



Some Müntz orthogonal systems¹

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Abstract

Let $A = \{\lambda_0, \lambda_1, \dots\}$ be a given sequence of complex numbers such that $|\lambda_i| > 1$ and $\operatorname{Re}(\lambda_i \bar{\lambda}_j - 1) > 0$ for each i and j . For Müntz polynomials $P(x) = \sum_{i=0}^n p_i x^{\lambda_i}$ and $Q(x) = \sum_{j=0}^m q_j x^{\lambda_j}$ we define an inner product $[P, Q]$ by

$$[P, Q] = \int_0^1 (P \odot \bar{Q})(x) \frac{dx}{x^2},$$

where $(P \odot \bar{Q})(x) = \sum_{i=0}^n \sum_{j=0}^m p_i \bar{q}_j x^{\lambda_i \bar{\lambda}_j}$. We obtain the Müntz polynomials $(Q_n(x))$ orthogonal with respect to this inner product and connect them with the Malmquist rational functions

$$W_n(z) = \frac{\prod_{v=0}^{n-1} (z - 1/\bar{\lambda}_v)}{\prod_{v=0}^n (z - \lambda_v)}$$

orthogonal on the unit circle with respect to the inner product

$$(u, v) = \frac{1}{2\pi i} \oint_{|z|=1} u(z) \overline{v(z)} \frac{dz}{z}.$$

Several interesting properties of polynomials $Q_n(x)$ are given.

Keywords: Müntz systems; Müntz polynomials; Malmquist systems; Rational functions; Orthogonality; Inner product; Recurrence formulas; Zeros

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1. Introduction

This paper is devoted to some classes of orthogonal Müntz polynomials, as well as to the associated Malmquist systems of orthogonal rational functions. The first papers for orthogonal rational functions on the unit circle whose poles are fixed were given by Djrbashian [7, 8]. These rational functions generalise the orthogonal polynomials of Szegő [19, pp. 287–295]. Recently, the paper of Djrbashian [8], which originally appeared in two parts, has been translated to English by Müller and Bultheel [14]. A survey on the theory of such orthogonal systems and some open problems was written also by Djrbashian [9]. Several papers in this direction have been appeared in the last period (cf. [4–6, 10, 15–18]).

The orthogonal Müntz systems were considered first by Armenian mathematicians Badalyan [1] and Taslakyán [20]. Recently, it was investigated by McCarthy et al. [11] and more completely by Borwein et al. [3] (see also the recent book [2]).

In this paper we consider a class of Müntz polynomials orthogonal on $(0, 1)$ with respect to an inner product introduced in an unusual way. The paper is organised as follows. Section 2 is devoted to one class of orthogonal Müntz systems on $(0, 1)$. The Malmquist systems of orthogonal rational functions and a connection with the orthogonal Müntz systems are considered in Section 3. Some recurrence formulae of orthogonal Müntz polynomials are given in Section 4. Finally, real zeros of the real Müntz polynomials are analysed in Section 5.

2. Orthogonal Müntz systems

Let $A = \{\lambda_0, \lambda_1, \dots\}$ be a given sequence of complex numbers. Taking the following definition for x^λ :

$$x^\lambda = e^{\lambda \log x}, \quad x \in (0, \infty), \quad \lambda \in \mathbb{C},$$

and the value at $x=0$ is defined to be the limit of x^λ as $x \rightarrow 0$ from $(0, \infty)$ whenever the limits exists, we will consider orthogonal Müntz polynomials as linear combinations of the Müntz system $\{x^{\lambda_0}, x^{\lambda_1}, \dots, x^{\lambda_n}\}$ (see [2, 3]). The set of all such polynomials we will denote by $M_n(A)$, i.e.,

$$M_n(A) = \text{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots, x^{\lambda_n}\},$$

where the linear span is over the complex numbers \mathbb{C} in general. The union of all $M_n(A)$ is denoted by $M(A)$.

For real numbers $0 \leq \lambda_0 < \lambda_1 < \dots \rightarrow \infty$, it is well known that the real Müntz polynomials of the form $\sum_{k=0}^n a_k x^{\lambda_k}$ are dense in $L^2[0, 1]$ if and only if $\sum_{k=1}^{\infty} \lambda_k^{-1} = +\infty$. In addition, if $\lambda_0 = 0$ this condition also characterises the denseness of the Müntz polynomials in $C[0, 1]$ in the uniform norm.

