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Extension of Mathieu series and alternating Mathieu series involving the Neumann function Y_ν

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Abstract
The main objective of this paper is to present a new extension of the familiar Mathieu series and the alternating Mathieu series $S(r)$ and $\widetilde{S}(r)$ which are denoted by $\mathbb{S}_{\mu,\nu}(r)$ and $\widetilde{\mathbb{S}}_{\mu,\nu}(r)$, respectively. The computable series expansions of their related integral representations are obtained in terms of the exponential integral E_1, and convergence rate discussion is provided for the associated series expansions. Further, for the series $\mathbb{S}_{\mu,\nu}(r)$ and $\widetilde{\mathbb{S}}_{\mu,\nu}(r)$, related expansions are presented in terms of the Riemann Zeta function and the Dirichlet Eta function, also their series built in Gauss' ${}_2F_1$ functions and the associated Legendre function of the second kind Q_μ^ν are given. Our discussion also includes the extended versions of the complete Butzer–Flocke–Hauss Omega functions. Finally, functional bounding inequalities are derived for the investigated extensions of Mathieu-type series.

Keywords Mathieu and alternating Mathieu series · Neumann function Y_ν · Euler–Abel transformation of series · Exponential integral E_1 · Gubler–Weber formula · Associated Legendre function of second kind · Riemann Zeta function · Dirichlet Eta function · Polylogarithm · Complete Butzer–Flocke–Hauss Ω function · Functional bounding inequality

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1 Introduction and preliminaries

During the study of elasticity of solid bodies, Émile Leonard Mathieu (1835–1890) introduced and investigated the famous infinite functional series called Mathieu series of the form [20]

$$S(r) = \sum_{n\geq 1} \frac{2n}{(n^2+r^2)^2}, \qquad r > 0.$$

The alternating version of Mathieu series, introduced and investigated by Pogány et al. in [27, p. 72, Eq. (2.7)], is

$$\widetilde{S}(r) = \sum_{n\geq 1} (-1)^{n-1} \frac{2n}{(n^2+r^2)^2}, \qquad r > 0.$$

Elegant integral forms of the Mathieu series $S(r)$ and the alternating Mathieu series $\widetilde{S}(r)$ was established by Emersleben [13]:

$$S(r) = \frac{1}{r} \int_0^\infty \frac{x \sin(rx)}{\mathrm{e}^x - 1}\, \mathrm{d}x, \tag{1.1}$$

and Pogány et al. [27, p. 72, Eq. (2.8)]:

$$\widetilde{S}(r) = \frac{1}{r} \int_0^\infty \frac{x \sin(rx)}{\mathrm{e}^x + 1}\, \mathrm{d}x. \tag{1.2}$$

Milovanović and Pogány [22] discovered other integral forms for Mathieu and alternating Mathieu series; Tomovski and Pogány [29] deduced Cauchy principal value integrals for these series; moreover, see [7, 9, 12] for this integral form, and [8, 25, 26] for another similarly focused study. The present authors studied and investigated a multi-parameter extension of the well-known Mathieu series and the alternating Mathieu series in a recent paper [24].

We emphasize the integral representations [22, pp. 185–186, Corollary 2.2]

$$S(r) = \pi \int_0^\infty \frac{r^2 - x^2 + \frac{1}{4}}{\left(x^2 - r^2 + \frac{1}{4}\right)^2 + r^2} \frac{\mathrm{d}x}{\cosh^2(\pi x)}, \tag{1.3}$$

$$\widetilde{S}(r) = \pi \int_0^\infty \frac{x}{\left(x^2 - r^2 + \frac{1}{4}\right)^2 + r^2} \frac{\sinh(\pi x)\, \mathrm{d}x}{\cosh^2(\pi x)}, \tag{1.4}$$

which will have a special treatment below.

Let $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{C}$ be the sets of positive integers, integers, and complex numbers, respectively. The Bessel function of the first kind of the order ν is defined by

$$J_\nu(z) = \sum_{k\geq 0} \frac{(-1)^k \left(\frac{z}{2}\right)^{\nu+2k}}{k!\, \Gamma(\nu+k+1)}, \qquad -z \notin \mathbb{N};\ \nu \in \mathbb{C}, \tag{1.5}$$

where the principal branch of $J_\nu(z)$ should be considered (it corresponds to the principal value of z^ν) and $J_\nu(z)$ is analytic in the z-plane cut along the interval $(-\infty, 0]$. Moreover, for $\nu \in \mathbb{Z}$, the Bessel function of the first kind is entire in z on the whole complex plane, see [15, p. 5].

The Bessel function of the second kind (Neumann function or Weber–Bessel function) of order ν is expressible in terms of the Bessel function of the first kind defined as [30, p. 64)]

$$Y_\nu(z) = \frac{\cos(\nu\pi)\, J_\nu(z) - J_{-\nu}(z)}{\sin(\nu\pi)} = \cot(\nu\pi)\, J_\nu(z) - \csc(\nu\pi)\, J_{-\nu}(z), \qquad \nu \notin \mathbb{Z}. \tag{1.6}$$

Also, Bessel functions of half-integer order have the connection or recurrence formula [17, p. 925, Eq. (8.465)]

$$Y_{n+\frac{1}{2}}(z) = (-1)^{n-1} J_{-n-\frac{1}{2}}(z)\,.$$

On the other hand, see [23, p. 228, Eq. (10.16.1)],

$$J_{\frac{1}{2}}(z) = Y_{-\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}}\,\sin(z)\,.$$

We can realize the extension of the Mathieu series by considering the related integral representation extending the integrand by a weight function. Namely, rewrite (1.1) into the form

$$S(r) = \sqrt{\frac{\pi}{2r}}\int_0^\infty \frac{x^{3/2}}{e^x-1}\sqrt{\frac{2}{\pi r x}}\,\sin(rx)\,dx = \sqrt{\frac{\pi}{2r}}\int_0^\infty \frac{x^{3/2}}{e^x-1}\,Y_{-\frac{1}{2}}(rx)\,dx\,. \tag{1.7}$$

The same can be done for the alternating Mathieu series, so

$$\widetilde{S}(r) = \sqrt{\frac{\pi}{2r}}\int_0^\infty \frac{x^{3/2}}{e^x+1}\,Y_{-\frac{1}{2}}(rx)\,dx\,.$$

2 Polylogarithmic approach to Mathieu and alternating Mathieu series

In the exposition we use the series definition of the Riemann Zeta function [28, p. 164, Eq. (1)]

$$\zeta(s) = \sum_{n\ge 1} n^{-s}, \qquad \Re(s) > 1\,,$$

and its integral representation

$$\zeta(s) = \frac{1}{\Gamma(s)}\int_0^\infty \frac{x^{s-1}}{e^x-1}\,dx, \qquad \Re(s) > 1. \tag{2.1}$$

The close relative of the Riemann Zeta function known as the Dirichlet Eta function (or the alternating Riemann Zeta function) $\eta(s)$ and its integral representation are given by [28, p. 384, Eq. (35)]

$$\eta(s) = \left(1-2^{1-s}\right)\zeta(s) = \sum_{n\ge 1}(-1)^{n-1} n^{-s}, \qquad \Re(s) > 0\,,$$

that is,

$$\eta(s) = \frac{1}{\Gamma(s)}\int_0^\infty \frac{x^{s-1}}{e^x+1}\,dx, \qquad \Re(s) > 0, \tag{2.2}$$

respectively.

The polylogarithm (de Jonquière's function) is the Dirichlet type power series in complex argument z, viz.

$$\mathrm{Li}_s(z) = \sum_{n\ge 1}\frac{z^n}{n^s};$$

here the defining series converges for the complex order $s \in \mathbb{C}$ for all $|z| < 1$, while by analytic continuation it can be extended to $|z| \ge 1$. There is extensive literature available for the polylogarithm and related topic; consult the standard references [1, 14, 19, 23, 31]. Obviously, $\mathrm{Li}_s(1) = \zeta(s),\ \Re(s) > 1$.

