NEGATION AND IMPOSSIBILITY

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In this paper we shall present a method for modelling negation inspired by modal logic. The gist of the method is to treat negation as an impossibilty operator added to negationless logic. To model this impossibility operator we use Kripke-style models which have an R_N relation on the set of "worlds" such that for every world x

$$x \models \neg A \Leftrightarrow \forall y(xR_N y \Rightarrow y \not\models A).$$

We shall concentrate here on propositional logics obtained by extending Heyting's positive logic with some negation axioms. It is easy to pass from models for these logics to models for logics obtained by extending the positive fragment of classical propositional logic, or of some intermediate propositional logics, as we indicate briefly at the end of this paper. As a matter of fact, the method we shall present is even more general: it can be used with relevant model structures in order to model various extensions of positive fragments of relevant propositional logics, as it is shown in [1].

We shall first consider the weakest extension of Heyting's positive logic captured by our models. This logic will be called N. In models of N the R_N relation is as general as possible, and hence N is in the same position as the minimal normal modal logic K. Next we shall consider what conditions concerning R_N correspond to various negation axioms. Completeness with respect to models satisfying these conditions can be proved along rather familiar lines, and we shall only indicate briefly how to obtain these completeness proofs.

Models for systems with negation still weaker than negation in N could be obtained by adapting the neighbourhood semantics for modal logic. (These models are treated in [9].)

This paper summarizes results which are given in more detail in

[6] and [5]. It is an offspring of the treatment of intuitionistic modal logic presented in [2] and [4].

<u>The system N.</u> The system N and its extensions are formulated in a standard propositional language which we shall call L. In L we have denumerably many propositional variables, for which we use the schematic letters p,q,...; the connectives of L are $\rightarrow$, $\wedge$, $\vee$ and $\neg$. We use A,B,C,... as schematic letters for formulae. The symbols $\forall$, $\exists$, $\Rightarrow$, $\Leftrightarrow$, &, <u>or</u>, <u>iff</u>, <u>not</u>, and set-theoretical symbols, will be used in the metalanguage with the usual meaning they have in classical logic.

The system N is obtained by extending a standard axiomatization of Heyting's positive logic in L (see [6]) with the rule

$$\frac{A \rightarrow B}{\neg B \rightarrow \neg A}$$

and the axiom schema

$$(\neg A \wedge \neg B) \rightarrow \neg (A \vee B).$$

Models with respect to which N can be shown sound and complete are defined as follows.

<u>Definition 1.</u> $Fr = \langle X, R_I, R_N \rangle$ is an N <u>frame</u> iff

(i) X is a nonempty set,

(ii) $R_I \subseteq X^2$ and R_I is reflexive and transitive,

(iii) $R_N \subseteq X^2$ and $R_I R_N \subseteq R_N R_I^{-1}$.

<u>Definition 2.</u> $M = \langle X, R_I, R_N, V \rangle$ is an N <u>model</u> iff

(i) $\langle X, R_I, R_N \rangle$ is an N frame,

(ii) V is a mapping from the set of propositional variables of L to the power set of X such that for every p and for every x and y in X we have $xR_I y \Rightarrow (x \in V(p) \Rightarrow y \in V(p))$.

If $M = \langle X, R_I, R_N, V \rangle$ and $x \in X$, the relation A <u>holds in</u> x, i.e. $x \models A$, is defined by the following equivalences:

(i) $x \models p \Leftrightarrow x \in V(p)$

(ii) $x \models B \wedge C \iff (x \models B \ \& \ x \models C)$

(iii) $x \models B \vee C \iff (x \models B$ or $x \models C)$

(iv) $x \models B \rightarrow C \iff \forall y(xR_I y \Rightarrow (y \models B \Rightarrow y \models C))$

(v) $x \models \neg B \iff \forall y(xR_N y \Rightarrow y \not\models B)$, where $y \not\models B$ means not $y \models B$.

A formula A holds in a model M, i.e. $M \models A$, iff A holds in every x in the X of M; A holds in a frame Fr, i.e. $Fr \models A$, iff A holds in every M with the frame Fr.

The only point which needs explanation in these definitions is the condition $R_I R_N \subseteq R_N R_I^{-1}$ which N frames must satisfy. This condition is sufficient to prove the following statement:

Intuitionistic Heredity. In every N model, for every x and y, and for every A of L, we have $xR_I y \Rightarrow (x \models A \Rightarrow y \models A)$.