The first considerations of orthogonal Müntz systems were made by Badalyan [1] and Taslakyán [20]. Recently, it was investigated by McCarthy et al. [11] and more completely by Borwein et al. [3].

Supposing that $\text{Re}(\lambda_k) > -\frac{1}{2}$ ($k=0, 1, \dots$) they introduced the *Müntz-Legendre polynomials* on $(0, 1]$ as (see [3, 20]):

$$P_n(\lambda_0, \dots, \lambda_n; x) = \frac{1}{2\pi i} \oint_{\Gamma} \prod_{\nu=0}^{n-1} \frac{s + \bar{\lambda}_\nu + 1}{s - \lambda_\nu} \frac{x^s ds}{s - \lambda_n} \quad (n=0, 1, \dots),$$

where the simple contour Γ surrounds all the zeros of the denominator in the integrand.

The polynomials $P_n(x) \equiv P_n(\lambda_0, \dots, \lambda_n; x)$ satisfy an orthogonality relation on $(0, 1)$

$$\int_0^1 P_n(x) \overline{P_m(x)} dx = \delta_{n,m} / (1 + \lambda_n + \bar{\lambda}_n),$$

for every $n, m = 0, 1, \dots$.

Definition 1. For $\alpha, \beta \in \mathbb{C}$ we have

$$x^\alpha \odot x^\beta = x^{\alpha\beta} \quad (x \in (0, \infty)). \quad (1)$$

Using (1) we can introduce an external operation for the Müntz polynomials from $M(\Lambda)$.

Definition 2. For polynomials $P \in M_n(\Lambda)$ and $Q \in M_m(\Lambda)$, i.e.,

$$P(x) = \sum_{i=0}^n p_i x^{\lambda_i} \quad \text{and} \quad Q(x) = \sum_{j=0}^m q_j x^{\lambda_j}, \quad (2)$$

we have

$$(P \odot Q)(x) = \sum_{i=0}^n \sum_{j=0}^m p_i q_j x^{\lambda_i \lambda_j}. \quad (3)$$

Under restrictions that for each i and j we have

$$|\lambda_i| > 1, \quad \operatorname{Re}(\lambda_i \bar{\lambda}_j - 1) > 0, \quad (4)$$

then we can introduce a new inner product for Müntz polynomials.

Definition 3. Let the conditions (4) be satisfied for the Müntz polynomials $P(x)$ and $Q(x)$ given by (2). Their inner product $[P, Q]$ is defined by

$$[P, Q] = \int_0^1 (P \odot \bar{Q})(x) \frac{dx}{x^2}, \quad (5)$$

where $(P \odot Q)(x)$ is determined by (3).

It is not clear immediately that (5) represents an inner product. Therefore, we prove the following result:

Theorem 4. Let $\Lambda = \{\lambda_0, \lambda_1, \dots\}$ be a sequence of the complex numbers such that the conditions (4) hold. Then

- (i) $[P, P] \geq 0$;
- (ii) $[P, P] = 0 \Leftrightarrow P(x) \equiv 0$;
- (iii) $[P + Q, R] = [P, R] + [Q, R]$;
- (iv) $[cP, Q] = c[P, Q]$;
- (v) $[P, Q] = \overline{[Q, P]}$

for each $P, Q, R \in M(\Lambda)$ and each $c \in \mathbb{C}$.

Proof. Let $P(x)$ and $Q(x)$ be given by (2). Using (5) and (3) we have

$$[P, Q] = \int_0^1 (P \odot Q)(x) \frac{dx}{x^2} - \sum_{i=0}^n \sum_{j=0}^m p_i \bar{q}_j \int_0^1 x^{\lambda_i \bar{\lambda}_j - 2} dx.$$

Because of (4), we find that $\int_0^1 x^{\lambda_i \bar{\lambda}_j - 2} dx = 1/(\lambda_i \bar{\lambda}_j - 1)$ for each i and j , so that we get

$$[P, Q] = \sum_{i=0}^n \sum_{j=0}^m \frac{p_i \bar{q}_j}{\lambda_i \bar{\lambda}_j - 1}. \quad (6)$$