Our interest in polylogarithm is drawn by the integral representation

$$\mathrm{Li}_s(z) = \frac{z}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}}{\mathrm{e}^t - z}\, \mathrm{d}t, \qquad \Re(s) > 0, z \in \mathbb{C} \setminus [1, \infty)\,. \tag{2.3}$$

This integral is closely connected with the Bose–Einstein distribution's integral [10]

$$G_k(x) = \frac{1}{\Gamma(k+1)} \int_0^\infty \frac{t^k}{\mathrm{e}^{t-x} - 1}\, \mathrm{d}t, \qquad k > -1\,.$$

Here $x \le 0$; in turn for $x > 0$ the Cauchy principal value integral should be used, see [10]. Obviously,

$$G_k(x) = \frac{\mathrm{e}^x}{\Gamma(k+1)} \int_0^\infty \frac{t^k}{\mathrm{e}^t - \mathrm{e}^x}\, \mathrm{d}t = \mathrm{Li}_{k+1}\big(\mathrm{e}^x\big)\,. \tag{2.4}$$

The Fermi–Dirac distribution integral (see also Clunie's note [10]) is

$$F_k(x) = \frac{1}{\Gamma(k+1)} \int_0^\infty \frac{t^k}{\mathrm{e}^{t-x} + 1}\, \mathrm{d}t, \qquad k > 0\,.$$

We point out, see [31], that

$$F_k(x) = -\mathrm{Li}_{k+1}\big(-\mathrm{e}^x\big)\,. \tag{2.5}$$

The similarity to Emersleben's integral expressions for the Mathieu series and the alternating Mathieu series $\widetilde{S}(r)$ is obvious, compare (1.1) and (1.2). Motivated by these 'similarities', our next goal is to establish inter-connection formulae between the polylogarithm, the series built from the Riemann Zeta function, the Fermi–Dirac and Bose–Einstein integrals from one, and Mathieu series and alternating Mathieu series from the other side.

Theorem 2.1 *For all* $|r| < 1$,

$$S(r) = 2 \sum_{n \ge 0} \frac{(-1)^n\, (2)_n\, r^{2n}}{n!}\, \zeta(2n+3), \tag{2.6}$$

$$\widetilde{S}(r) = 2 \sum_{n \ge 0} \frac{(-1)^n\, (2)_n\, r^{2n}}{n!}\, \eta(2n+3)\,. \tag{2.7}$$

Proof Consider the integral representation (1.1). By the Taylor expansion of the sine function in the integrand we conclude

$$S(r) = \frac{1}{r} \int_0^\infty \frac{x}{\mathrm{e}^x - 1} \sum_{n \ge 0} \frac{(-1)^n (rx)^{2n+1}}{(2n+1)!}\, \mathrm{d}x = \sum_{n \ge 0} \frac{(-1)^n r^{2n}}{(2n+1)!} \int_0^\infty \frac{x^{2n+2}}{\mathrm{e}^x - 1}\, \mathrm{d}x\,.$$

In turn, by (2.3) and (2.4) we confirm that

$$\int_0^\infty \frac{x^{2n+2}}{\mathrm{e}^x - 1}\, \mathrm{d}x = \Gamma(2n+3)\, G_{2n+2}(0) = \Gamma(2n+3)\, \mathrm{Li}_{2n+3}(1) = (2n+2)!\, \zeta(2n+3)\,, \tag{2.8}$$

which results in

$$S(r) = 2 \sum_{n \ge 0} (-1)^n r^{2n} (n+1)\, \zeta(2n+3)\,,$$

getting (2.6). Next, starting now from (1.2) we infer similarly the second formula which holds for the alternating Mathieu series $\widetilde{S}(r)$. Indeed, applying (2.5), we conclude

$$\begin{aligned}\widetilde{S}(r) &= \frac{1}{r}\int_0^\infty \frac{x}{\mathrm{e}^x+1}\sum_{n\geq 0}\frac{(-1)^n(rx)^{2n+1}}{(2n+1)!}\,\mathrm{d}x = 2\sum_{n\geq 0}(-1)^n r^{2n}(n+1)\int_0^\infty \frac{x^{2n+2}}{\mathrm{e}^x+1}\,\mathrm{d}x\\ &= 2\sum_{n\geq 0}(-1)^n r^{2n}(n+1)F_{2n+2}(0) = -2\sum_{n\geq 0}(-1)^n r^{2n}(n+1)\,\mathrm{Li}_{2n+3}(-1)\\ &= 2\sum_{n\geq 0}(-1)^n r^{2n}(n+1)\,\eta(2n+3)\,,\end{aligned}$$

which completes the proof. □

Remark 2.2 We point out that (2.6) and (2.7) are not new; in fact these relations coincide with the series representations [24, Eqs. (1.7-8)], also see [27, p. 72, Proposition 1] for (2.7). We also point out that there are no reasons to consider the series $S(r)$ and $\widetilde{S}(r)$ exclusively for $r > 0$; the exception can be Mathieu's original mathematical model in which he described the vibration of clamped rectangular plates and membranes, see the discussion in the memoir [24, §8.3]. Hence the importance of the previously presented results.

3 Series expansions of integrals (1.3) and (1.4)

The derivation of the integral expressions (1.3) and (1.4) associated to $S(r)$ and $\widetilde{S}(r)$ is realized by complex analytical and integral transformation methods, see [22]. Then, since their integrands include reciprocals of hyperbolic functions, we explore other series expansions of these integrals.

First, we introduce the exponential integral of the first order [1, p. 228, Eq. 5.1.1]

$$E_1(z) = \fint_z^\infty x^{-1}\mathrm{e}^{-x}\,\mathrm{d}x\,, \qquad |\arg(z)| < \pi,$$

whose mirror symmetry property reads $E_1(\overline{z}) = \overline{E_1(z)}$, see [1, p. 229, Eq. 5.1.13]. Obviously, we consider here the principal value of the integral when $z \neq 0$, consult [23, p. 150, Eq. 6.2.1]. Moreover, in the `Mathematica` package the exponential integral is defined also as the principal value of the integral [23, p. 150, Eq. 6.2.5]

$$\mathrm{Ei}(x) = -\fint_{-x}^\infty t^{-1}\,\mathrm{e}^{-t}\,\mathrm{d}t\,, \qquad x > 0.$$

However, the inter-connection $E_1(x) = -\mathrm{Ei}(-x)$ holds true, see [23, p. 150, Eq. 6.2.6].

Theorem 3.1 *For all $r > 0$, the following series expansions hold:*

$$S(r) = \frac{1}{r}\sum_{n\geq 0} s\left\{\mathrm{e}^{-rs}\Re\big[E_1\big((-r+\tfrac{\mathrm{i}}{2})s\big)\big] - \mathrm{e}^{rs}\Re\big[E_1\big((r+\tfrac{\mathrm{i}}{2})s\big)\big]\right\}\Big|_{s=2\pi(n+1)}, \tag{3.1}$$

$$\widetilde{S}(r) = \frac{1}{r}\sum_{n\geq 0} s\left\{\mathrm{e}^{rs}\Re\big[E_1\big((r+\tfrac{\mathrm{i}}{2})s\big)\big] - \mathrm{e}^{-rs}\Re\big[E_1\big((-r+\tfrac{\mathrm{i}}{2})s\big)\big]\right\}\Big|_{s=\pi(2n+1)}, \tag{3.2}$$

where $\Re[z]$ denotes the real part of $z \in \mathbb{C}$.

Proof Expanding the secant hyperbolic kernel in the integrand of (1.3), for all $x > 0$ we have

$$\begin{aligned}\frac{1}{\cosh^2(\pi x)} &= \frac{4\,\mathrm{e}^{-2\pi x}}{\left(1+\mathrm{e}^{-2\pi x}\right)^2}\\ &= 4\sum_{n\geq 0}(-1)^n(n+1)\,\mathrm{e}^{-2\pi(n+1)x} = 4\,\mathrm{e}^{-2\pi x}\,{}_1F_0\big[2;-;-\mathrm{e}^{-2\pi x}\big]. \quad (3.3)\end{aligned}$$

Let $\mathscr{L}_x[f](s)$ denote the Laplace transform of a suitable function f with respect to the input variable x of the output variable s. By the expansion (3.3), the integral (1.3) becomes a series of Laplace transforms which reads

$$S(r) = 4\pi\sum_{n\geq 0}\frac{(-1)^n(2)_n}{n!}\mathscr{L}_x\left[\frac{r^2-x^2+\frac14}{\left(x^2-r^2+\frac14\right)^2+r^2}\right]\big(2\pi(n+1)\big). \qquad (3.4)$$

Next, we need the related Laplace integral property [1, p. 230, Eq. 5.1.28]

$$\int_0^\infty \frac{\mathrm{e}^{-sx}}{x+a}\,\mathrm{d}x = \mathscr{L}_x[(x+a)^{-1}](s) = \mathrm{e}^{as}\,E_1(as)\,, \qquad s>0,\ a>0\,.$$

The partial fraction decomposition of the integrand is

$$\frac{r^2-x^2+\frac14}{\left(x^2-r^2+\frac14\right)^2+r^2} = \frac{1}{4r}\left\{\frac{1}{x+r+\frac{\mathrm{i}}{2}}+\frac{1}{x+r-\frac{\mathrm{i}}{2}}-\frac{1}{x-r+\frac{\mathrm{i}}{2}}-\frac{1}{x-r-\frac{\mathrm{i}}{2}}\right\}.$$