In fact, $R_I R_N \subseteq R_N R_I^{-1}$ is both necessary and sufficient for Intuitionistic Heredity (see [6]). There is a certain regularity in this condition. Suppose that we extend L with the modal operators $\Box, \Diamond, \not\Box$ and $\not\Diamond$, and that in corresponding models of the type $\langle X, R_I, R_M, V \rangle$ we require that:

$$x \models \Box A \iff \forall y(xR_M y \Rightarrow y \models A)$$
$$x \models \Diamond A \iff \exists y(xR_M y \ \& \ y \models A)$$
$$x \models \not\Box A \iff \exists y(xR_M y \ \& \ y \not\models A)$$
$$x \models \not\Diamond A \iff \forall y(xR_M y \Rightarrow y \not\models A).$$

Then necessary and sufficient conditions for Intuitionistic Heredity in these models will be respectively:

$$R_I R_M \subseteq R_M R_I$$
$$R_I^{-1} R_M \subseteq R_M R_I^{-1}$$
$$R_I^{-1} R_M \subseteq R_M R_I .$$
$$R_I R_M \subseteq R_M R_I^{-1}.$$

The minimal systems captured by these models are obtained by extending the Heyting propositional calculus with respectively:

$$\frac{A \rightarrow B}{\Box A \rightarrow \Box B}, \quad (\Box A \wedge \Box B) \rightarrow \Box(A \wedge B), \quad \Box(A \rightarrow A)$$

$$\frac{A \to B}{\Diamond A \to \Diamond B}, \quad \Diamond(A \vee B) \to (\Diamond A \vee \Diamond B), \quad \neg\Diamond\neg(A \to A)$$

$$\frac{A \to B}{\emptyset B \to \emptyset A}, \quad \emptyset(A \wedge B) \to (\emptyset A \vee \emptyset B), \quad \neg\emptyset(A \to A)$$

$$\frac{A \to B}{\oslash B \to \oslash A}, \quad (\oslash A \wedge \oslash B) \to \oslash(A \vee B), \quad \oslash\neg(A \to A).$$

(Proofs of these results are in [2] and [5] .) As $\oslash$ corresponds to $\neg$, so $\emptyset$ corresponds to an operator dual to intuitionistic negation, which is called Brouwerian negation. (Brouwerian negation is investigated in [7] and [8].)

An N frame (model) will be called <u>condensed</u> if $R_I R_N = R_N$, and it will be called <u>strictly condensed</u> if $R_I R_N = R_N R_I^{-1} = R_N$. The conditions of condensed and strictly condensed N frames are sufficient for Intuitionistic Heredity, but they are not necessary. A further "condensation" of N models would be made by requiring that R_I is not only reflexive and transitive, but a partial-ordering relation. The completeness results which follow would also hold for such an R_I. (Similar "condensations" can be achieved for models connected with $\Box$, $\Diamond$ and $\emptyset$, as it is shown in [2] and [5] .)

Next, it is possible to prove the following theorem:

<u>Theorem 1</u>. The formula A is provable in N iff A holds in every (condensed, strictly condensed) N frame.

This completeness theorem is proved by using a rather standard technique of canonical models (see [6]). The canonical N frame will be made of <u>theories</u>, i.e. sets of formulae which are deductively closed and have the disjunction property, but are <u>not</u> necessarily consistent. If Γ and Δ are theories, the R_I relation on the canonical N frame is defined as usual by $\Gamma R_I \Delta \Leftrightarrow \Gamma \subseteq \Delta$, and the R_N relation by:

$$\Gamma R_N \Delta \Leftrightarrow \forall A(\neg A \in \Gamma \Rightarrow A \notin \Delta).$$

Let us explain where this definition of R_N comes from. For the canonical model we must show that

$$\Gamma \models A \Leftrightarrow A \in \Gamma.$$

In order to prove this equivalence for canonical models of systems with $\Box, \Diamond, \not\Box$ and $\not\Diamond$ we must have (see [2] and [5]):

$$\Box A \in \Gamma \Rightarrow (\Gamma R_M \Delta \Rightarrow A \in \Delta)$$
$$\Gamma R_M \Delta \ \& \ A \in \Delta \Rightarrow \Diamond A \in \Gamma$$
$$\Gamma R_M \Delta \ \& \ A \notin \Delta \Rightarrow \not\Box A \in \Gamma$$
$$\not\Diamond A \in \Gamma \Rightarrow (\Gamma R_M \Delta \Rightarrow A \notin \Delta).$$

From these implications we can extrapolate respectively the equivalences:

$$\Gamma R_M \Delta \Leftrightarrow \forall A(\Box A \in \Gamma \Rightarrow A \in \Delta)$$
$$\Gamma R_M \Delta \Leftrightarrow \forall A(A \in \Delta \Rightarrow \Diamond A \in \Gamma)$$
$$\Gamma R_M \Delta \Leftrightarrow \forall A(A \notin \Delta \Rightarrow \not\Box A \in \Gamma)$$
$$\Gamma R_M \Delta \Leftrightarrow \forall A(\not\Diamond A \in \Gamma \Rightarrow A \notin \Delta).$$

The last equivalence then corresponds to the definition of R_N. The canonical N model is a strictly condensed N model.

<u>Extensions of N</u>. In this section we shall consider a number of negation axioms which correspond to first-order conditions on N frames. Using N frames which satisfy these conditions we can prove completeness theorems for a number of familiar extensions of N.

