MODIFIED WEIERSTRASS METHOD WITH GAUSS–SEIDEL APPROACH FOR FINDING MULTIPLE POLYNOMIAL ZEROS*

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Abstract. It is well-known that the convergence of Weierstrass method for simultaneous finding polynomial zeros is quadratic in the case of simple zeros but only linear in the case of multiple zeros. Using Gauss-Seidel approach the convergence can be accelerated. Introducing the mean of the components converging to the same zero, in this paper we give a modified method which enables the multiplicity order of the zeros to be determined during the algorithm. In addition, the method presented has a fast convergence. Comparison with a similar method [5] is performed in a few numerical examples.

1. Introduction

Let

$$P(z) = \sum_{i=0}^{n} a_i z^{n-i} \ , \quad a_i \in \mathbb{R} \ (i = 0, 1, \dots, n) \ , \ a_n \neq 0 \ ,$$

be a monic complex polynomial of degree n and let $w_1, \dots, w_n$ be exact zeros of this polynomial, and $z_1, \dots, z_n$ their approximations, respectively. Since, for a fixed $j \in \{1, \dots, n\}$,

$$P(z) = \prod_{i=1}^{n} (z - w_i) = (z - w_j) \prod_{\substack{i=1 \\ i \neq j}}^{n} (z - w_i) \ ,$$

we have

$$w_j = z - \frac{P(z)}{\prod\limits_{\substack{i=1 \\ i \neq j}}^{n} (z - w_i)} \ .$$

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Then, for $z = z_j$ and $w_i \approx z_i$, we obtain

$$w_j \approx z_j - \frac{P(z_j)}{\prod\limits_{\substack{i=1 \\ i \neq j}} (z_j - z_i)} \quad (j = 1, \dots, n) \ .$$

This suggests us to construct the following iterative method for the simulataneous finding of all polynomial zeros,

$$(1) \qquad z_j^{(m+1)} = z_j^{(m)} - \frac{P\big(z_j^{(m)}\big)}{\prod\limits_{\substack{i=1 \\ i \neq j}} \big(z_j^{(m)} - z_i^{(m)}\big)} \quad (j = 1, \dots, n \ ; \ m = 0, 1, \dots) \ ,$$

which is known as Weierstrass' method ([11]). The same iterative method was derived in many different ways by several authors (see, for example, [3], [2], [7], [8], [4]). In the case when all zeros w_i are simple and if z_i are reasonably close approximations of these zeros w_i $(i = 1, \dots, n)$, the method (1) has quadratic convergence. In the case of multiple zeros the method (1) converges only linearly (cf. [10]).

Consider a monic polynomial P of degree n,

$$P(z) = \sum_{i=0}^{n} a_i z^{n-i} = \prod_{s=1}^{k} (z - \xi_s)^{\nu_s} \quad (a_i \in \mathbb{R})$$

with real or complex zeros $\xi_1, \dots, \xi_k$ $(k \leq n)$ having the order of multiplicity $\nu_1, \dots, \nu_k$ respectively, where $\nu_1 + \cdots + \nu_k = n$. Let $Y = [y_1 \dots y_n]^T$ be a vector, so-called *vector of means*, defined by

$$(2) \qquad y_j = \frac{1}{\nu_h} \sum_{i \in I_h} z_i \ , \quad j \in I_h = \{i | w_i = \xi_h\} \ , \ h = 1, \dots, k \ .$$

Using the vector Y, Fraigniaud [5] gave a modification of the iterative method (1) and showed that the sequence $Y^{(m)} = \big[y_1^{(m)} \dots y_n^{(m)}\big]^T$ $(m = 0, 1, \dots)$ satisfies the equality

$$(3) \qquad \big\|Y^{(m+1)} - w\big\| = O\Big(\big\|Z^{(m)} - w\big\|^2\Big) \ ,$$

where $w = [w_1 \dots w_n]^T$ i $Z^{(m)} = \big[z_1^{(m)} \dots z_n^{(m)}\big]^T$. In the paper [5] the property given by (3) was called the Quadratic-Like Convergence of the Mean, and was noted QLCM.

Starting from the Weierstrass method (1), with the so-called Gauss-Seidel approach, and using a modification by the vector Y, in this paper we give an algorithm

for finding polynomial zeros. Numerical examples are included in order to compare results obtained by this method and by the method from [5].