Hence, applying the previously listed results, we have

$$\begin{aligned}&\mathscr{L}_x\left[\frac{r^2-x^2+\frac14}{\left(x^2-r^2+\frac14\right)^2+r^2}\right](s)\\ &\quad= \frac{1}{4r}\Big[\mathrm{e}^{(r+\frac{\mathrm{i}}{2})s}\,E_1\big((r+\tfrac{\mathrm{i}}{2})s\big)+\mathrm{e}^{(r-\frac{\mathrm{i}}{2})s}\,E_1\big((r-\tfrac{\mathrm{i}}{2})s\big)\\ &\qquad -\mathrm{e}^{-(r-\frac{\mathrm{i}}{2})s}\,E_1\big(-(r-\tfrac{\mathrm{i}}{2})s\big)-\mathrm{e}^{-(r+\frac{\mathrm{i}}{2})s}\,E_1\big((-(r+\tfrac{\mathrm{i}}{2})s\big)\Big].\end{aligned}$$

By the mirror symmetry of the exponential integral we readily conclude

$$\begin{aligned}&\mathscr{L}_x\left[\frac{r^2-x^2+\frac14}{\left(x^2-r^2+\frac14\right)^2+r^2}\right](s)\\ &\quad= \frac{1}{2r}\Big\{\mathrm{e}^{rs}\,\Re\big[\mathrm{e}^{\frac{\mathrm{i}}{2}s}\,E_1\big((r+\tfrac{\mathrm{i}}{2})s\big)\big]-\mathrm{e}^{-rs}\,\Re\big[\mathrm{e}^{\frac{\mathrm{i}}{2}s}\,E_1\big((-r+\tfrac{\mathrm{i}}{2})s\big)\big]\Big\},\end{aligned}$$

whose right-hand side for $s = 2\pi(n+1)$ reduces to

$$\frac{(-1)^{n+1}}{2r}\Big\{\mathrm{e}^{2r\pi(n+1)}\Re\big[E_1\big(2\pi(r+\tfrac{\mathrm{i}}{2})(n+1)\big)\big]-\mathrm{e}^{-2r\pi(n+1)}\Re\big[E_1\big(2\pi(-r+\tfrac{\mathrm{i}}{2})(n+1)\big)\big]\Big\}.$$

Inserting the last expression into (3.4) we arrive at the series expansion (3.1).

Next, as to (3.2), since

$$\begin{aligned}\frac{\sinh(\pi x)}{\cosh^2(\pi x)} &= -\frac{1}{\pi}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{\cosh(\pi x)}\right) = 2\sum_{n\geq 0}(-1)^n(2n+1)\mathrm{e}^{-(2n+1)\pi x}\\ &= 2\mathrm{e}^{-\pi x}\sum_{n\geq 0}\frac{(1)_n(\frac32)_n}{(\frac12)_n\,n!}\big(-\mathrm{e}^{-2\pi x}\big)^n = 2\mathrm{e}^{-\pi x}\,{}_2F_1\left[{1,\ \frac32 \atop \frac12}\,\Big|\,-\mathrm{e}^{-2\pi x}\right],\end{aligned}$$

the integral expression (1.4) becomes the following series of Laplace transforms:

$$\widetilde{S}(r) = 2\pi \sum_{n\geq 0}(-1)^n \frac{(1)_n(\frac{3}{2})_n}{(\frac{1}{2})_n\, n!}\mathscr{L}_x\left[\frac{x}{(x^2-r^2+\frac{1}{4})^2+r^2}\right]\big((2n+1)\pi\big)\,.$$

The partial fraction decomposition of the Laplace transform input function reads

$$\frac{x}{(x^2-r^2+\frac{1}{4})^2+r^2} = -\frac{\mathrm{i}}{4r}\left\{\frac{1}{x+r+\frac{\mathrm{i}}{2}} - \frac{1}{x+r-\frac{\mathrm{i}}{2}} - \frac{1}{x-r+\frac{\mathrm{i}}{2}} + \frac{1}{x-r-\frac{\mathrm{i}}{2}}\right\},$$

therefore

$$\begin{aligned}&\mathscr{L}_x\left[\frac{x}{(x^2-r^2+\frac{1}{4})^2+r^2}\right](s)\\ &= -\frac{\mathrm{i}}{4r}\Big[\mathrm{e}^{(r+\frac{\mathrm{i}}{2})s}\, E_1\big((r+\tfrac{\mathrm{i}}{2})s\big) - \mathrm{e}^{(r-\frac{\mathrm{i}}{2})s}\, E_1\big((r-\tfrac{\mathrm{i}}{2})s\big)\\ &\quad - \mathrm{e}^{-(r-\frac{\mathrm{i}}{2})s}\, E_1\big(-(r-\tfrac{\mathrm{i}}{2})s\big) + \mathrm{e}^{-(r+\frac{\mathrm{i}}{2})s}\, E_1\big((-(r+\tfrac{\mathrm{i}}{2})s\big)\Big].\end{aligned}$$

Again, by the mirror symmetry of the exponential integral $E_1(z)$, inserting $s=\pi(2n+1)$, we conclude that

$$\begin{aligned}\mathscr{L}_x\left[\frac{x}{(x^2-r^2+\frac{1}{4})^2+r^2}\right]\big(\pi(2n+1)\big) &= \frac{(-1)^n}{2r}\Big\{\mathrm{e}^{r\pi(2n+1)}\Re\big[E_1\big(\pi(r+\tfrac{\mathrm{i}}{2})(2n+1)\big)\big]\\ &\quad - \mathrm{e}^{-r\pi(2n+1)}\Re\big[E_1\big(\pi(-r+\tfrac{\mathrm{i}}{2})(2n+1)\big)\big]\Big\}.\end{aligned}$$

The rest is obvious. □

Unfortunately, the series (3.1) for the sum $S(r)$ is slowly convergent. Denote its general term by $u_n(r)$, i.e.,

$$u_n(r) = \frac{s}{r}\Big\{\mathrm{e}^{-rs}\Re\big[E_1\big((-r+\tfrac{\mathrm{i}}{2})s\big)\big] - \mathrm{e}^{rs}\Re\big[E_1\big((r+\tfrac{\mathrm{i}}{2})s\big)\big]\Big\}\Big|_{s=2\pi(n+1)},$$

and consider another auxiliary series:

$$T(r) = \frac{1}{2}u_0(r) + \frac{1}{2}\sum_{n\geq 0}\big(u_n(r)+u_{n+1}(r)\big), \tag{3.5}$$

where $n\geq 0$ and $r>0$. Let $S_n(r)$ and $T_n(r)$ be nth partial sums of the series $S(r)$ and $T(r)$, respectively. Since the series (3.1) is convergent, $\lim_{n\to+\infty} u_n(r)=0$, and according to $T_n(r)-S_n(r)=\frac{1}{2}u_{n+1}(r)$ we conclude that $T(r)$ is also a convergent series with the same sum $S(r)$.

Remark 3.2 Numerical calculations show that for fixed values of r, $u_n(r)>0$ for even n, and negative for odd n, so the transformation of the series (3.1) given by (3.5) is, in fact, the well-known *Euler–Abel transformation*. The series (3.1) is extremely slowly convergent and is practically not usable for numerical calculations. On the other hand, the transformed series (3.5) shows a relatively fast convergence, so a reasonable number of initial terms is enough to approximate the sum $S(r)$ with the required accuracy. The following examples illustrate these properties.

Example 3.3 In Fig. 1 (left) we present the errors

$$E_{S,n}(r) := T_n(r) - S(r) = \frac{1}{2}u_0(r) + \frac{1}{2}\sum_{k=0}^{n}\big(u_k(r)+u_{k+1}(r)\big), \tag{3.6}$$

with only $n = 0, 1, 2$, and 5. As the exact value $S(r)$ we take a very precise approximation obtained by using the Gaussian quadrature formula with respect to the hyperbolic weight function (see [21, 22]), applied directly to the integral (1.3). As we can see, only for small values of r, the errors $E_{S,n}(r)$ are significant if $n \leq 5$. In the same figure (right) we present the corresponding relative errors $R_{S,n}(r) = |E_{S,n}(r)/S(r)|$, taking the partial sums in (3.6) for $n = 5, 10, 50$ and 100 terms. For example, with $n = 100$, the relative error for $r \in [0, 1]$ is less than 10^{-6}, and for larger $r > 1$ this error is less than 10^{-8}, which means that we obtain the values of $S(r)$ with at least 6 and 8 exact decimal digits, respectively.

A series with faster convergence can be obtained by repeating the previous transformation to the series (3.5). Then we get

$$\frac{1}{4}\Bigl(3u_0(r) + u_1(r) + \sum_{n\geq 0}(u_n(r) + 2u_{n+1}(r) + u_{n+2}(r))\Bigr). \tag{3.7}$$

The corresponding errors in the partial sums are denoted by $\mathcal{E}_{S,n}(r)$ and presented in Fig. 2 (left), as well as the relative errors $\mathcal{R}_{S,n}(r)$ in the same figure (right).

