11) gives the inequality (1.8).

Let us prove some fundamental results that will fill the long-standing gaps in the literature on the inequalities for polar derivatives of complex polynomials in L^p environment. To begin with, let us present our first result on the L^p version of the inequality (1.7) not only for $|\alpha|\ge 1$, as given in the hypotheses of (1.7), but also for $|\alpha|<1$. The result states that

Theorem 1.3. *For any complex number α and $p\ge 0$, we have*

$$\mathcal{K}_p(D_\alpha,\ \mathcal{P}_n)\le n\min\left\{1+|\alpha|,\frac{2\max\{1,|\alpha|\}}{M_p(E)}\right\} \tag{1.12}$$

whenever $E(z)=1+z$. When $p=\infty$, the bound in (1.12) is best possible and consequently the equality in (1.5) holds for $P(z)=z^n$ whenever $|\alpha|\ge 1$ and $\pi_n=\mathcal{P}_n$.

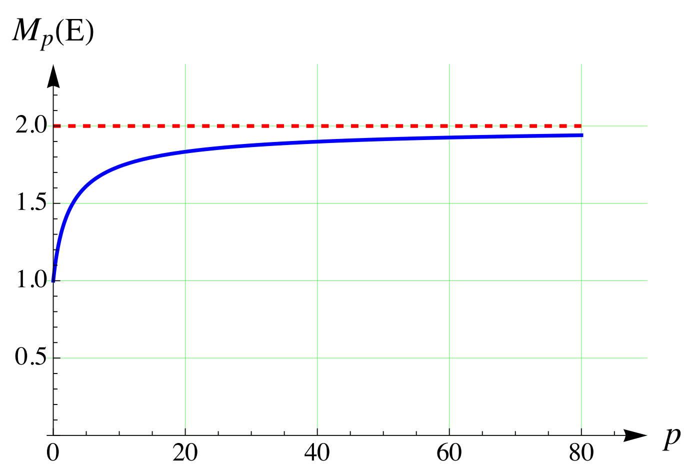

FIGURE 1. The constant $M_p(E)$ for $0\le p\le 80$

It is known that

$$M_p(E) = \begin{cases} 1, & \text{if } p = 0, \\ \left(\dfrac{2^p\,\Gamma(\frac{p+1}{2})}{\sqrt{\pi}\,\Gamma(\frac{p}{2}+1)} \right)^{1/p}, & \text{if } 0 < p < \infty, \\ 2, & \text{if } p = \infty, \end{cases}$$

where Γ denotes the gamma function (see Figure 1). Note that $M_1(E) = 4/\pi$ and $M_2(E) = \sqrt{2}$.

It may be remarked that for any non-negative values of p, the equality $M_p(E) = M_p(z^n + c)$, holds whenever $|c| = 1$. Since $M_0(E) = 1$, the exact bound in the inequality (1.5) for the case $p = 0$ is quite interesting and significant, and stated below.

Corollary 1.4. *For any complex number* α,

$$\mathcal{K}_0(D_\alpha,\ \mathcal{P}_n) = n(1 + |\alpha|)$$

and the inequality (1.5) *for* $p = 0$ *and* $\pi_n = \mathcal{P}_n$, *turns into equality on polynomials* $c(z + a)^n$ *whenever* $|a| = 1$ *and* $\arg(a) = \arg(\alpha)$.

Apparently, on $\mathcal{P}_n$, the inequality (1.5) for $p = 0$ is sharp and the equality is attained for $P(z) = (1+z)^n$ when $\alpha = 0$ and for $P(z) = (1 + \overline{\alpha}z/|\alpha|)^n$ when $\alpha \neq 0$.

Since $M_\infty(E) = 2$, we obtain the inequality

$$M_\infty(D_\alpha P(z)) \leq n \max\{1, |\alpha|\} M_\infty(P) \tag{1.13}$$

that extends the inequality due to Aziz [3, p. 188] given by

$$M_\infty(D_\alpha P(z)) \leq n|\alpha| M_\infty(P) \quad \text{for} \ \ |\alpha| \geq 1$$

to all values of α. The inequality (1.13) is best possible for any complex number α. As mentioned earlier, the equality is attained for $P(z) = z^n$ whenever $|\alpha| \geq 1$. When $|\alpha| < 1$, the sequence of polynomials $P_k(z) = 1 + z^n/k^n$ shows that for $k \to \infty$ the π_n−constant cannot be replaced by a smaller number in this case.

Remark 1.5. It is quite natural to think about the lower estimate for $\mathcal{K}_p(D_\alpha,\ \mathcal{P}_n)$. We can conclude from the examples $z^n + c$ and $(z + c)^n$ that

$$\mathcal{K}_p(D_\alpha,\ \mathcal{P}_n) \geq n \max\left\{ \frac{M_p(\alpha z + 1)}{M_P(E)},\ \ \frac{(1 + |\alpha|)M_{(n-1)p}(E)}{M_{np}(E)} \right\}$$

for any complex values of α and $p \geq 0$.

As explained earlier, the exact constant $\mathcal{K}_p(D_\alpha,\ \mathcal{P}_n(1))$ of the inequality for the classical derivative for $\alpha = \infty$, and $0 \leq p \leq \infty$ (see [12], [5], [17] and [2]) is expressed by

$$\lim_{|\alpha|\to\infty} \frac{\mathcal{K}_p(D_\alpha,\ \mathcal{P}_n(1))}{|\alpha|} = \frac{n}{M_p(E)}$$

for $0 \leq p \leq \infty$. The inequality (1.5) on $\mathcal{P}_n(1)$ turns into equality on the polynomials $c(z^n + a)$, with $|a| = 1$, and in the case of $p = \infty$, on any arbitrary polynomial having all its zeros on the unit circle.