In order to prove (i) and (ii) it is enough to conclude that the quadratic form

$$[P, P] = \sum_{i=0}^n \sum_{j=0}^n \frac{1}{\lambda_i \bar{\lambda}_j - 1} p_i \bar{p}_j,$$

i.e., its matrix $H_n = [1/(\lambda_i \bar{\lambda}_j - 1)]_{i,j=0}^n$, is positive definite. Therefore, we use the Sylvester's necessary and sufficient conditions (cf. [12, p. 214])

$$D_k = \det H_k = \det [1/(\lambda_i \bar{\lambda}_j - 1)]_{i,j=0}^k > 0 \quad (k = 0, 1, \dots, n).$$

In order to evaluate the determinants D_k , we use Cauchy's formula (see [13, p. 345])

$$\det \left[\frac{1}{a_i + b_j} \right]_{i,j=0}^k = \frac{\prod_{i>j=0}^k (a_i - a_j)(b_i - b_j)}{\prod_{i,j=0}^k (a_i + b_j)}$$

with $a_i = \lambda_i$ and $b_j = -1/\bar{\lambda}_j$. Thus, we obtain

$$\begin{aligned} D_k &= \frac{1}{\prod_{j=0}^k \bar{\lambda}_j} \det \left[\frac{1}{\lambda_i - 1/\bar{\lambda}_j} \right]_{i,j=0}^k \\ &= \frac{1}{\prod_{j=0}^k \bar{\lambda}_j} \cdot \frac{\prod_{i>j=0}^k (\lambda_i - \lambda_j)((-1/\bar{\lambda}_i) + (1/\bar{\lambda}_j))}{\prod_{i,j=0}^k (\lambda_i - (1/\bar{\lambda}_j))}, \end{aligned}$$

i.e.,

$$D_k = \frac{1}{\prod_{j=0}^k \bar{\lambda}_j} \cdot \frac{\prod_{i>j=0}^k |\lambda_i - \lambda_j|^2 / \bar{\lambda}_i \bar{\lambda}_j}{\prod_{i,j=0}^k (\lambda_i \bar{\lambda}_j - 1) / \bar{\lambda}_j}.$$

Since $D_0 = 1/(|\lambda_0|^2 - 1) > 0$ (because of (4) and

$$D_k = \frac{D_{k-1}}{|\lambda_k|^2 - 1} \prod_{i=0}^{k-1} \frac{|\lambda_k - \lambda_i|^2}{|\lambda_i \bar{\lambda}_k - 1|^2},$$

by induction we conclude that $D_k > 0$ for all k .

The properties (iii)–(v) follow directly from (5) or (6). \square

Now we are ready to define the Müntz polynomials

$$Q_n(x) \equiv Q_n(\lambda_0, \lambda_1, \dots, \lambda_n; x) \quad (n=0, 1, \dots) \quad (7)$$

orthogonal with respect to the inner product (5).

Definition 5. Let $\Lambda = \{\lambda_0, \lambda_1, \dots\}$ be a sequence of the complex numbers such that the conditions (4) hold and let

$$W_n(s) = \frac{\prod_{v=0}^{n-1} (s - 1/\bar{\lambda}_v)}{\prod_{v=0}^n (s - \lambda_v)}. \quad (8)$$

The n th Müntz polynomial $Q_n(x)$, associated to the rational function (8), is defined by

$$Q_n(x) = \frac{1}{2\pi i} \oint_{\Gamma} W_n(s) x^s ds, \quad (9)$$

where the simple contour Γ surrounds all the points λ_v ($v=0, 1, \dots, n$).

Using (9), (16), (17) and Cauchy's residue theorem we get a representation of (7) in the form

$$Q_n(x) = \sum_{k=0}^n A_{n,k} x^{\lambda_k}, \quad (10)$$

where

$$A_{n,k} = \frac{\prod_{v=0}^{n-1} (\lambda_k - 1/\bar{\lambda}_v)}{\prod_{v \neq k} (\lambda_k - \lambda_v)} \quad (k=0, 1, \dots, n), \quad (11)$$

and where we assumed that $\lambda_i \neq \lambda_j$ ($i \neq j$).

The following theorem gives an orthogonality relation for polynomials $Q_n(x)$.

Theorem 6. Under conditions (4) on the sequence Λ , the Müntz polynomials $Q_n(x)$, $n=0, 1, \dots$, defined by (9), are orthogonal with respect to the inner product (5), i.e.,

$$[Q_n, Q_m] = \frac{1}{(|\lambda_n|^2 - 1)|\lambda_0 \lambda_1 \cdots \lambda_{n-1}|^2} \delta_{n,m}. \quad (12)$$

The proof of this theorem will be given in the next section using the orthogonality of a Malmquist system of rational functions.