2. Iterative method

Applying the Gauss-Seidel approach to the acceleration of the method (1) we obtain the following iterative method

$$(4)\qquad z_i^{(m+1)} = z_i^{(m)} - \frac{P(z_i^{(m)})}{\prod_{j=1}^{i-1}(z_i^{(m)} - z_j^{(m+1)}) \prod_{j=i+1}^{n}(z_i^{(m)} - z_j^{(m)})} = z_i^{(m)} - \Delta z_i^{(m)} ,$$

which was considered in the papers [1] (real case) and [9] (interval case). This method has R-order of convergence greater than 2 for simple zeros and greater than 1 for multiple zeros.

As in [5], suppose that the approximations $z_1^{(m)}, \dots, z_n^{(m)}$ are pairwise distinct. Using the vector of means, we will modify the method (4) in the following sense. If $i_h = \min_{i \in I_h} i$, in each iterative step m we determine the index $i(h,m) = i_h + (m \mod \nu_h)$ for every $h = 1, \dots, k$. Then, after finding $z_1^{(m+1)}, \dots, z_n^{(m+1)}$ by (4), we change the component $z_{i(h,m)}^{(m+1)}$ by $(1/\nu_h)\sum_{i \in I_h} z_i^{(m+1)}$. Evidently, for this modification we need multiplicities ν_i $(i = 1, \dots, k)$ of the zeros $\xi_1, \dots, \xi_k$. A practical way for estimating these multiplicities was given in [5] and based on the fact that

$$\lim_{m \to \infty} \frac{\Delta z_i^{(m+1)}}{\Delta z_i^{(m)}} = \frac{\nu_i - 1}{\nu_i} \quad (i = 1, \dots, n) .$$

Therefore, our algorithm can be expressed in the following form:

1° *Find some starting approximations* $z_1^{(0)}, \dots, z_n^{(0)}$ *of zeros* $w_1, \dots, w_n$.

2° *Use the method* (4) *and in each iteration calculate the ratio*

$$\frac{\Delta z_i^{(m+1)}}{\Delta z_i^{(m)}} \quad (i = 1, \dots, n \ ; \ m = 0, 1, \dots) .$$

Stop this process for $m = M$ *if*

$$(5)\qquad \left| \frac{\Delta z_i^{(M)}}{\Delta z_i^{(M-1)}} - \frac{\Delta z_i^{(M-1)}}{\Delta z_i^{(M-2)}} \right| < \varepsilon , \quad i = 1, \dots, n ,$$

where ε *is a given required accuracy. From ratios* $\Delta z_i^{(M)} / \Delta z_i^{(M-1)}$ *which converge to the same value* r_h*, determine* $\lfloor 1/(1 - r_h) \rfloor$ ($\lfloor x \rfloor$ *is the closest integer to* x)*, and take it as the multiplicity* ν_h $(h = 1, \dots, k)$.

3° *Split the components* $z_i^{(M)}$ $(i = 1, \dots, n)$ *converging towards the same zero into groups, and set them as starting approximations* $z_i^{(0)}$ $(i = 1, \dots, n)$ *for the next two steps.*

4° *For each* $h = 1, \dots, k$, *calculate the index* $i(h, m) = i_h + (m \mod \nu_h)$ *and replace the corresponding component* $z_{i(h,m)}^{(m)}$ *by* $(1/\nu_h) \sum_{i \in I_h} z_i^{(m)}$ $(m = 0, 1, \dots)$.

5° *Apply the method* (4) *and Step* 4° (*in the scope of one iteration*) *to the previous obtained vector* $\left[z_1^{(m)} \dots z_n^{(m)}\right]^T$ $(m = 0, 1, \dots)$, *until all components* $z_i^{(m)}$ $(i = 1, \dots, n)$ *achieve a given required accuracy.*

This modification of the accelerated Weierstrass method has some adventages over the method given in [5] (F-method in the sequel). Namely, because of faster convergence of method with Gauss–Seidel approach, inequality (5) achieves here with a smaller M, so that Step 2° requires less iterations than the corresponding step in [5]. Apart from the faster convergence, this modified method is more suitable for programming and, also, occupy less computer storage space because the new approximations take positions of the previous ones. The number of all operations which are necessary in realization of one iteration of this method is the same as in the F-method, so that the computational efficiency cannot be smaller than one of F-method. On the other hand, the proposed method converges also faster than accelerated Weierstrass method, which is obviously based on the arguments from [5].

3. Numerical results

In a few examples we will illustrate the previous consideration and compare our results with ones obtained by the method (4) and F-method. In these examples we take the starting approximations obtained by a simple algorithm from [6]. All computations were performed in double precision arithmetic using FORTRAN 77.