The exact inequality (1.5) for polynomials in $\mathcal{P}_n(1)$ for $|\alpha| \geq 1$ in the uniform norm is obtained by Aziz in 1988 [3]. For $|\alpha| < 1$ in the inequality (1.5) without any restrictions on the zeros of polynomials, the extremal polynomial has all its zeros at the point $z = \infty$ and therefore, the exact constant in this case coincides with the constant in the inequality without restrictions. As a result, the equality

$$\mathcal{K}_\infty(D_\alpha,\ \mathcal{P}_n(1)) = \frac{n}{2}(1 + \max\{1, |\alpha|\})$$

is true for all complex values of α. In this case the inequality (1.5) turns into equality for $|\alpha| \geq 1$ on the polynomials $z^n + c$ with $|c| = 1$, and for $|\alpha| \leq 1$ on constants.

Thus the inequality (1.8), and its various generalizations in general L^p settings have been proved for $|\alpha| \geq 1$ (see [7]) and the inequality (1.5) with the π_n- constant given in (1.9) is one such result and as mentioned earlier, the case $|\alpha| < 1$ is still open. So let us establish the consolidated bound for $M_p(D_\alpha P)$ for any values of α as follows.

Theorem 1.6. *For any $p \geq 0$ and any complex number α, we have*

$$\mathcal{K}_p(D_\alpha,\ \mathcal{P}_n(1)) \leq \begin{cases} \dfrac{n(1+|\alpha|)}{M_p(E)}, & \text{if } |\alpha| \geq 1, \\[2ex] \dfrac{n(1+|\alpha|)}{M_p(E_{1,\alpha})}, & \text{if } |\alpha| < 1, \end{cases}$$

where $E(z)$ is as defined in Theorem 1.3, and $E_{1,\alpha}(z) := 1 + \alpha z$. In the case $p = \infty$, the above bound for π_n-constant is best possible and equality in (1.5) on $\mathcal{P}_n(1)$ holds for $P(z) = 1 + z^n$ whenever $|\alpha| \geq 1$.

In fact, the case $p = \infty$ in the above inequality extends the inequality (1.8) to all complex values of α as given below.

Corollary 1.7. *For any complex number α, we have*

$$\mathcal{K}_\infty(D_\alpha,\ \mathcal{P}_n(1)) \leq \begin{cases} \dfrac{n}{2}(1+|\alpha|), & \text{if } |\alpha| \geq 1, \\[2ex] n, & \text{if } |\alpha| < 1. \end{cases}$$

This π_n-constant when $\pi_n = \mathcal{P}_n(1)$ is best possible and equality in this case holds in (1.5) for $P(z) = z^n + 1$ whenever $|\alpha| \geq 1$ and for $|\alpha| < 1$, the sequence of polynomials $P_k(z) = 1 + z^n/k^n$ shows that as $k \to \infty$, the bound n cannot be replaced by any smaller value.

The corresponding result for the case $p = 0$ is also quite interesting and given below.

Corollary 1.8. *For any complex values of α, the equality*

$$\mathcal{K}_0(D_\alpha,\ \mathcal{P}_n(1)) = \mathcal{K}_0(D_\alpha,\ \mathcal{P}_n) = n(1 + |\alpha|)$$

hold.

The above equality in Corollary 1.8 is a consequence of the fact that the inequality for polynomials without restrictions turns into equality for polynomials with zeros on the unit circle.

Remark 1.9. For any $P(z) \in \mathcal{P}_n(1)$ of degree n, the Mahler measure is $M_0(P) = |P(0)|$. Hence for $p = 0$, and $|\alpha| \geq 1$ the inequality (1.5) with π_n-constant given in Theorem 1.6 for $|\alpha| \geq 1$ may be rewritten as

$$\frac{1}{2\pi}\int_0^{2\pi} \log\left|D_\alpha P(e^{i\theta})\right|\, d\theta \,\leq\, \log\bigl(n(1+|\alpha|)\,|P(0)|\bigr).$$

Remark 1.10. As a factor depending on α but not on p, the term $1+|\alpha|$ in Theorem 1.6 is best possible. Indeed, equality is attained when $p=\infty$ and $P(z)=1+z^n$ with $|\alpha|\geq 1$. Also it may be a challenge to find the unique smallest number $\mathcal{K}_p(D_\alpha,\ \pi_n)$ depending on α and p that can replace the terms $1+|\alpha|$ given in Theorems 1.3 and 1.6.

Remark 1.11. The lower estimate for $\mathcal{K}_p(D_\alpha,\ \mathcal{P}_n(1))$ would also be useful and again from the examples z^n+c and $(z+c)^n$ we can arrive that

$$\mathcal{K}_p(D_\alpha,\ \mathcal{P}_n(1))\geq n\max\left\{\frac{M_p(\alpha z+1)}{M_P(E)},\ \ \frac{(1+|\alpha|)M_{(n-1)p}(E)}{M_{np}(E)}\right\}$$

for any complex values of α.

On the class of polynomials $\mathcal{P}_n(K),\quad K\geq 1$, the sharp π_n-constants in the inequality (1.5) for the classical derivative was given by M. A. Malik [14]. In view of this, Theorem 1.6 can be generalized to the class of polynomials having no zeros in a disc $|z|<K$, $K\geq 1$, as follows.

Theorem 1.12. *Let $K\geq 1$. Then for any $p\geq 0$, and for every complex number α, we have*

$$\mathcal{K}_p(D_\alpha,\ \mathcal{P}_n(K))\leq\begin{cases}\dfrac{n(K+|\alpha|)}{M_p(E_{K,1})}, & \text{if } |\alpha|\geq 1,\\[2ex] \dfrac{n(K+|\alpha|)}{M_p(E_{K,\alpha})}, & \text{if } |\alpha|<1,\end{cases}$$

where $E_{K,1}(z):=K+z$ and $E_{K,\alpha}(z):=K+\alpha z$. In the case $p=\infty$, the above π_n-constant and the inequality (1.5) *in this case is best possible and equality holds for the polynomial $(z+K)^n$ with real $\alpha\geq 1$, and $K\geq 1$.*

In the lines of the inequality (1.9), the above result can be further sharpened as follows.