One particular result of Theorem 6, when $\lambda_v \rightarrow \lambda$ for each v , can be interesting:

Corollary 7. Let $Q_n(x)$ be defined by (9) and let $\lambda_0 = \lambda_1 = \cdots = \lambda_n = \lambda$. Then

$$Q_n(x) = x^\lambda L_n(-(\lambda - 1/\bar{\lambda}) \log x), \quad (13)$$

where $L_n(x)$ is the Laguerre polynomial orthogonal with respect to e^{-x} on $[0, \infty)$ and such that $L_n(0) = 1$.

Proof. For $\lambda_0 = \lambda_1 = \dots = \lambda_n = \lambda$, (9) reduces to

$$Q_n(x) = \frac{1}{2\pi i} \oint_{\Gamma} \frac{(s - 1/\bar{\lambda})^n}{(s - \lambda)^{n+1}} x^s ds,$$

where $\lambda \in \text{int } \Gamma$. Since

$$\text{Res}_{z=\lambda} \left[\frac{(s - 1/\bar{\lambda})^n}{(s - \lambda)^{n+1}} x^s \right] = \frac{1}{n!} \lim_{z \rightarrow \lambda} \frac{d^n}{dz^n} [(s - 1/\bar{\lambda})^n x^s],$$

we obtain by Cauchy's residue theorem

$$\begin{aligned} Q_n(x) &= \frac{1}{n!} \sum_{k=0}^n \binom{n}{k} n(n-1) \dots (k+1) (\lambda - 1/\bar{\lambda})^k x^\lambda (\log x)^k \\ &= x^\lambda \sum_{k=0}^n \frac{1}{k!} \binom{n}{k} (\lambda - 1/\bar{\lambda})^k (\log x)^k, \end{aligned}$$

which gives (13). \square

3. Orthogonal Malmquist systems and a connection with an orthogonal Müntz system

Let $A = \{a_0, a_1, \dots\}$ be an arbitrary sequence of complex numbers in the unit circle ($|a_v| < 1$). The Malmquist system of rational functions (see [7–9],[21]) is defined in the following way:

$$\phi_n(s) = \frac{(1 - |a_n|^2)^{1/2}}{1 - \bar{a}_n s} \prod_{v=0}^{n-1} \frac{a_v - s}{1 - \bar{a}_v s} \cdot \frac{|a_v|}{a_v} \quad (n = 0, 1, \dots), \tag{14}$$

where for $a_v = 0$ we put $|a_v|/a_v = \bar{a}_v/|a_v| = -1$. In the case $n = 0$, an empty product should be taken to be equal 1. Such system of functions was intensively investigated in several papers by Djrbashian [7–9], Bultheel et al. [4–6], Pan [15–18], etc.

In this section we want to prove some auxiliary results in order to connect this system of orthogonal functions with some Müntz orthogonal system of polynomials.

Excluding the normalisation constants, the system (14) can be represented in the form

$$W_n(s) = \frac{\prod_{v=0}^{n-1} (s - a_v)}{\prod_{v=0}^n (s - a_v^*)}, \tag{15}$$

where $a_v^* = 1/\bar{a}_v$. For $a_v = 0$ we put only s instead of $(s - a_v)/(s - a_v^*)$.

Suppose now that $a_v \neq a_\mu$ for $v \neq \mu$. Then (15) can be written in the form

$$W_n(s) = \sum_{k=0}^n \frac{A_{n,k}}{s - a_k^*}, \tag{16}$$

where

$$A_{n,k} = \frac{\prod_{v=0}^{n-1} (a_k^* - a_v)}{\prod_{v=0, v \neq k}^n (a_k^* - a_v^*)} \quad (k = 0, 1, \dots, n). \tag{17}$$

The case when $a_v = a_\mu$ can be considered as a limiting process $a_v \rightarrow a_\mu$.