Example 3.4 In the case of the alternating Mathieu series $\widetilde{S}(r)$ we study the auxiliary series

$$\widetilde{T}(r) = \frac{1}{2}v_0(r) + \frac{1}{2}\sum_{n\geq 0}\bigl(v_n(r) + v_{n+1}(r)\bigr), \tag{3.8}$$

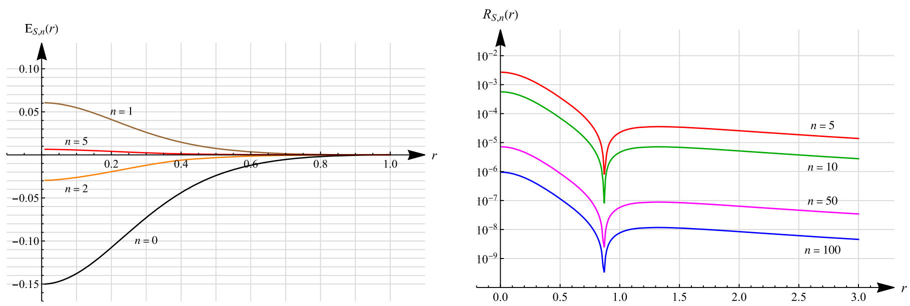

Fig. 1 Errors $E_{S,n}(r)$ for $n = 0, 1, 2$ and 5, when r runs over $[0, 1]$ (left); relative errors $R_{S,n}(r)$ for $0 \leq r \leq 3$, for $n = 5, 10, 50$, and 100 (right)

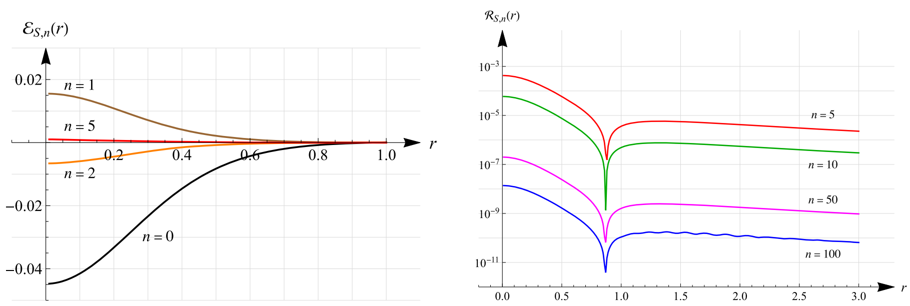

Fig. 2 Errors $\mathcal{E}_{S,n}(r)$ for $n = 0, 1, 2$ and 5, when r runs over $[0, 1]$ (left); relative errors $\mathcal{R}_{S,n}(r)$ for $0 \leq r \leq 3$, for $n = 5, 10, 50$, and 100 (right)

with the general term

$$v_n(r) = \frac{s}{r}\left\{\mathrm{e}^{rs}\Re\big[E_1\big((r+\tfrac{\mathrm{i}}{2})s\big)\big] - \mathrm{e}^{-rs}\Re\big[E_1\big((-r+\tfrac{\mathrm{i}}{2})s\big)\big]\right\}\Big|_{s=\pi(2n+1)}.$$

The repeated Euler–Abel transformation, in this case, gives the accelerated series in the following form:

$$\frac{1}{4}\Big(3v_0(r) + v_1(r) + \sum_{n\geq 0}(v_n(r) + 2v_{n+1}(r) + v_{n+2}(r))\Big). \tag{3.9}$$

The corresponding diagrams are presented in Figs. 3 and 4 with the same notations as the ones in the previous case for the sum $S(r)$ (Example 3.3).

Remark 3.5 As we can see, there exist certain oscillations in the graphics for the relative errors $\mathcal{R}_{S,n}(r)$ (Fig. 2 (right)) and $\mathcal{R}_{\tilde{S},n}(r)$ (Fig. 4 (right)) for larger r and sufficiently large n ($n = 100$), because of unstable calculations in such cases. Namely, the values $S(r)$ and $\tilde{S}(r)$, as well as their approximations, i.e., the partial sums of series (3.7) and (3.9), respectively, are close to zero in such cases.

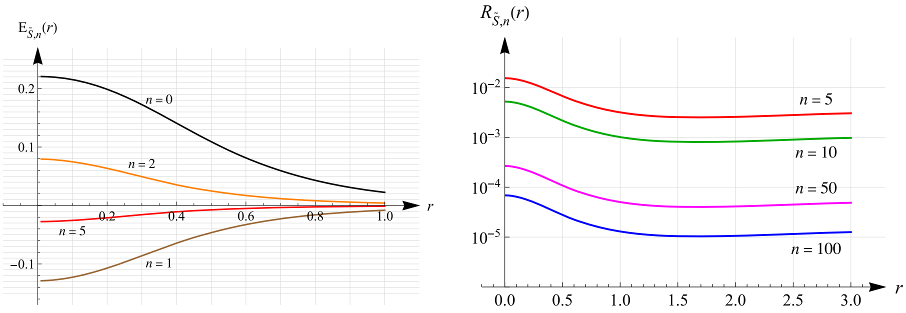

Fig. 3 Errors $E_{\tilde{S},n}(r)$ for $n = 0, 1, 2$ and 5, when r runs over $[0, 1]$ (left); relative errors $R_{\tilde{S},n}(r)$ for $0 \leq r \leq 3$, for $n = 5, 10, 50$, and 100 (right)

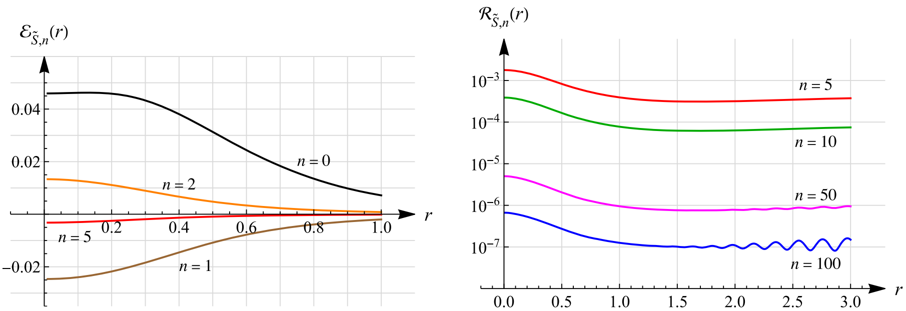

Fig. 4 Errors $\mathcal{E}_{\tilde{S},n}(r)$ for $n = 0, 1, 2$ and 5, when r runs over $[0, 1]$ (left); relative errors $\mathcal{R}_{\tilde{S},n}(r)$ for $0 \leq r \leq 3$, for $n = 5, 10, 50$, and 100 (right)

4 The extended Mathieu series $\mathbb{S}_{\mu,\nu}(r)$ and $\widetilde{\mathbb{S}}_{\mu,\nu}(r)$

Motivated by (1.7), replacing there the kernel function $Y_{-\frac{1}{2}}$ with the general Bessel function of the second kind of order ν, we introduce the extended Mathieu series $S(r)$ and its alternating variant $\widetilde{S}(r)$ in the following forms:

$$\mathbb{S}_{\mu,\nu}(r) = \sqrt{\frac{\pi}{2r}} \int_0^\infty \frac{x^{\mu-1}}{\mathrm{e}^x - 1}\, Y_\nu(rx)\, \mathrm{d}x, \qquad \mu+\nu \geq 1, \tag{4.1}$$

$$\widetilde{\mathbb{S}}_{\mu,\nu}(r) = \sqrt{\frac{\pi}{2r}} \int_0^\infty \frac{x^{\mu-1}}{\mathrm{e}^x + 1}\, Y_\nu(rx)\, \mathrm{d}x, \qquad \mu+\nu \geq 0, \tag{4.2}$$

where in both cases $r > 0,\ \mu > 0$. Clearly $\mathbb{S}_{\frac{5}{2},-\frac{1}{2}}(r) = S(r)$ and $\widetilde{\mathbb{S}}_{\frac{5}{2},-\frac{1}{2}}(r) = \widetilde{S}(r)$.