Theorem 1.13. *If $P(z)$ is a polynomial of degree n and z_m are its zeros such that $|z_m|\geq K_m\geq 1,\ \ 1\leq m\leq n$, then for any $p\geq 0$, and for every complex number α,*

$$M_p(D_\alpha(P))\leq\begin{cases}\dfrac{n(t_0+|\alpha|)}{M_p(E_{t_0,1})}M_p(P), & \text{if } |\alpha|\geq 1,\\[2ex] \dfrac{n(t_0+|\alpha|)}{M_p(E_{t_0,\alpha})}M_p(P), & \text{if } |\alpha|<1,\end{cases}$$

where $E_{t_0,1}(z):=t_0+z$ and $E_{t_0,\alpha}(z):=t_0+\alpha z$ and $t_0=1+n\left(\sum_{m=1}^{n}(K_m-1)^{-1}\right)^{-1}$ if $K_m>1$ $(1\leq m\leq n)$, and $t_0=1$ if $K_m=1$ for some m, $1\leq m\leq n$.

In the case $p=\infty$, the above inequality is best possible if $K_m=K\geq 1, 1\leq m\leq n$, and equality holds for the polynomial $(z+K)^n$ with real $\alpha\geq 1$, and $K\geq 1$.

Remark 1.14. Since Theorem 1.13 contains Theorems 1.6 and 1.12, we prove only Theorem 1.13. Also observe that the case $|\alpha|\geq 1$ in Theorem 1.13 is already established as given in (1.9), and therefore we present the proof only for $|\alpha|<1$.

As mentioned earlier, surprisingly, the corresponding result for $p=\infty$ in Theorems 1.6, 1.12 and 1.13 for the case $|\alpha|<1$ are same, and they are independent of the location of zeros as given in Corollary 1.7. But it may be observed that for $|\alpha|\geq 1$, the results behave differently (see also [7]) as functions of α, and K or t_0 accordingly.

The next result which seems to be of independent interest, and it can be obtained using the above Theorem 1.13.

Theorem 1.15. *If $P(z)$ is a polynomial of degree n, and z_m are its zeros such that $0 < |z_m| \le K_m \le 1$, $1 \le m \le n$, then for any $p \ge 0$, and for every complex number α,*

$$M_p(D_\alpha(P)) \le \begin{cases} \dfrac{n|\alpha|(1+|\alpha|s_0)}{M_p(E_{\overline{\alpha}s_0,1})} M_p(P), & \text{if } |\alpha| > 1, \\ \dfrac{n(1+|\alpha|s_0)}{M_p(E_{s_0,1})} M_p(P), & \text{if } |\alpha| \le 1, \end{cases}$$

where $E_{\overline{\alpha}s_0,1}(z) := \overline{\alpha}s_0 + z$, and $s_0 = 1 + n\left(\sum_{m=1}^{n} K_m/(1-K_m)\right)^{-1}$ if $K_m < 1$ $(1 \le m \le n)$, and $s_0 = 1$ if $K_m = 1$ for some m, $1 \le m \le n$. Again in the case $p = \infty$, the above inequality is sharp in the case $K_m = K \le 1$, $1 \le m \le n$ and equality holds for the polynomial $(z+K)^n$ with non-negative real $\alpha \le 1$ and $K \le 1$.

Remark 1.16. For the case $|\alpha| \le 1$, Theorem 1.15 is already established by Govil and Kumar [8], and therefore we will be proving only the case $|\alpha| > 1$.

Before closing this section, let us make a remark on lower bound for $M_\infty(D_\alpha P)$ in Corollary 1.7 whenever $|\alpha| < 1$.

From the definition of the polar derivative of a polynomial $P(z)$ of degree n, it follows that, for $|\alpha| < 1$, and for any real values of θ,

$$|D_\alpha P(e^{i\theta})| \ge |nP(e^{i\theta})| - (1+|\alpha|)|P'(e^{i\theta})|$$

and therefore

$$M_\infty(D_\alpha P) \ge nM_\infty(P) - (1+|\alpha|)M_\infty(P').$$

If $P(z)$ has no zeros in $|z| < 1$ then the inequality (1.2) applies here and therefore we get

$$M_\infty(D_\alpha P) \ge \frac{n}{2}(1-|\alpha|)M_\infty(P),$$

which brings us to the following result.

Proposition 1.17. *If $P(z)$ is a polynomial of degree n having all its zeros outside the open unit disc, then for any α with $|\alpha| < 1$, we have*

$$M_\infty(D_\alpha P)) \ge \frac{n}{2}(1-|\alpha|)M_\infty(P). \tag{1.14}$$

The result is sharp and equality holds in the above inequality whenever $\alpha = 0$ for the polynomial $z^n + 1$.

It may be observed that when $\alpha = 0$, the above inequality (1.14) gives the well known Turán's inequality [19], a generalization of which was recently proved by Kumar [10].

2. Lemmas

We need the following lemmas to prove our results. First lemma is a simple exercise [8].

Lemma 2.1. *Let $z_1, z_2 \in \mathbb{C}$ and $0 \le \alpha \le 2\pi$. Then for any $p \ge 0$, we have*

$$\int_0^{2\pi} \left| z_1 + z_2 e^{i\alpha} \right|^p d\alpha = \int_0^{2\pi} \left| |z_1| + |z_2| e^{i\alpha} \right|^p d\alpha.$$

Next lemma is contained in the paper due to de Bruijn [5, inequality (19)], but we present a different proof over here.