Example 1. We consider the polynomial

$$P(z) = z^4 - 2z^2 + 1 = (z-1)^2(z+1)^2$$

with exact zeros $\xi_1 = w_1 = w_2$ and $\xi_2 = w_3 = w_4$ having the order of multiplicity $\nu_1 = \nu_2 = 2$. Regarding to [6], for starting values we take:

$$\begin{aligned} z_1^{(0)} &= (0.0, 0.35669)\,, \qquad z_2^{(0)} = (-0.35669, 0.0)\,, \\ z_3^{(0)} &= (0.0, -0.35669)\,, \quad z_4^{(0)} = (0.35669, 0.0)\,. \end{aligned} \tag{6}$$

Step 2°, with an accuracy $\varepsilon = 10^{-2}$, is realized in 10 iterations, where

$$\begin{aligned} z_1^{(10)} &= (0.99989, 0.00127)\,, \qquad z_2^{(10)} = (-1.00014, 0.00164)\,, \\ z_3^{(10)} &= (-0.99991, -0.00101)\,, \quad z_4^{(10)} = (1.00006, -0.00079)\,. \end{aligned}$$

Starting from these approximations in order to obtain the zeros $\xi_{1,2}$ with 5 exact decimals we need 3 additional iterations (i.e., three applications of Step 5°). Thus, the total number of iterations is 13. On the other hand, starting from the same values (6), the method (4) gives $\xi_{1,2}$ (with the same accuracy) after 17, while the F-method even after 65 iterations.

With the starting values $z_1^{(0)} = (0.9, 0.3)$, $z_2^{(0)} = (1.1, -0.2)$, $z_3^{(0)} = (-0.8, 0.3)$, $z_4^{(0)} = (-1.2, 0.1)$, which were taken in [5], the F-method needs 6, the method (4) 12, and our method 8 iterations, wherefrom one can see that these methods are sensitive to the choice of starting values. Notice that it is not clear how to select the previous starting values (i.e., which algorithm to use), because the author of [5] mentioned only reference [6] in that context.

EXAMPLE 2. The polynomial

$$P(z) = z^5 + z^4 - 2z^3 - 2z^2 + z + 1 = (z-1)^2(z+1)^3$$

has exact zeros $\xi_1 = w_1 = w_2 = 1$ and $\xi_2 = w_3 = w_4 = w_5 = -1$ with their multiplicities $\nu_1 = 2$ and $\nu_2 = 3$. Set

$$\begin{aligned} &z_1^{(0)} = (0.32189, 0.99069)\,, && z_2^{(0)} = (-0.84273, 0.61228)\,, \\ &z_3^{(0)} = (-0.84273, -0.61228)\,, && z_4^{(0)} = (0.32189, -0.99069)\,, \\ &z_5^{(0)} = (1.04167, 0.0)\,. \end{aligned} \tag{7}$$

Under conditions same as in Example 1, the F-method gives requested results in 84 iterations, method (4) in 23, and our method in 21 iterations. In this case, it is interesting that the basic Weierstrass method (1) converges faster than the F-method. Namely, it gives the corresponding results in 79 iterations. As in Example 1, the starting values from [5] give an advantage over our method.

EXAMPLE 3. Consider now the polynomial

$$\begin{aligned} P(z) = {} & z^6 + (-4+4i)z^5 + (2-16i)z^4 + (12+24i)z^3 + (-23-16i)z^2 \\ & + (16+4i)z - 4 = (z-1)^4(z+2i)^2 \end{aligned}$$

with a double complex zero $\xi_1 = w_1 = w_2 = -2i$ and a real zero of multiplicity four, $\xi_2 = w_3 = w_4 = w_5 = w_6 = 1$. Set $\varepsilon = 10^{-2}$ and take starting approximations regarding to [6]. Now, Step 2° is realized in 15 and 22 iterations in our and the F-method, respectively. For an accuracy of 3 exact decimals, Step 5° needs only 2 iterations in our method, and 15 in the second one.

If we take starting approximations (similar to the choice in [5])

$$\begin{aligned} &z_1^{(0)} = (1.1, 0.1)\,, && z_2^{(0)} = (1.2, -0.2)\,, && z_3^{(0)} = (0.9, 0.3)\,, \\ &z_4^{(0)} = (0.8, -0.1)\,, && z_5^{(0)} = (0.1, -2.2)\,, && z_6^{(0)} = (-0.2, -1.9)\,, \end{aligned}$$

the F-method needs 18, and our method 13 iterations.

All examples point out that F-method is much more sensitive to the choice of starting values than our method. Since in [5] the algorithm for the starting values was not given, the method presented is, in general case, more effective. However, the starting values problem for method in [5] is still open.

Finally, based on the practical tests, one can notice that in both methods (F-method and our one) the number of iterations increases with the order of multiplicity.

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