Using the recurrence formula [30, p. 66, Eq. (1)]

$$Y_{\nu-1}(z) - Y_{\nu+1}(z) = \frac{2\,\nu}{z} Y_\nu(z),$$

we obtain the following recurrence formulae:

$$\frac{2\,\nu}{r}\, \mathbb{S}_{\mu,\nu}(r) = \mathbb{S}_{\mu+1,\nu-1}(r) + \mathbb{S}_{\mu+1,\nu+1}(r),$$
$$\frac{2\,\nu}{r}\, \widetilde{\mathbb{S}}_{\mu,\nu}(r) = \widetilde{\mathbb{S}}_{\mu+1,\nu-1}(r) + \widetilde{\mathbb{S}}_{\mu+1,\nu+1}(r)\,.$$

Theorem 4.1 *If* $\mu,\ \nu+1 > 0$, $n \in \mathbb{N}$ *and* $\mu > |\nu| > 0$, *then*

$$\begin{aligned}\mathbb{S}_{\mu,\nu}(r) &= \kappa_1(\mu,\nu) \sum_{n\geq 1} \frac{1}{(n^2+r^2)^{\frac{\mu+\nu}{2}}}\, {}_2F_1\left[\begin{matrix}\frac{1}{2}(\mu+\nu),\, \frac{1}{2}(1-\mu+\nu)\\ \nu+1\end{matrix}\,\Big|\, \frac{r^2}{n^2+r^2}\right]\\ &\quad - \kappa_2(\mu,\nu) \sum_{n\geq 1} \frac{1}{(n^2+r^2)^{\frac{\mu-\nu}{2}}}\, {}_2F_1\left[\begin{matrix}\frac{1}{2}(\mu-\nu),\, \frac{1}{2}(1-\mu-\nu)\\ 1-\nu\end{matrix}\,\Big|\, \frac{r^2}{n^2+r^2}\right].\end{aligned}$$

Moreover, if $|\nu| < 1$ *and* $\mu+\nu+1 > 0$, *then*

$$\begin{aligned}\widetilde{\mathbb{S}}_{\mu,\nu}(r) &= \kappa_1(\mu,\nu) \sum_{n\geq 1} \frac{(-1)^{n-1}}{(n^2+r^2)^{\frac{\mu+\nu}{2}}}\, {}_2F_1\left[\begin{matrix}\frac{1}{2}(\mu+\nu),\, \frac{1}{2}(1-\mu+\nu)\\ \nu+1\end{matrix}\,\Big|\, \frac{r^2}{n^2+r^2}\right]\\ &\quad - \kappa_2(\mu,\nu) \sum_{n\geq 1} \frac{(-1)^{n-1}}{(n^2+r^2)^{\frac{\mu-\nu}{2}}}\, {}_2F_1\left[\begin{matrix}\frac{1}{2}(\mu-\nu),\, \frac{1}{2}(1-\mu-\nu)\\ 1-\nu\end{matrix}\,\Big|\, \frac{r^2}{n^2+r^2}\right],\end{aligned}$$

where

$$\kappa_1(\mu,\nu) = \cot(\nu\pi) \frac{\sqrt{\pi}\, r^{\nu-\frac{1}{2}}\, \Gamma(\mu+\nu)}{2^{\nu+\frac{1}{2}}\, \Gamma(\nu+1)}; \quad \kappa_2(\mu,\nu) = \csc(\nu\pi) \frac{\sqrt{\pi}\, r^{-\nu-\frac{1}{2}}\, \Gamma(\mu-\nu)}{2^{\frac{1}{2}-\nu}\, \Gamma(1-\nu)}.$$

Proof Insert the binomial series expansion $(\mathrm{e}^x-1)^{-1} = \sum_{n\geq 1} \mathrm{e}^{-nx},\ x > 0$ into (4.1). The legitimate integral-sum interchange which can be proved, e.g., by the dominated convergence theorem results in

$$\mathbb{S}_{\mu,\nu}(r) = \sqrt{\frac{\pi}{2r}} \sum_{n\geq 1} \int_0^\infty x^{\mu-1}\, \mathrm{e}^{-nx}\, Y_\nu(r\,x)\, \mathrm{d}x\,.$$

Making use of the integral representation [30, p. 385, Eq. (4)] or, in other words, the Laplace–Mellin transform of the Bessel function Y_ν, i.e., $\mathcal{L}_a[x^{\mu-1}\, Y_\nu(bx)]$ and $\mathcal{M}_\mu[\mathrm{e}^{-ax}\, Y_\nu(bx)]$, respectively, we infer that

$$\begin{aligned}\int_0^\infty & x^{\mu-1}\mathrm{e}^{-ax}Y_\nu(bx)\,\mathrm{d}x\\ &= \frac{\cot(\nu\pi)\cdot\left(\frac{b}{2}\right)^\nu\Gamma(\mu+\nu)}{(a^2+b^2)^{\frac12(\mu+\nu)}\,\Gamma(\nu+1)}\,{}_2F_1\Big[\begin{matrix}\frac12(\mu+\nu),\,\frac12(1-\mu+\nu)\\ \nu+1\end{matrix}\Big|\frac{b^2}{a^2+b^2}\Big]\\ &\quad-\frac{\csc(\nu\pi)\cdot\left(\frac{b}{2}\right)^{-\nu}\Gamma(\mu-\nu)}{(a^2+b^2)^{\frac12(\mu-\nu)}\,\Gamma(1-\nu)}\,{}_2F_1\Big[\begin{matrix}\frac12(\mu-\nu),\,\frac12(1-\mu-\nu)\\ 1-\nu\end{matrix}\Big|\frac{b^2}{a^2+b^2}\Big],\end{aligned}$$

whose parameter space consists of $\Re(\mu) > |\Re(\nu)|$ and $\Re(a\pm ib) > 0$, taken above $a = n$ and $b = r$, we conclude the first asserted formula. □

In the sequel we need the associated Legendre function of the second kind of a real argument [23, Eq. 14.3.7]

$$\begin{aligned}Q_q^p(x) = \mathrm{e}^{\pi\mathrm{i}p}&\,\frac{\sqrt{\pi}\,\Gamma(p+q+1)\,(x^2-1)^{\frac{q}{2}}}{2^{p+1}\,\Gamma(p+\frac32)\,x^{p+q+1}}\\ &\times{}_2F_1\Big[\begin{matrix}\frac12(p+q)+1,\ \frac12(p+q+1)\\ p+\frac32\end{matrix}\Big|\frac{1}{x^2}\Big],\qquad x>1\,,\end{aligned}$$

provided the parameter range consists of $p, q \in \mathbb{C}$ and $-(p+q) \notin \mathbb{N}$.

Theorem 4.2 *If* $\mu,\ \nu+1>0$, $n\in\mathbb{N}$, *and* $\mu>|\nu|>0$, *then*

$$\mathbb{S}_{\mu,\nu}(r) = -\sqrt{\frac{2}{\pi r}}\,\Gamma(\mu+\nu)\sum_{n\ge1}\frac{1}{(n^2+r^2)^{\frac12\mu}}\,Q_{\mu-1}^{-\nu}\Big[\frac{n}{\sqrt{n^2+r^2}}\Big],$$

$$\widetilde{\mathbb{S}}_{\mu,\nu}(r) = \sqrt{\frac{2}{\pi r}}\,\Gamma(\mu+\nu)\sum_{n\ge1}\frac{(-1)^n}{(n^2+r^2)^{\frac12\mu}}\,Q_{\mu-1}^{-\nu}\Big[\frac{n}{\sqrt{n^2+r^2}}\Big].$$

Proof The same binomial expansion as in the previous proof and a change of the order of integration and summation gives

$$\mathbb{S}_{\mu,\nu}(r) = \sqrt{\frac{\pi}{2r}}\sum_{n\ge1}\int_0^\infty x^{\mu-1}\,\mathrm{e}^{-nx}\,Y_\nu(r\,x)\,\mathrm{d}x.$$

By virtue of the integral [17, p. 700, Eq. 6.621. 2]

$$\int_0^\infty x^{\mu-1}\mathrm{e}^{-ax}\,Y_\nu(bx)\,\mathrm{d}x = -\frac{2}{\pi}\,\frac{\Gamma(\mu+\nu)}{(a^2+b^2)^{\frac12\mu}}\,Q_{\mu-1}^{-\nu}\Big[\frac{a}{\sqrt{a^2+b^2}}\Big],$$

whose parameter space consists of $a>0,\ b>0,\ \Re(\mu)>|\Re(\nu)|$, for $a=n$ and $b=r$ we obtain the first asserted formula.