Lemma 2.2. *Let $A, B \in \mathbb{C}$. Then for any real values of γ, and $p \geq 0$*

$$\int_0^{2\pi} \left|A + Be^{i\gamma}\right|^p d\gamma \ \geq \ 2\pi \max\{|A|^p, |B|^p \ \}.$$

Proof. In view of the Lemma 2.1, it suffices to prove Lemma 2.2 for positive values of A and B. Without loss of the generality, assume that $A \geq B > 0$, and set $r = B/A \in (0,1]$. Then consider the function

$$f(p) = \int_0^{2\pi} |1 + re^{i\theta}|^p \, \mathrm{d}\theta.$$

Since the map $p \mapsto |1+re^{i\theta}|^p$ is convex for each choice of r and θ, it follows that $f(p)$ is also convex. Moreover,

$$f'(0) = \mathrm{Re}\left[\int_0^{2\pi} \log(1 + re^{i\theta}) \, \mathrm{d}\theta\right] = \mathrm{Re}\left[\int_{|z|=r} \frac{\log(1+z)}{iz} \, \mathrm{d}z\right] = 0.$$

Hence $f(0) = 2\pi$ is the global minimum of $f(p)$ or equivalently

$$f(p) \geq f(0) = 2\pi,$$

which completes the proof. $\square$

The following lemma is motivated by a proof in [16, p. 554].

Lemma 2.3. *Let $P(z) \in \mathcal{P}_n(1)$ be of degree n. Then for any $\gamma \in \mathbb{R}$, we have*

$$zP'(z) + e^{i\gamma}\big(nP(z) - zP'(z)\big) \in \mathcal{P}_n(1). \tag{2.1}$$

Proof. Suppose that $P(z) = c\prod_{\mu=1}^{m}(z - z_\mu)$, where $c \neq 0$, $m \leq n$ and $|z_\mu| \geq 1$ for $\mu = 1, \ldots, m$. For any given z and ζ in the open unit disk, we have

$$n - (z - \zeta)\,\frac{P'(z)}{P(z)} \ = \ \sum_{\nu=1}^{n} A_\nu,$$

where

$$A_\nu \ := \begin{cases} \dfrac{\zeta - z_\nu}{z - z_\nu}, & \text{for } \nu = 1, \ldots, m, \\ 1, & \text{for } \nu = m+1, \ldots, n. \end{cases}$$

Now consider the image of $K := \{w \in \mathbb{C} \ : \ |w| \geq 1\} \cup \{\infty\}$ under the Möbius transform

$$\psi \ : \ w \mapsto \frac{\zeta - w}{z - w}\,.$$

Since $\psi(\infty) = 1$ and $\psi(w) \notin \{0, \infty\}$ for $w \in K$, we conclude that $\psi(K)$ is a closed disk that contains all the numbers $A_1, \ldots, A_n$ but not the point 0. Hence, by a convexity argument,

$$0 \neq \frac{1}{n}\sum_{\nu=1}^{n} A_\nu \ \in \ \psi(K),$$

which implies that $nP(z) - (z - \zeta)P'(z) \neq 0$. This holding for any $|z| < 1$ and $|\zeta| < 1$, it also holds for $|z| < 1$ and $\zeta = ze^{-i\gamma}$. Thus, (2.1) holds true. $\square$

Now we shall employ a rather deep result of Arestov [1]. First we introduce a notation. For $\boldsymbol{l} = (l_0, \ldots, l_n) \in \mathbb{C}^{n+1}$, let $\Lambda_{\boldsymbol{l}}$ be the linear operator that associates with each polynomial $P(z) = \sum_{k=0}^{n} a_k z^k$ of degree at most n the polynomial

$$\left(\Lambda_{\boldsymbol{l}} P\right)(z) := \sum_{k=0}^{n} l_k a_k z^k .$$

Definition 2.4. We call the operator $\Lambda_{\boldsymbol{l}}$ *admissible* if it has the following property: whenever $P(z)$ is a polynomial of degree at most n not vanishing for $|z| < 1$, then either $(\Lambda_{\boldsymbol{l}} P)(z) \equiv 0$ or $(\Lambda_{\boldsymbol{l}} P)(z) \neq 0$ for $|z| < 1$.

The following lemma is part of a result of Arestov [1, Theorem 4]; also see [16, Theorem 13.2.11].

Lemma 2.5. *Let $\phi(t) = \psi(\log t)$, where ψ is a convex nondecreasing function on $\mathbb{R}$. Then for any $P(z) \in \mathcal{P}_n$ and any admissible operator $\Lambda_{\boldsymbol{l}}$, we have*

$$\int_0^{2\pi} \phi\left(\left|\left(\Lambda_{\boldsymbol{l}} P\right)(e^{i\theta})\right|\right) d\theta \leq \int_0^{2\pi} \phi\left(c(\boldsymbol{l}, n)\left|P(e^{i\theta})\right|\right) d\theta, \tag{2.2}$$

where $c(\boldsymbol{l}, n) = \max(|l_0|, |l_n|)$.

Lemma 2.6. *Let a, b be any complex numbers and suppose that $|a| \geq |b| > 0$. Then for any $p > 0$, the function*

$$f(x) := \int_0^{2\pi} |a + xb\, e^{i\gamma}|^p \, d\gamma \tag{2.3}$$

is increasing on $(0, 1)$ and $f(x) < f(1)$ for all $x \in (0, 1)$.

Proof. In view of Lemma 2.1, it suffices to prove (2.3) for non-negative values of a, b with $a \geq b > 0$.

In fact the result can be verified using subordination property. Since the function $g(z) = a + bz$ is analytic in the open unit disc $|z| < 1$, the function $g_p(z) = |g(z)|^p$ is subharmonic in the open unit disc for any $p > 0$. Therefore

$$f(x) := \int_0^{2\pi} g_p(xe^{i\theta}) \, d\theta$$

is an increasing function of $x \in (0, 1)$, which establishes (2.3).