Alternatively, for $|a_v| < 1$, (17) can be reduced to

$$A_{n,k} = \frac{\prod_{v=0}^{n-1} \left(\frac{1}{\bar{a}_k} - a_v \right)}{\prod_{v/k}^n \left(\frac{1}{\bar{a}_k} - \frac{1}{\bar{a}_v} \right)} = \frac{\bar{a}_0 \cdots \bar{a}_n}{\bar{a}_k} \cdot \frac{\prod_{v=0}^{n-1} (\bar{a}_k a_v - 1)}{\prod_{v/k}^n (\bar{a}_k - \bar{a}_v)}. \tag{18}$$

It is well known that system of functions (14) is orthonormal on the unit circle $|s| = 1$ with respect to the inner product

$$(u, v) = \frac{1}{2\pi i} \oint_{|s|=1} u(s) \overline{v(s)} \frac{ds}{s} = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(e^{i\theta}) \overline{v(e^{i\theta})} d\theta. \tag{19}$$

Namely, $(\phi_n, \phi_m) = \delta_{nm}$ ($n, m = 0, 1, \dots$). This generalises the Szegő class of polynomials orthogonal on the unit circle (see [19]).

We note also that

$$(u, v) = \frac{1}{2\pi i} \oint_{|s|=1} u(s) \overline{v(s^*)} \frac{ds}{s},$$

where $s^* = 1/\bar{s}$ on the unit circle $|s| = 1$.

For the sake of completeness we mention the following result (cf. [21]):

Theorem 8. *If the system of rational functions $\{W_n\}_{n=0}^{+\infty}$ defined by (15) and the inner product (\cdot, \cdot) by (19), then*

$$(W_n, W_m) = \|W_n\|^2 \delta_{nm},$$

where

$$\|W_n\|^2 = \frac{|a_0 a_1 \cdots a_n|^2}{1 - |a_n|^2}.$$

In order to prove Theorem 6 we need the following auxiliary result:

Lemma 9. *Let $-1 \leq t \leq 1$ and let F be defined by*

$$F(t) = \frac{1}{2\pi i} \oint_{|s|=1} W_n(s) \overline{W_m(ts)} \frac{ds}{s}, \tag{20}$$

where the system functions $\{W_n(s)\}$ is defined by (15) with mutually different numbers a_v ($v = 0, 1, \dots$) in the unit circle $|s| = 1$. Then

$$F(t) = \sum_{i=0}^n \sum_{j=0}^m \frac{A_{n,i} \bar{A}_{m,j}}{a_i^* \bar{a}_j^* - t}, \tag{21}$$

where the numbers $A_{n,k}$ are given in (17).

Proof. For $-1 \leq t \leq 1$ we conclude that the integrand in

$$\begin{aligned} F(t) &= \frac{1}{2\pi i} \oint_{|s|=1} \frac{\prod_{v=0}^{n-1} (s - a_v)}{\prod_{v=0}^n (s - a_v^*)} \cdot \frac{\prod_{v=0}^{m-1} \left(\frac{t}{s} - \frac{1}{a_v^*} \right)}{\prod_{v=0}^m \left(\frac{t}{s} - \frac{1}{a_v} \right)} \cdot \frac{ds}{s} \\ &= \frac{(-1)a_0 \cdots a_m}{a_0^* \cdots a_{m-1}^*} \cdot \frac{1}{2\pi i} \oint_{|s|=1} \frac{\prod_{v=0}^{n-1} (s - a_v)}{\prod_{v=0}^n (s - a_v^*)} \cdot \frac{\prod_{v=0}^{m-1} (s - a_v^* t)}{\prod_{v=0}^m (s - a_v t)} ds \end{aligned}$$

has $(m+1)$ poles inside the circle $|s|=1$: $a_j t$ ($j=0, 1, \dots, m$). By Cauchy's residue theorem we find that

$$F(t) = \frac{(-1)a_0 \cdots a_m}{a_0^* \cdots a_{m-1}^*} \sum_{j=0}^m \frac{\prod_{v=0}^{n-1} (a_j t - a_v)}{\prod_{v=0}^n (a_j t - a_v^*)} \cdot \frac{\prod_{v=0}^{m-1} (a_j t - a_v^* t)}{\prod_{v \neq j}^m (a_j t - a_v t)}.$$

Define $G_{m,j}$ by

$$G_{m,j} = \frac{\prod_{v=0}^{m-1} (a_j t - a_v^* t)}{\prod_{v \neq j}^m (a_j t - a_v t)}.$$