The derivation of the series expansion for $\widetilde{\mathbb{S}}_{\mu,\nu}(r)$ gives

$$(1+\mathrm{e}^x)^{-1} = \sum_{n\ge1}(-1)^{n-1}\mathrm{e}^{-nx},\quad x>0.$$

Now, the path to the final formula is obvious. □

5 Functional bounding inequalities

Recall the Gubler–Weber formula [30, p. 165, Eq. (5)]

$$Y_\nu(z) = \frac{2\,\left(\frac{z}{2}\right)^\nu}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})}\,\left\{\int_0^1 (1-t^2)^{\nu-\frac{1}{2}}\,\sin(zt)\,\mathrm{d}t - \int_0^\infty (1+t^2)^{\nu-\frac{1}{2}}\,\mathrm{e}^{-zt}\,\mathrm{d}t\right\},$$

which holds for $\Re(z) > 0$ and $\nu > -1/2$. Splitting the ν-domain into three disjoint intervals

$$(-1/2,\ \infty) = (-1/2,\ 1/2] \cup (1/2,\ 3/2) \cup (3/2,\ \infty) = U_1 \cup U_2 \cup U_3\,,$$

Baricz et al. [3, pp. 957–958] obtained the functional bounding inequality for the real argument Neumann function $Y_\nu(x)$ (see also [18, pp. 7–8], [11, p. 76]):

$$\big|Y_\nu(x)\big| + \frac{x^\nu}{2^\nu\,\Gamma(\nu+1)} \le \begin{cases} \dfrac{\left(\frac{x}{2}\right)^{\nu-1}}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})}, & -\frac{1}{2} < \nu \le \frac{1}{2}, \\[2ex] \dfrac{\left(\frac{x}{2}\right)^{\nu-1}}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})} + \dfrac{2^\nu\,\Gamma(\nu)}{\pi x^\nu}, & \frac{1}{2} < \nu < \frac{3}{2}, \\[2ex] \dfrac{x^{\nu-1}}{\sqrt{2\pi}\,\Gamma(\nu+\frac{1}{2})} + \dfrac{2^{2\nu-\frac{3}{2}}\,\Gamma(\nu)}{\pi x^\nu}, & \nu > \frac{3}{2}\,. \end{cases} \tag{5.1}$$

Theorem 5.1 *If* $\mu,\ \nu+\frac{1}{2} > 0$, $n \in \mathbb{N}$, *and* $\mu > |\nu| > 0$, *then*

$$\big|\mathbb{S}_{\mu,\nu}(r)\big| \le \begin{cases} c_1(\mu,\nu)\,\zeta(\mu+\nu) + c_2(\mu,\nu)\,\zeta(\mu+\nu-1), & -\frac{1}{2} < \nu \le \frac{1}{2}, \\ c_1(\mu,\nu)\,\zeta(\mu+\nu) + c_2(\mu,\nu)\,\zeta(\mu+\nu-1) + c_3(\mu,\nu)\,\zeta(\mu-\nu), & \frac{1}{2} < \nu < \frac{3}{2}, \\ c_1(\mu,\nu)\,\zeta(\mu+\nu) + c_4(\mu,\nu)\,\zeta(\mu+\nu-1) + c_5(\mu,\nu)\,\zeta(\mu-\nu), & \nu > \frac{3}{2}\,. \end{cases}$$

Moreover, if $\mu+\nu+1 > 0$, *then*

$$\big|\widetilde{\mathbb{S}}_{\mu,\nu}(r)\big| \le \begin{cases} c_1(\mu,\nu)\,\eta(\mu+\nu) + c_2(\mu,\nu)\,\eta(\mu+\nu-1), & -\frac{1}{2} < \nu \le \frac{1}{2}, \\ c_1(\mu,\nu)\,\eta(\mu+\nu) + c_2(\mu,\nu)\,\eta(\mu+\nu-1) + c_3(\mu,\nu)\,\eta(\mu-\nu), & \frac{1}{2} < \nu < \frac{3}{2}, \\ c_1(\mu,\nu)\,\eta(\mu+\nu) + c_4(\mu,\nu)\,\eta(\mu+\nu-1) + c_5(\mu,\nu)\,\eta(\mu-\nu), & \nu > \frac{3}{2}, \end{cases}$$

where

$$c_1(\mu,\nu) = \frac{\sqrt{\pi}\,r^{\nu-\frac{1}{2}}\,\Gamma(\mu+\nu)}{2^{\nu+\frac{1}{2}}\,\Gamma(\nu+1)},\ c_2(\mu,\nu) = \frac{\sqrt{\pi}\,r^{\nu-\frac{3}{2}}\,\Gamma(\mu+\nu-1)}{2^{\nu+\frac{1}{2}}\,\Gamma(\nu+\frac{1}{2})},$$

$$c_3(\mu,\nu) = \frac{2^{\nu-\frac{1}{2}}\,\Gamma(\nu)\Gamma(\mu-\nu)}{\sqrt{\pi}\,r^{\nu+\frac{1}{2}}\,\Gamma(\nu+1)},$$

$$c_4(\mu,\nu) = \frac{\sqrt{\pi}\,r^{\nu-\frac{3}{2}}\,\Gamma(\mu+\nu-1)}{2\,\Gamma(\nu+\frac{1}{2})},\ c_5(\mu,\nu) = \frac{2^{2\nu-2}\,\Gamma(\nu)\Gamma(\mu-\nu)}{\sqrt{\pi}\,r^{\nu+\frac{1}{2}}}.$$

Proof Starting with (4.1) and splitting the range of ν into three disjoint intervals

$$(-1/2,\ \infty) = (-1/2,\ 1/2] \cup (1/2,\ 3/2) \cup (3/2,\ \infty) = U_1 \cup U_2 \cup U_3,$$

and using the estimates (5.1), we conclude

$$|\mathbb{S}_{\mu,\nu}(r)| \le \sqrt{\frac{\pi}{2r}} \int_0^\infty \frac{x^{\mu-1}}{\mathrm{e}^x-1}\, |Y_\nu(rx)|\, \mathrm{d}x \le \begin{cases} \mathbb{S}_{\mu,U_1}(r), & -\frac{1}{2} < \nu \le \frac{1}{2}, \\ \mathbb{S}_{\mu,U_2}(r), & \frac{1}{2} < \nu < \frac{3}{2}, \\ \mathbb{S}_{\mu,U_3}(r), & \nu > \frac{3}{2}, \end{cases}$$

where

$$\mathbb{S}_{\mu,U_1}(r) \le \sqrt{\frac{\pi}{2r}} \left\{ \frac{\left(\frac{r}{2}\right)^\nu}{\Gamma(\nu+1)} \int_0^\infty \frac{x^{\mu+\nu-1}}{\mathrm{e}^x-1}\, \mathrm{d}x + \frac{\left(\frac{r}{2}\right)^{\nu-1}}{\sqrt{\pi}\, \Gamma(\nu+\frac{1}{2})} \int_0^\infty \frac{x^{\mu+\nu-2}}{\mathrm{e}^x-1}\, \mathrm{d}x \right\},$$

$$\begin{aligned}\mathbb{S}_{\mu,U_2}(r) \le \sqrt{\frac{\pi}{2r}} \Bigg\{ & \frac{\left(\frac{r}{2}\right)^\nu}{\Gamma(\nu+1)} \int_0^\infty \frac{x^{\mu+\nu-1}}{\mathrm{e}^x-1}\, \mathrm{d}x + \frac{\left(\frac{r}{2}\right)^{\nu-1}}{\sqrt{\pi}\, \Gamma(\nu+\frac{1}{2})} \int_0^\infty \frac{x^{\mu+\nu-2}}{\mathrm{e}^x-1}\, \mathrm{d}x \\ & + \frac{2^\nu\, \Gamma(\nu)}{\pi\, r^\nu\, \Gamma(\nu+1)} \int_0^\infty \frac{x^{\mu+\nu-1}}{\mathrm{e}^x-1}\, \mathrm{d}x \Bigg\}\end{aligned}$$

and

$$\begin{aligned}\mathbb{S}_{\mu,U_3}(r) \le \sqrt{\frac{\pi}{2r}} \Bigg\{ & \frac{\left(\frac{r}{2}\right)^\nu}{\Gamma(\nu+1)} \int_0^\infty \frac{x^{\mu+\nu-1}}{\mathrm{e}^x-1}\, \mathrm{d}x + \frac{r^{\nu-1}}{\sqrt{2\pi}\, \Gamma(\nu+\frac{1}{2})} \int_0^\infty \frac{x^{\mu+\nu-2}}{\mathrm{e}^x-1}\, \mathrm{d}x \\ & + \frac{2^{2\nu-\frac{3}{2}}\, \Gamma(\nu)}{\pi\, r^\nu\, \Gamma(\nu+1)} \int_0^\infty \frac{x^{\mu+\nu-1}}{\mathrm{e}^x-1}\, \mathrm{d}x \Bigg\},\end{aligned}$$

which is equivalent to the first statement of this theorem. In the derivation procedure we apply the integral representation (2.1) of the Riemann Zeta function.