But let us present an alternative proof in detail. The function $(1+z)^{p/2}$ has an analytic branch $\phi(z)$ in the open unit disc such that $\phi(x) > 0$ for all $x \in (-1,\ 1)$. This branch has the binomial series as power series expansion given by

$$\phi(z) = \sum_{j=0}^{\infty} \binom{p/2}{j} z^j,$$

where

$$\binom{p/2}{j} = \frac{1}{j!} \cdot \frac{p}{2}\left(\frac{p}{2} - 1\right) \cdots \left(\frac{p}{2} - (j-1)\right).$$

Note that the series converges absolutely for $|z| < 1$. Using this function $\phi(z)$ and and its expansion, we can write the integrand in (2.3) as

$$\begin{aligned}|a + xbe^{i\gamma}|^p &= a^p \phi\left(\frac{xb}{a}e^{i\gamma}\right)\phi\left(\frac{xb}{a}e^{-i\gamma}\right)\\ &= a^p \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\binom{p/2}{j}\binom{p/2}{k}\left(\frac{xb}{a}\right)^{j+k} e^{i\gamma(j-k)}.\end{aligned}$$

Since the series is absolutely convergent, we can perform the integration termwise with respect to γ over $[0,\ 2\pi]$. Observe that for the terms with $j \neq k$, the integral vanishes and thus we obtain

$$f(x) = a^p \sum_{j=0}^{\infty}\binom{p/2}{j}^2\left(\frac{xb}{a}\right)^{2j}.$$

This shows that $f(x_1) < f(x_2)$ for $0 < x_1 < x_2 < 1$, which proves the first part. The second part on the comparison of $f(x)$ with $f(1)$ follows from the continuity. $\square$

Lemma 2.7. *Let a, b be any two positive real numbers and $l \geq 1$ such that $a \geq bl$. Suppose γ is any real such that $0 \leq \gamma \leq 2\pi$. Then*

$$|l + me^{i\gamma}|(a + mb) \leq (l + m)|a + e^{i\gamma}mb|$$

for any $0 \leq m < 1$.

Proof. We need to show that

$$\frac{a + mb}{l + m} \leq \left|\frac{a + e^{i\gamma}mb}{l + e^{i\gamma}m}\right|,$$

which is equivalent to

$$b + \frac{a - bl}{l + m} \leq \left|b + \frac{a - bl}{l + e^{i\gamma}m}\right|.$$

Observe that $\mathrm{Re}\left(1/(l + e^{i\gamma}m)\right)$ is an increasing function of γ which increases from $1/(l+m)$ to $1/(l-m)$ in $[0, \pi]$ and decreases in the reverse order in $[\pi, 2\pi]$. Therefore, $\mathrm{Re}\left(1/(l + e^{i\gamma}m)\right) \geq 1/(l + m)$ for any real value of γ, and therefore

$$b + \frac{a - bl}{l + m} \leq \mathrm{Re}\left(b + \frac{a - bl}{l + e^{i\gamma}m}\right) \leq \left|b + \frac{a - bl}{l + e^{i\gamma}m}\right|.$$

Hence the proof is complete. $\square$

Finally we need the following result due to Gardner and Govil [6].

Lemma 2.8. *If $P(z) = a_n \prod_{m=1}^{n}(z - z_m) \in \mathcal{P}_n(1)$ is of degree n such that $|z_m| \geq K_m \geq 1$, $1 \leq m \leq n$, then on $|z| = 1$,*

$$t_0|P'(z)| \leq |nP(z) - zP'(z)|,$$

where t_0 is as given in Theorem 1.13.

3. Proofs of Theorems

Proof of Theorem 1.3. Let $R = \max\{|\alpha|, 1\}$. For $p > 0$, we consider the term

$$T := \int_0^{2\pi} \left|(1+e^{i\gamma})\right|^p d\gamma \int_0^{2\pi} \left|D_\alpha\{P(e^{i\theta})\}\right|^p d\theta. \tag{3.1}$$

Obviously it may be rewritten and estimated as

$$\begin{aligned}
T &= \int_0^{2\pi}\int_0^{2\pi} \left|(1+e^{i\gamma})\right|^p \left|D_\alpha\{P(e^{i\theta})\}\right|^p \, d\gamma \, d\theta \\
&= \int_0^{2\pi}\int_0^{2\pi} \left|(1+e^{i\gamma})\right|^p \left|nP(e^{i\theta}) - e^{i\theta}P'(e^{i\theta}) + \alpha P'(e^{i\theta})\right|^p \, d\gamma \, d\theta \\
&\leq \int_0^{2\pi}\int_0^{2\pi} \left|(1+e^{i\gamma})\right|^p \Big[\left|nP(e^{i\theta}) - e^{i\theta}P'(e^{i\theta})\right| + \left|\alpha P'(e^{i\theta})\right|\Big]^p d\gamma \, d\theta \\
&\leq R^p \int_0^{2\pi}\int_0^{2\pi} \Big|(1+e^{i\gamma})\Big[\left|nP(e^{i\theta}) - e^{i\theta}P'(e^{i\theta})\right| + \left|P'(e^{i\theta})\right|\Big]\Big|^p \, d\gamma \, d\theta \\
&\leq (2R)^p \int_0^{2\pi}\int_0^{2\pi} \Big|\Big[\left|P'(e^{i\theta})\right| + e^{i\gamma}\left|nP(e^{i\theta}) - e^{i\theta}P'(e^{i\theta})\right|\Big]\Big|^p \, d\gamma \, d\theta \\
&= (2R)^p \int_0^{2\pi}\int_0^{2\pi} \left|\left|e^{i\theta}P'(e^{i\theta})\right| + e^{i\gamma}\left|nP(e^{i\theta}) - e^{i\theta}P'(e^{i\theta})\right|\right|^p \, d\gamma \, d\theta, \qquad (3.2)
\end{aligned}$$

where we have used the fact

$$\Big|\left|P'(e^{i\theta})\right| + e^{i\gamma}\left|nP(e^{i\theta}) - e^{i\theta}P'(e^{i\theta})\right|\Big| = \Big|\left|nP(e^{i\theta}) - e^{i\theta}P'(e^{i\theta})\right| + e^{i\gamma}\left|P'(e^{i\theta})\right|\Big|. \tag{3.3}$$