Since

$$G_{m,j} = \frac{\prod_{v=0}^{m-1} (a_j \bar{a}_v - 1)}{\prod_{v \neq j}^m (a_j - a_v)} \cdot \frac{1}{\bar{a}_0 \cdots \bar{a}_{m-1}},$$

because of (18), we have

$$G_{m,j} = \frac{a_j}{a_0 \cdots a_m} \cdot \bar{A}_{m,j} \cdot \frac{1}{\bar{a}_0 \cdots \bar{a}_{m-1}} = \frac{a_j \bar{a}_m \bar{A}_{m,j}}{|\bar{a}_0 \cdots \bar{a}_m|^2}.$$

Thus,

$$\frac{a_0 \cdots a_m}{a_0^* \cdots a_{m-1}^*} G_{m,j} = a_j \bar{A}_{m,j},$$

and we obtain

$$F(t) = - \sum_{j=0}^m \bar{A}_{m,j} \frac{\prod_{v=0}^{n-1} \left(t - \frac{a_v}{a_j} \right)}{\prod_{v=0}^n \left(t - \frac{a_v^*}{a_j} \right)}.$$

On the other hand, expanding $Q_j(t) = \prod_{v=0}^{n-1} (t - a_v/a_j) / \prod_{v=0}^n (t - a_v^*/a_j)$ in partial fractions, we get

$$Q_j(t) = \sum_{i=0}^n \frac{\prod_{v=0}^{n-1} \left(\frac{a_i^*}{a_j} - \frac{a_v}{a_j} \right)}{\prod_{v \neq i}^n \left(\frac{a_i^*}{a_j} - \frac{a_v^*}{a_j} \right)} \cdot \frac{1}{t - \frac{a_i^*}{a_j}} = \sum_{i=0}^n \frac{\prod_{v=0}^{n-1} (a_i^* - a_v)}{\prod_{v \neq i}^n (a_i^* - a_v^*)} \cdot \frac{1}{t - a_i^* \bar{a}_j^*}.$$

Because of (17), the right hand side of the last equality becomes

$$\sum_{i=0}^n \frac{A_{n,i}}{t - a_i^* \bar{a}_j^*},$$

so that we obtain

$$F(t) = \sum_{j=0}^m \bar{A}_{m,j} \sum_{i=0}^n A_{n,i} \frac{1}{a_i^* \bar{a}_j^* - t},$$

i.e., (21). \square

For $t=1$, from (20) we see that $F(1) = (W_n, W_m)$. Thus,

$$(W_n, W_m) = \sum_{i=0}^n \sum_{j=0}^m \frac{A_{n,i} \bar{A}_{m,j}}{a_i^* \bar{a}_j^* - 1}. \quad (22)$$

Now, we give a connection between the Malmquist system of rational functions (15) and a Müntz system, which is orthogonal with respect to the inner product (5).

At first, we assume that $A = \{\lambda_0, \lambda_1, \dots\}$ is a complex sequence satisfying (4) and define $a_v = 1/\bar{\lambda}_v$. Then we use the Malmquist system (15) with these a_v . The corresponding numbers a_v^* ($= 1/\bar{a}_v = \lambda_v$) which appear in (15) are outside the unit circle and they form the Müntz system $\{x^{a_0^*}, x^{a_1^*}, \dots, x^{a_n^*}\}$. In order to shorten our notation and to be consistent with the previous section, we write $a_v^* = \lambda_v$ ($v=0, 1, \dots$)

Finally, we can prove Theorem 6:

Proof of Theorem 6. According to (6) and (10), we have

$$[Q_n, Q_m] = \sum_{i=0}^n \sum_{j=0}^m \frac{A_{n,i} \bar{A}_{m,j}}{\lambda_i \bar{\lambda}_j - 1},$$

where $A_{n,k}$ is given by (11).

Using Lemma 9 with $t=1$, i.e., equality (22), we conclude that

$$[Q_n, Q_m] = (W_n, W_m),$$

where $W_n(s)$ is determined by (8).

Since $a_v = 1/\bar{\lambda}_v$ ($v=0, 1, \dots$), (12) follows from Theorem 8. \square

For the norm of the polynomial $Q_n(x)$ we obtain

$$\|Q_n\| = \sqrt{[Q_n, Q_n]} = \frac{1}{|\lambda_0 \lambda_1 \cdots \lambda_{n-1}|} \cdot \frac{1}{\sqrt{|\lambda_n|^2 - 1}},$$

where the complex numbers satisfy the condition (4).