Similarly, if we start with the expression (4.2), we obtain the second formula with the aid of the Dirichlet Eta function's integral form (2.2). In both cases the parameter constraints are controlled by the convergence conditions (2.1) and (2.2), respectively. □

6 Extended Mathieu series in terms of the Riemann Zeta and Dirichlet Eta functions

The Bessel function of the second kind Y_ν has two kinds of power series expansions depending on the nature of the order parameter. Firstly, when $\nu = n \in \mathbb{Z}$, we have [1, p. 360, Eq. 9.1.11]

$$\begin{aligned} Y_n(z) = \frac{2}{\pi}\, J_n(z)\, \log\frac{z}{2} - \frac{1}{\pi}\left(\frac{2}{z}\right)^n \sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!} \left(\frac{z^2}{4}\right)^k \\ - \frac{1}{\pi}\left(\frac{z}{2}\right)^n \sum_{k\ge 0} \frac{\psi(k+1)+\psi(n+k+1)}{(n+k)!k!} \left(-\frac{z^2}{4}\right)^k, \end{aligned} \tag{6.1}$$

which immediately follows from (1.5) and (1.6). Here ψ is the digamma function defined by

$$\psi(x) = (\log \Gamma(x))' = \frac{\Gamma'(x)}{\Gamma(x)}\,.$$

For a noninteger order $\nu \notin \mathbb{Z}$ there exist several equivalent series representations; we work with the reformulated (1.6), viz.

$$Y_\nu(z) = \cot(\nu\pi) \sum_{n\ge 0} \frac{(-1)^n (\frac{z}{2})^{2n+\nu}}{\Gamma(n+\nu+1)\, n!} - \csc(\nu\pi) \sum_{n\ge 0} \frac{(-1)^n (\frac{z}{2})^{2n-\nu}}{\Gamma(n-\nu+1)\, n!}. \tag{6.2}$$

Theorem 6.1 *If $\mu, r > 0$ and $n \in \mathbb{N}$, then*

$$\begin{aligned}
\mathbb{S}_{\mu,n}(r) &= \sqrt{\frac{2}{\pi r}}\, \Gamma(\mu+n) \sum_{k\ge 0} \frac{(-1)^k (\mu+n)_{2k}}{(k+n)!\, k!} \\
&\quad \times \Big[\log\frac{r}{2} + \psi(\mu+2k+n)\Big]\Big(\frac{r}{2}\Big)^{2k+n} \zeta(\mu+2k+n) \\
&\quad + \sqrt{\frac{2}{\pi r}}\, \Gamma(\mu+n) \sum_{k\ge 0} \frac{(-1)^k (\mu+n)_{2k}}{(k+n)!\, k!} \Big(\frac{r}{2}\Big)^{2k+n} \zeta'(\mu+2k+n) \\
&\quad - \frac{\Gamma(\mu-n)}{\sqrt{2\pi r}} \sum_{k=0}^{n-1} \frac{(n-k-1)!(\mu-n)_{2k}}{k!} \Big(\frac{r}{2}\Big)^{2k-n} \zeta(\mu+2k-n) \\
&\quad - \frac{\Gamma(\mu+n)}{\sqrt{2\pi r}} \sum_{k\ge 0} (-1)^k \frac{\psi(k+1)+\psi(n+k+1)}{(n+k)!k!} (\mu+n)_{2k} \\
&\quad \times \Big(\frac{r}{2}\Big)^{2k+n} \zeta(\mu+2k+n)\,.
\end{aligned}$$

Proof Consider (4.1) for $\nu = n \in \mathbb{N}$. By the series (6.1) and by legitimate transformations we get

$$\begin{aligned}
\mathbb{S}_{\mu,n}(r) &= \sqrt{\frac{2}{\pi r}} \log\frac{r}{2} \sum_{k\ge 0} \frac{(-1)^k}{\Gamma(k+n+1)\, k!} \Big(\frac{r}{2}\Big)^{2k+n} \int_0^\infty \frac{x^{\mu+2k+n-1}}{\mathrm{e}^x - 1}\, \mathrm{d}x \\
&\quad + \sqrt{\frac{2}{\pi r}} \sum_{k\ge 0} \frac{(-1)^k}{\Gamma(k+n+1)\, k!} \Big(\frac{r}{2}\Big)^{2k+n} \int_0^\infty \frac{x^{\mu+2k+n-1}\log x}{\mathrm{e}^x - 1}\, \mathrm{d}x \\
&\quad - \frac{1}{\sqrt{2\pi r}} \sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!} \Big(\frac{r}{2}\Big)^{2k-n} \int_0^\infty \frac{x^{\mu+2k-n-1}}{\mathrm{e}^x - 1}\, \mathrm{d}x \\
&\quad - \frac{1}{\sqrt{2\pi r}} \sum_{k\ge 0} (-1)^k \frac{\psi(k+1)+\psi(n+k+1)}{(n+k)!k!} \Big(\frac{r}{2}\Big)^{2k+n} \int_0^\infty \frac{x^{\mu+2k+n-1}}{\mathrm{e}^x - 1}\, \mathrm{d}x.
\end{aligned} \tag{6.3}$$

The first, third and fourth integrals are already known by virtue of (2.8), however, the second one is more challenging. Since

$$I_p = \int_0^\infty \frac{x^{p-1}\log x}{\mathrm{e}^x - 1}\, \mathrm{d}x = \sum_{m\ge 0} \int_0^\infty x^{p-1} \mathrm{e}^{-(m+1)x} \log x\, \mathrm{d}x =: \sum_{m\ge 0} \mathscr{I}_m,$$

by the Mellin transform [16, p. 315, Eq. (9)]

$$\int_0^\infty x^{p-1} \mathrm{e}^{-qx} \log x\, \mathrm{d}x = \frac{\Gamma(p)}{q^p}\big[\psi(p) - \log q\big], \qquad \Re(q) > 0,\ \Re(p) > 0,$$

and having in mind

$$\sum_{n\ge 1}\frac{\log n}{n^p} = -\zeta'(p), \qquad \Re(p) > 1,$$

setting $p = \mu + 2k + n$ and $q = m + 1$, we infer

$$\begin{aligned} I_{\mu+2k+n} &= \Gamma(\mu+2k+n)\,\psi(\mu+2k+n)\,\zeta(\mu+2k+n) - \Gamma(\mu+2k+n) \\ &\quad\times \sum_{m\ge 0}\frac{\log(m+1)}{(m+1)^{\mu+2k+n}} \\ &= \Gamma(\mu+2k+n)\big[\psi(\mu+2k+n)\,\zeta(\mu+2k+n) + \zeta'(\mu+2k+n)\big]. \end{aligned} \tag{6.4}$$

Finally, applying (2.8) and (6.4) to the expression (6.3), after certain transformations and reduction, we arrive at the statement. □

Theorem 6.2 *If $\mu, r > 0$ and $n \in \mathbb{N}_0$, then*

$$\begin{aligned} \widetilde{\mathbb{S}}_{\mu,n}(r) &= \sqrt{\frac{2}{\pi r}}\,\Gamma(\mu+n)\sum_{k\ge 0}\frac{(-1)^k(\mu+n)_{2k}}{(k+n)!\,k!} \\ &\quad\times\left[\log\frac{r}{2} + \psi(\mu+2k+n)\right]\left(\frac{r}{2}\right)^{2k+n}\eta(\mu+2k+n) \\ &\quad+ \sqrt{\frac{2}{\pi r}}\,\Gamma(\mu+n)\sum_{k\ge 0}\frac{(-1)^k(\mu+n)_{2k}}{(k+n)!\,k!}\left(\frac{r}{2}\right)^{2k+n}\eta'(\mu+2k+n) \\ &\quad- \frac{\Gamma(\mu-n)}{\sqrt{2\pi r}}\sum_{k=0}^{n-1}\frac{(n-k-1)!(\mu-n)_{2k}}{k!}\left(\frac{r}{2}\right)^{2k-n}\eta(\mu+2k-n) \\ &\quad- \frac{\Gamma(\mu+n)}{\sqrt{2\pi r}}\sum_{k\ge 0}(-1)^k\frac{\psi(k+1)+\psi(n+k+1)}{(n+k)!k!}(\mu+n)_{2k} \\ &\quad\times\left(\frac{r}{2}\right)^{2k+n}\eta(\mu+2k+n). \end{aligned}$$

Proof Applying the Mellin transform

$$\int_0^\infty \frac{x^{p-1}}{e^x+1}\,dx = \Gamma(p)\,\eta(p), \qquad \Re(p) > 0,$$

for all integrals which we derive by the lines of the previous proof, we clearly deduce the claimed result. □ □

Now, we present the Riemann Zeta building blocks series presentation of the extended Mathieu $\mathbb{S}_{\mu,\nu}(r)$ and Dirichlet Eta function terms for extended alternating Mathieu series $\widetilde{\mathbb{S}}_{\mu,\nu}(r)$ by using the noninteger ν parameter case.