Now employing Lemma 2.1 to the integral with respect to γ in (3.2) and interchanging the order of integration afterwards, we find that

$$T \leq (2R)^p \int_0^{2\pi}\int_0^{2\pi} \left|e^{i\theta}P'(e^{i\theta}) + e^{i\gamma}\big(nP(e^{i\theta}) - e^{i\theta}P'(e^{i\theta})\big)\right|^p \, d\theta \, d\gamma. \tag{3.4}$$

It is easily verified that the linear mapping

$$P(z) \mapsto zP'(z) + e^{i\gamma}\big(nP(z) - zP'(z)\big)$$

is an operator $\Lambda_{\boldsymbol{l}}$ with the vector $\boldsymbol{l} \in \mathbb{C}^{n+1}$ given by

$$l_k := k\big(1 - e^{i\gamma}\big) + ne^{i\gamma} \qquad (k = 0, 1, \ldots, n).$$

By Lemma 2.3, this operator is admissible in the sense of Definition 2.4. Furthermore, for $t > 0$ and $p > 0$, we have $t^p = \psi(\log t)$, where ψ is a nondecreasing, convex function on $\mathbb{R}$. Hence Lemma 2.5 applies to the inner integral in (3.4). Since $\max(|l_0|, |l_n|) = n$, we obtain

$$\int_0^{2\pi} \left|e^{i\theta}P'(e^{i\theta}) + e^{i\gamma}\big(nP(e^{i\theta}) - e^{i\theta}P'(e^{i\theta})\big)\right|^p d\theta \leq n^p \int_0^{2\pi} \left|P(e^{i\theta})\right|^p d\theta. \tag{3.5}$$

Finally, integrating both sides with respect to γ over $[0, 2\pi]$ and combining the resulting inequality with (3.4) and the definition of T in (3.1), we arrive at

$$\int_0^{2\pi} |1 + e^{i\gamma}|^p \, d\gamma \int_0^{2\pi} \left|D_\alpha\{P(e^{i\theta})\}\right|^p d\theta \;\le\; (2R)^p 2\pi n^p \int_0^{2\pi} \left|P(e^{i\theta})\right|^p d\theta,$$

which proves one part of the requirement with the fact that $R = |\alpha|$ if $|\alpha| \ge 1$, and $R = 1$ if $|\alpha| < 1$ by obtaining the inequality (1.5), where

$$\mathcal{K}_p(D_\alpha,\ \mathcal{P}_n) = \begin{cases} \dfrac{2n|\alpha|}{M_p(E)}, & \text{if } |\alpha| \ge 1, \\ \dfrac{2n}{M_p(E)}, & \text{if } |\alpha| < 1, \end{cases}$$

and $E(z) = 1 + z$.

Let us establish the remaining part of our claim. For any $p > 0$, and using the inequality (3.5), we get

$$\begin{aligned} S = n^p(1+|\alpha|)^p \int_0^{2\pi} |P(e^{i\theta})|^p d\theta &= \frac{1}{2\pi} n^p (1+|\alpha|)^p \int_0^{2\pi}\!\!\int_0^{2\pi} |P(e^{i\theta})|^p \, d\theta \, d\gamma \\ &\ge \frac{1}{2\pi}(1+|\alpha|)^p \int_0^{2\pi}\!\!\int_0^{2\pi} \left|e^{i\theta}P'(e^{i\theta}) + e^{i\gamma}(nP(e^{i\theta}) - e^{i\theta}P'(e^{i\theta})\right|^p d\theta \, d\gamma \end{aligned}$$

for any γ with $0 \le \gamma \le 2\pi$. Therefore, using Lemma 2.1, we have

$$\begin{aligned} S &\ge \frac{1}{2\pi}(1+|\alpha|)^p \int_0^{2\pi}\!\!\int_0^{2\pi} \left||P'(e^{i\theta})| + e^{i\gamma}|nP(e^{i\theta}) - e^{i\theta}P'(e^{i\theta})|\right|^p d\gamma \, d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi}\!\!\int_0^{2\pi} \Big[\left||P'(e^{i\theta})| + e^{i\gamma}|nP(e^{i\theta}) - e^{i\theta}P'(e^{i\theta})|\right| \\ &\qquad\qquad + |\alpha| \left||nP(e^{i\theta}) - e^{i\theta}P'(e^{i\theta})| + e^{i\gamma}|P'(e^{i\theta})|\right|\Big]^p d\theta \, d\gamma, \end{aligned}$$

where we used again the property given in (3.3). Therefore using the triangle inequality and then by doing some rearrangements, we will have

$$\begin{aligned} S \;\ge\; \frac{1}{2\pi} \int_0^{2\pi}\!\!\int_0^{2\pi} \Big| &|P'(e^{i\theta})| + |\alpha||nP(e^{i\theta}) - e^{i\theta}P'(e^{i\theta})| \\ &+ e^{i\gamma}\left(|nP(e^{i\theta}) - e^{i\theta}P'(e^{i\theta})| + |\alpha||P'(e^{i\theta})|\right)\Big|^p d\gamma \, d\theta. \end{aligned}$$