4. Recurrence formulae

Some recurrence formulae for polynomials $Q_n(x)$ are given in the following theorem. Similar relations were obtained in [3] for Müntz–Legendre polynomials.

Theorem 10. Suppose that Λ is a complex sequence satisfying (4). Then the polynomials $Q_n(x)$, defined by (9), satisfy the following relations:

$$xQ'_n(x) = xQ'_{n-1}(x) + \lambda_n Q_n(x) - (1/\bar{\lambda}_{n-1})Q_{n-1}(x), \quad (23)$$

$$xQ'_n(x) = \lambda_n Q_n(x) + \sum_{k=0}^{n-1} (\lambda_k - 1/\bar{\lambda}_k) Q_k(x), \quad (24)$$

$$xQ''_n(x) = (\lambda_n - 1)Q'_n(x) + \sum_{k=0}^{n-1} (\lambda_k - 1/\bar{\lambda}_k) Q'_k(x), \quad (25)$$

$$Q_n(1) = 1, \quad Q'_n(1) = \lambda_n + \sum_{k=0}^{n-1} (\lambda_k - 1/\bar{\lambda}_k), \quad (26)$$

$$Q_n(x) = Q_{n-1}(x) - (\lambda_n - 1/\bar{\lambda}_{n-1})x^{\lambda_n} \int_x^1 t^{-\lambda_n-1} Q_{n-1}(t) dt. \quad (27)$$

In proving this theorem we use the same method as in [3]. Because of that, we omit the proof.

5. Real zeros of Müntz polynomials

Now we consider the orthogonal real Müntz polynomials when the sequence Λ is real and such that

$$1 < \lambda_0 < \lambda_1 < \dots \quad (28)$$

Theorem 11. Let Λ be a real sequence satisfying (28). Then the polynomial $Q_n(x)$, defined by (9), has exactly n simple zeros in $(0, 1)$ and no other zeros in $[1, \infty)$.

Proof. Because of orthogonality we have

$$\int_0^1 x^{\lambda_v} \odot Q_n(x) \frac{dx}{x^2} = 0 \quad (v=0, 1, \dots, n-1).$$

Since $x^{\lambda_v} \odot Q_n(x) = Q_n(x^{\lambda_v})$, these orthogonality conditions reduce to

$$\int_0^1 Q_n(x^{\lambda_v}) \frac{dx}{x^2} = 0 \quad (v=0, 1, \dots, n-1),$$

or, after changing variable $x^{\lambda_v} = t$,

$$\int_0^1 Q_n(t)t^{-\mu_v} dt = 0 \quad (v=0, 1, \dots, n-1), \quad (29)$$

where $\mu_v = 1 + 1/\lambda_v$. Note that $1 < \mu_v < 2$ and μ_v are mutually different under condition (28).

According to (29) the polynomial $Q_n(t)$ has at least one zero in $(0, 1)$ in which $Q_n(t)$ changes the sign. On the other side, it is easy to prove by induction (cf. [11, Theorem. 4.1]) that every real Müntz polynomial from $M_n(\Lambda)$ has at most n zeros in $(0, \infty)$.

In order to prove our theorem, we suppose that the number of sign changes of $Q_n(t)$ in $(0, 1)$ is $k < n$. Let τ_1, \dots, τ_k be the points where $Q_n(t)$ changes sign. Now, we can construct a polynomial $P_k(t) = \sum_{i=0}^k b_i t^{-\mu_i}$, such that $P_k(\tau_v) = 0$ for each $v = 1, \dots, k$, and, for example, $b_0 = 1$. We note also that $P_k(t)$ has no other zeros in $(0, \infty)$. For existence and uniqueness of such a polynomial see [2, Chap. 3].

As we can see now, $P_k(t)Q_n(t)$ has a constant sign in $(0, 1)$, and therefore

$$\int_0^1 P_k(t)Q_n(t) dt \neq 0.$$

On the other side, using (29), we have

$$\int_0^1 P_k(t)Q_n(t) dt = \int_0^1 \left(\sum_{i=0}^k b_i t^{-\mu_i} \right) Q_n(t) dt = \sum_{i=0}^k b_i \int_0^1 t^{-\mu_i} Q_n(t) dt = 0$$

for $k < n$, which gives a contradiction. Thus, $k = n$. The points τ_1, \dots, τ_n are simple zeros of $Q_n(t)$ in $(0, 1)$ and this polynomial has no other positive zeros. \square

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