Theorem 6.3 *For all $\mu, r > 0$ and for $|\nu| < 1$, if $\mu \pm \nu > 1$, then*

$$\begin{aligned} \mathbb{S}_{\mu,n}(r) &= \cot(\nu\pi)\,\Gamma(\mu+\nu)\sqrt{\frac{\pi}{2r}}\sum_{k\ge 0}\frac{(-1)^k(\mu+\nu)_{2k}}{\Gamma(k+\nu+1)\,k!}\left(\frac{r}{2}\right)^{2k+\nu}\zeta(\mu+2k+\nu) \\ &\quad- \csc(\nu\pi)\,\Gamma(\mu-\nu)\sqrt{\frac{\pi}{2r}}\sum_{k\ge 0}\frac{(-1)^k(\mu-\nu)_{2k}}{\Gamma(k-\nu+1)\,k!}\left(\frac{r}{2}\right)^{2k-\nu}\zeta(\mu+2k-\nu). \end{aligned}$$

Moreover, for $\mu, r > 0$ *and for* $|\nu| < 1$*, if* $\mu \pm \nu > 0$*, then*

$$\widetilde{\mathbb{S}}_{\mu,n}(r) = \cot(\nu\pi)\,\Gamma(\mu+\nu)\sqrt{\frac{\pi}{2r}}\sum_{k\geq 0}\frac{(-1)^k(\mu+\nu)_{2k}}{\Gamma(k+\nu+1)\,k!}\Big(\frac{r}{2}\Big)^{2k+\nu}\eta(\mu+2k+\nu)$$
$$-\csc(\nu\pi)\,\Gamma(\mu-\nu)\sqrt{\frac{\pi}{2r}}\sum_{k\geq 0}\frac{(-1)^k(\mu-\nu)_{2k}}{\Gamma(k-\nu+1)\,k!}\Big(\frac{r}{2}\Big)^{2k-\nu}\eta(\mu+2k-\nu)\,. \quad (6.5)$$

Proof We start again with the integral (4.1) when $\nu \in (-1, 1)$. The series representation (6.2) implies

$$\begin{aligned}\mathbb{S}_{\mu,n}(r) &= \sqrt{\frac{\pi}{2r}}\int_0^\infty \frac{x^{\mu-1}}{e^x-1}\Bigg\{\cot(\nu\pi)\sum_{k\geq 0}\frac{(-1)^k(\frac{rx}{2})^{2k+\nu}}{\Gamma(k+\nu+1)\,k!}\\
&\quad -\csc(\nu\pi)\sum_{k\geq 0}\frac{(-1)^k(\frac{rx}{2})^{2k-\nu}}{\Gamma(k-\nu+1)\,k!}\Bigg\}\,dx\\
&= \cot(\nu\pi)\sqrt{\frac{\pi}{2r}}\sum_{k\geq 0}\frac{(-1)^k}{\Gamma(k+\nu+1)\,k!}\Big(\frac{r}{2}\Big)^{2k+\nu}\int_0^\infty\frac{x^{\mu+2k+\nu-1}}{e^x-1}\,dx\\
&\quad -\csc(\nu\pi)\sqrt{\frac{\pi}{2r}}\sum_{k\geq 0}\frac{(-1)^k}{\Gamma(k-\nu+1)\,k!}\Big(\frac{r}{2}\Big)^{2k-\nu}\int_0^\infty\frac{x^{\mu+2k-\nu-1}}{e^x-1}\,dx\\
&= \cot(\nu\pi)\sqrt{\frac{\pi}{2r}}\sum_{k\geq 0}\frac{(-1)^k}{\Gamma(k+\nu+1)\,k!}\Big(\frac{r}{2}\Big)^{2k+\nu}\Gamma(\mu+2k+\nu)\,\zeta(\mu+2k+\nu)\\
&\quad -\csc(\nu\pi)\sqrt{\frac{\pi}{2r}}\sum_{k\geq 0}\frac{(-1)^k}{\Gamma(k-\nu+1)\,k!}\Big(\frac{r}{2}\Big)^{2k-\nu}\Gamma(\mu+2k-\nu)\,\zeta(\mu+2k-\nu)\,,\end{aligned}$$

which is equivalent to the stated formula. The proof of (6.5) is now straightforward. □

7 Extending the Butzer–Flocke–Hauss (complete) Omega function $\Omega(z)$ via Neumann functions

The notation $\Omega(z)$, $z \in \mathbb{C}$, stands for the so-called complete Butzer–Flocke–Hauss (BHF) Omega function introduced in [4, Definition 7.1], [5] in the form

$$\Omega(z) := 2\int_{0+}^{\frac{1}{2}}\sinh(zu)\cot(\pi u)\,du, \qquad z \in \mathbb{C}.$$

It is the Hilbert transform $\mathscr{H}_1[e^{-zx}](0)$ at zero of the 1-periodic function $\big(e^{-zx}\big)_1$ defined by the periodic extension of the exponential function e^{-zx}, $|x| < \frac{1}{2}$, $z \in \mathbb{C}$, thus

$$\Omega(z) = \mathscr{H}_1\big[e^{-zx}\big](0) = \fint_{-\frac{1}{2}}^{\frac{1}{2}} e^{zu}\cot(\pi u)\,du.$$

Another expressions for the complete BHF Omega function $\Omega(x)$ are given by Butzer et al. [6]:

$$\Omega(x) = \frac{2}{\pi}\sinh\Big(\frac{x}{2}\Big)\int_0^\infty\frac{1}{e^t+1}\cos\Big(\frac{xt}{2\pi}\Big)\,dt, \qquad x \in \mathbb{R}, \quad (7.1)$$

while the real argument complete BHF $\varOmega$ function's integral form by Tomovski and Pogány reads [29, p. 10, Theorem 3.3]

$$\varOmega(x) = 2\sqrt{\frac{2}{\pi}}\,\sinh\left(\frac{x}{2}\right)\,⨍_0^\infty \sinh\left(\frac{xt}{\pi}\right)\tan t\,\mathrm{d}t.$$

By extensions in the integrand of the Butzer–Flocke–Hauss Omega function which is intimately connected to the generalized Mathieu series (consult the extensive study by Butzer and Pogány [5]) we are faced with a new territory of ideas and series/integral conclusion upon the structure of these kinds of generalizations.

Inspired by (7.1), we can write

$$\varOmega(x) = -\frac{\sqrt{x}}{\pi}\,\sinh\left(\frac{x}{2}\right)\int_0^\infty \frac{\sqrt{t}}{\mathrm{e}^t+1}\,Y_{\frac12}\left(\frac{xt}{2\pi}\right)\mathrm{d}t$$

having in mind that $\cos(z) = -\sqrt{\pi z/2}\,Y_{\frac12}(z)$ implementing the Neumann function of the general order ν instead of $Y_{-\frac12}$ in the kernel in the following way:

$$\varOmega_{\mu,\nu}(x) = -\frac{\sqrt{x}}{\pi}\,\sinh\left(\frac{x}{2}\right)\int_0^\infty \frac{t^{\mu-1}}{\mathrm{e}^t+1}\,Y_{\nu}\left(\frac{xt}{2\pi}\right)\mathrm{d}t. \tag{7.2}$$

The parameter range derivation will be our first goal. In turn, recognizing that the same integral consist both $\varOmega_{\mu,\nu}(x)$ and $\mathbb{S}_{\mu,\nu}(r)$ in (7.2) and (4.2), respectively, we deduce the relation

$$\varOmega_{\mu,\nu}(x) = -\frac{x}{\pi^2}\,\sinh\left(\frac{x}{2}\right)\widetilde{\mathbb{S}}_{\mu,\nu}\left(\frac{x}{2\pi}\right). \tag{7.3}$$

Therefore the parameter spaces coincide for any $x > 0$.

Next, the power series form of the complete BHF $\varOmega$ function whose coefficients are built by finite sums containing Dirichlet Eta function terms is reported in [5, p. 901, Theorem 5.4. (ii)]

$$\varOmega(z) = \frac{z}{\pi}\sum_{n\ge 0}\sum_{k=0}^{n}\frac{(-1)^k\,\eta(2k+1)}{\pi^{2k}(2(n-k)+1)!}\left(\frac{z}{2}\right)^{2n}, \qquad |z| < 2\pi,$$

which shows that $\varOmega$ is intimately connected with the Eta function. In [5] the authors discussed the relations of the Mathieu-, and the alternating Mathieu series and its generalized variants from one, and the $\varOmega(z)$ function from other side by the Taylor expansion of the Hilbert–Eisenstein series $\mathfrak{h}_1(z)$ of the first order and the polygamma function $\psi^{(r)}$ of order r (see also [2]).

However, our recent considerations are developed in another direction, according to the series representation of the expanded complete BHF $\varOmega_{\mu,\nu}(x)$ in terms of the Dirichlet Eta function. In turn, bearing in mind (7.3), the counterpart results valid for $\varOmega_{\mu,\nu}(x)$ exposed in Eq. (2.7) of Theorem 2.1, Theorem 6.2 and finally in Eq. (6.5) of Theorem 6.3 turn out to be their immediate consequences. So, we leave the formulation of these functional bound results to the interested reader.

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