Now, applying Lemma 2.2, we get

$$\begin{aligned} S \;&\ge \int_0^{2\pi} \left||nP(e^{i\theta}) - e^{i\theta}P'(e^{i\theta})| + |\alpha P'(e^{i\theta})|\right|^p d\theta \\ &\ge \int_0^{2\pi} \left|nP(e^{i\theta}) - e^{i\theta}P'(e^{i\theta}) + \alpha P'(e^{i\theta})\right|^p d\theta, \end{aligned}$$

and therefore we have

$$S \ge \int_0^{2\pi} \left|D_\alpha P(e^{i\theta})\right|^p d\theta.$$

Now raising to the power $1/p$ on both sides of the above inequality we get

$$M_p(D_\alpha P) \le n(1+|\alpha|)M_p(P). \tag{3.6}$$

Thus, the second estimate for π_n-constant in this case is

$$\mathcal{K}_p(D_\alpha,\ \mathcal{P}_n) \le n(1+|\alpha|).$$

The bound $n(1+|\alpha|)$ in (3.6) was first observed by Aziz and Rather [4]. Combining the above two estimates of $\mathcal{K}_p(D_\alpha,\ \mathcal{P}_n)$ we get the desired bound for $p>0$. The cases $p=0$ and $p=\infty$ are deduced by a limiting process. Thus the proof is complete. □

Proof of Theorem 1.13. We have for any $p>0$ and $|\alpha|<1$,

$$\begin{aligned}
&\int_0^{2\pi} \big|D_\alpha\big[P(e^{i\theta})\big]\big|^p\,d\theta \int_0^{2\pi} |t_0+\alpha e^{i\gamma}|^p\,d\gamma\\
&\quad= \int_0^{2\pi}\int_0^{2\pi} |t_0+\alpha e^{i\gamma}|^p\big|D_\alpha\big[P(e^{i\theta})\big]\big|^p\,d\theta\,d\gamma\\
&\quad= \int_0^{2\pi}\int_0^{2\pi} |t_0+\alpha e^{i\gamma}|^p\big|nP(e^{i\theta})-e^{i\theta}P'(e^{i\theta})+\alpha P'(e^{i\theta})\big|^p\,d\theta\,d\gamma\\
&\quad\le \int_0^{2\pi}\int_0^{2\pi} |t_0+|\alpha|e^{i\gamma}|^p\,\big[|nP(e^{i\theta})-e^{i\theta}P'(e^{i\theta})|+|\alpha|\,|P'(e^{i\theta})|\big]^p\,d\gamma\,d\theta,
\end{aligned}$$

where we used Lemma 2.1. Therefore

$$\begin{aligned}
&\int_0^{2\pi} \big|D_\alpha\big[P(e^{i\theta})\big]\big|^p\,d\theta \int_0^{2\pi} |t_0+\alpha e^{i\gamma}|^p\,d\gamma\\
&\quad\le \int_0^{2\pi}\int_0^{2\pi} \Big|(t_0+|\alpha|e^{i\gamma})\,\big[|nP(e^{i\theta})-e^{i\theta}P'(e^{i\theta})|+|\alpha||P'(e^{i\theta})|\big]\Big|^p\,d\gamma\,d\theta. \quad (3.7)
\end{aligned}$$

Since the zeros z_m of $P(z)$ satisfy $|z_m|\ge K_m\ge 1$, it follows by Lemma 2.8 that

$$t_0|P'(e^{i\theta})| \le |nP(e^{i\theta})-e^{i\theta}P'(e^{i\theta})|.$$

The above by using Lemma 2.7, we have for every complex number α with $|\alpha|<1$,

$$\begin{aligned}
&\big[|nP(e^{i\theta})-e^{i\theta}P'(e^{i\theta})|+|\alpha||P'(e^{i\theta})|\big]|t_0+|\alpha|e^{i\gamma}|\\
&\qquad\le (t_0+|\alpha|)\,\big||nP(e^{i\theta})-e^{i\theta}P'(e^{i\theta})|+e^{i\gamma}|\alpha||P'(e^{i\theta})|\big|. \quad (3.8)
\end{aligned}$$

Now, if we use the above inequality (3.8) in (3.7), and again using Lemma 2.1, we get

$$\begin{aligned}
&\int_0^{2\pi} \big|D_\alpha\big[P(e^{i\theta})\big]\big|^p\,d\theta \int_0^{2\pi} |t_0+\alpha e^{i\gamma}|^p d\gamma\\
&\quad\le (|\alpha|+t_0)^p\int_0^{2\pi}\int_0^{2\pi} \big||nP(e^{i\theta})-e^{i\theta}P'(e^{i\theta})|+e^{i\gamma}|\alpha||P'(e^{i\theta})|\big|^p\,d\gamma\,d\theta.
\end{aligned}$$

By applying Lemma 2.6 to the inner integral above and, then using the inequality (3.3) and Lemma 2.1 we get

$$\begin{aligned}
&\int_0^{2\pi} \big|D_\alpha\big[P(e^{i\theta})\big]\big|^p\,d\theta \int_0^{2\pi} |t_0+\alpha e^{i\gamma}|^p d\gamma\\
&\quad\le (|\alpha|+t_0)^p\int_0^{2\pi}\int_0^{2\pi} \big|e^{i\theta}P'(e^{i\theta})+e^{i\gamma}\big(nP(e^{i\theta})-e^{i\theta}P'(e^{i\theta})\big)\big|^p\,d\gamma\,d\theta. \quad (3.9)
\end{aligned}$$

Again using the inequality that appeared in the proof of Theorem 1.3 given by

$$\int_0^{2\pi} \left|e^{i\theta}P'(e^{i\theta}) + e^{i\gamma}\big(nP(e^{i\theta}) - e^{i\theta}P'(e^{i\theta})\big)\right|^p \, d\theta \,\leq\, n^p \int_0^{2\pi} \left|P(e^{i\theta})\right|^p \, d\theta$$

in (3.9), we obtain

$$\int_0^{2\pi} \left|D_\alpha\big[P(e^{i\theta})\big]\right|^p d\theta \int_0^{2\pi} |t_0+\alpha e^{i\gamma}|^p d\gamma \leq (|\alpha|+t_0)^p 2\pi n^p \int_0^{2\pi} |P(e^{i\theta})|^p d\theta, \quad (3.10)$$

and the desired inequality can now be obtained by raising to the power $1/p$ on both sides of (3.10), and then doing some rearrangement of terms. Again the cases $p=0$ and $p=\infty$ are deduced by a limiting process. Thus the proof is complete. □

Proof of Theorem 1.15. This result is a consequence of Theorem 1.12 and Proposition 1.1. For the sake of completeness, we present the proof over here. Since $P(z)$ has all its zeros z_m such that $0 < |z_m| \leq K_m$, $K_m \leq 1$, the polynomial $\overline{I}P(z) = Q(z) = z^n\overline{P(1/\overline{z})}$ has zeros $1/\overline{z_m}$ with $|1/z_m| \geq 1/K_m \geq 1$. Therefore, applying Theorem 1.13 to the polynomial $Q(z)$, we get for any complex number $|\alpha| < 1$ and $p \geq 0$,

$$M_p(D_\alpha Q) \,\leq\, \frac{n(|\alpha| + s_0)}{M_p(E_{s_0,\alpha})} M_p(Q).$$

If $|\alpha| > 1$, then $1/|\alpha| < 1$, and hence by replacing α by $1/\overline{\alpha}$ in the above inequality, we get for $|\alpha| > 1$,

$$M_p(D_{1/\overline{\alpha}}\, Q) \,\leq\, \frac{n(s_0, |1/\alpha|)}{M_p(E_{1/\overline{\alpha},s_0})} M_p(Q).$$

Using the Proposition 1.1 with operator $\overline{I}$ in the above inequality we get

$$M_p(D_\alpha P) \,\leq\, \frac{n|\alpha|(|1 + |\alpha|s_0)}{M_p(E_{1,\overline{\alpha}s_0})} M_p(Q).$$

Since $|Q(e^{i\theta})| = |P(e^{i\theta})|$, the above inequality reduces to desired inequality, and therefore the proof is complete. □

Acknowledgments. The authors are deeply grateful to the referees for their valuable comments and constructive suggestions for improvements of this paper and its better presentation. The second author was supported in part by the Serbian Academy of Sciences and Arts (Φ-96).

References

1. V. V. Arestov, On integral inequalities for trigonometric polynomials and their derivatives, *Izv. Akad. Nauk SSSR Ser. Math.*, **45** (1981), 3–22 (in Russian); English translation: *Math. USSR-Izv.*, **18** (1982), 1–17.
2. V. V. Arestov, Integral inequalities for algebraic polynomials with a restriction on their zeros, *Anal. Math.*, **17** (1991), 11–20.
3. A. Aziz, Inequalities for the polar derivative of a polynomial, *J. Approx. Theory* **55** (1988), 183–193.
4. A. Aziz and N. A. Rather, On an inequality concerning the polar derivative of a polynomial, *Proc. Indian Acad. Sci. (Math. Sci.)*, **117** (2007), 349–357.
5. N. G. de Bruijn, Inequalities concerning polynomials in the complex domain, *Nederl. Akad. Wetensch. Proc. Ser. A*, **50** (1947), 1265–1272.
6. R. B. Gardner and N. K. Govil, Inequalities concerning the L^p norm of a polynomial and its derivative, *J. Math. Anal. Appl.*, **179** (1993), 208–213.

7. R. B. Gardner, N. K. Govil and G. V. Milovanović, *Extremal problems and inequalities of Markov-Bernstein type for algebraic polynomial,* Academic Press/Elsevier (London, 2022).
8. N. K. Govil and P. Kumar, On L^p inequalities involving polar derivative of a polynomial, *Acta Math. Hungar.*, **152** (2017), 130–139.
9. G. H. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities,* Cambridge University Press (Cambridge, 1934).
10. P. Kumar, On the inequalities concerning polynomials, *Complex Anal. Oper. Theory*, **14** (2020), no. 6, Paper No. 65, 11 pp.; DOI:10.1007/s11785-020-01023-0.
11. P. Kumar, On the growth of polynomials, *Acta Math. Hungar.*, **168** (2022), 443–453.
12. P. D. Lax, Proof of a conjecture due to Erdös on the derivative of a polynomial, *Bull. Amer. Math. Soc.*, **50** (1944), 509–513.
13. K. Mahler, An application of Jensen's formula to polynomials, *Mathematica*, **7** (1960), 98–100.
14. M. A. Malik, On the derivative of a polynomial, *J. London Math. Soc.*, (2) **1** (1969), 57–60.
15. G. V. Milovanović, D. S. Mitrinović and Th. M. Rassias, *Topics in Polynomials: Extremal Problems, Inequalities, Zeros,* World Scientific Publishing Co. Inc. (River Edge, NJ, 1994).
16. Q. I. Rahman and G. Schmeisser, *Analytic Theory of Polynomials,* London Math. Soc. Monogr. (N.S.), 26, The Clarendon Press, Oxford University Press (Oxford, 2002).
17. Q. I. Rahman and G. Schmeisser, L^p inequalities for polynomials, *J. Appox. Theory*, **53** (1988), 26–32.
18. A. C. Schaeffer, Inequalities of A. Markov and S. Bernstein for polynomials and related functions, *Bull. Amer. Math. Soc.*, **47** (1941), 565–579.
19. P. Turán, Über die Ableitung von Polynomen, *Compositio Mathematica*, **7** (1939), 89–95.
20. A. Zygmund, A remark on conjugate series, *Proc. London Math. Soc.*, **34** (1932), 392–400.

[1] Ural Federal University, Yekaterinburg, 620000 Russia
[2]Krasovskii Institute of Mathematics and Mechanics of the Ural Branch of the Russian Academy of Sciences, Yekaterinburg, 620108 Russia
email: RRAkopyan@mephi.ru

[3] Department of Mathematics, Birla Institute of Technology and Science Pilani K K Birla Goa Campus, Goa, India 403726
email: prasannak@goa.bits-pilani.ac.in

[4]Serbian Academy of Sciences and Arts, 11000 Beograd, Serbia
[5]University of Niš, Faculty of Sciences and Mathematics, 18000 Niš, Serbia
email: gvm@mi.sanu.ac.rs