Let $R_\neg$ be an abbreviation for $R_N R_I^{-1}$ and let Fr be an N frame; then we can show the following equivalences (see [6]):

(1) $Fr \models A \rightarrow \neg\neg A \Leftrightarrow R_\neg$ is symmetric

(2) $Fr \models (A \rightarrow B) \rightarrow (\neg B \rightarrow \neg A) \Leftrightarrow \forall x \forall y(xR_N y \Rightarrow \exists z(xR_I z \ \& \ yR_I z \ \& \ xR_N z))$

(3) $Fr \models A \vee \neg A \Leftrightarrow Fr \models (\neg A \rightarrow A) \rightarrow A \Leftrightarrow R_\neg \subseteq R_I^{-1}$

(4) $Fr \models \neg A \vee \neg\neg A \Leftrightarrow R_\neg^{-1} R_\neg \subseteq R_\neg$

(5) $Fr \models \neg(A \rightarrow A) \rightarrow B \Leftrightarrow \forall x \exists y \ xR_\neg y$

(6) $Fr \models (A \wedge \neg A) \rightarrow B \Leftrightarrow R_\neg$ is reflexive

(7) $Fr \models \neg\neg A \rightarrow A \Leftrightarrow \forall x \exists y(xR_N y \ \& \ \forall z(yR_N z \Rightarrow zR_I x))$.

The Johansson propositional calculus J can be obtained by extending N with the axiom schemata of (1) and (2). The Heyting propositional calculus H can be obtained by extending J with the axiom schema of (6). Curry's system D (see [3], Chapter 6) can be obtained by extending J with either of the axiom schemata of (3). All these systems are complete with respect to (condensed, strictly condensed) N frames which satisfy the corresponding conditions in the equivalences above.

It is well known that J is complete with respect to Q models of the form $\langle X, R_I, Q, V\rangle$ where X, R_I and V are as before, and $Q \subseteq X$ is such that for every x and y in X we have $xR_Iy \Rightarrow (x \in Q \Rightarrow y \in Q)$; $x \models A$ is defined as before save that we have $x \models \neg B \Leftrightarrow \forall y(xR_Iy \Rightarrow (y \models B \Rightarrow y \in Q))$. As we have remarked above, J is also complete with respect to N models which satisfy the conditions of (1) and (2). Let us call these N models, J models. Now, Q models are intertranslatable with strictly condensed J models, as the following theorems of [6] show:

Theorem 2.1. Let $M_Q = \langle X, R_I, Q, V\rangle$ be a Q model, and let R_N be defined over X by

(i) $xR_Ny \Leftrightarrow \exists z(xR_Iz \;\&\; yR_Iz \;\&\; z \notin Q)$.

Then $M_N = \langle X, R_I, R_N, V\rangle$ is a strictly condensed J model such that

(ii) $z \in Q \Leftrightarrow \exists x \exists y(xR_Iz \;\&\; yR_Iz \;\&\; \underline{\text{not}}\; xR_Nz)$

(iii) $x \models A$ in $M_Q \Leftrightarrow x \models A$ in M_N.

Theorem 2.2. Let $M_N = \langle X, R_I, R_N, V\rangle$ be a strictly condensed J model, and let $Q \subseteq X$ be defined by (ii) of Theorem 2.1. Then $M_Q = \langle X, R_I, Q, V\rangle$ is a Q model such that (i) and (iii) of Theorem 2.1 hold.

As R_N becomes definable in terms of Q and R_I in strictly condensed J models, so it is definable in terms of R_I alone in strictly condensed H models, i.e. strictly condensed J models in which R_N is reflexive. Namely, it is possible to prove that a strictly condensed N frame is an H frame iff $R_N = R_IR_I^{-1}$. This is connected with the fact that in ordinary Kripke models for H, of the form $\langle X, R_I, V\rangle$, we have $x \models \neg A \Leftrightarrow \forall y(xR_IR_I^{-1}y \Rightarrow y \not\models A)$. This also points towards a certain connection between intuitionistic negation and the Brouwersche modal logic B (based on classical propositional logic), for which Kripke frames $\langle X, R_M\rangle$ where R_M is reflexive and symmetric are characteristic.

Curry's system E is obtained by extending J with $((A \rightarrow B) \rightarrow A) \rightarrow A$ (see [3], Chapter 6). It is easy to conclude that E is sound and complete with respect to (condensed, strictly condensed) J frames in which R_I is an equivalence relation, or identity. In general, N models with R_I an equivalence relation, or identity, can serve to study systems which like E are obtained by extending the negationless fragment of classical propositional logic with some negation axioms. Similarly, systems related to Dummett's intermediate logic LC, which are obtained by extending the negationless fragment of Heyting's propositional logic with $(A \rightarrow B) \vee (B \rightarrow A)$ and some negation axioms, could be studied with N models where R_I is a linear-ordering relation.

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REFERENCES

[1] M.Božić, A Contribution to the Semantics of Relevant Logics (in Serbo-Croatian), doctoral dissertation, University of Belgrade, 1983.

[2] M.Božić and K.Došen, Models for normal intuitionistic modal logics, Studia Logica 43(1984).

[3] H.B.Curry, Foundations of Mathematical Logic, McGraw-Hill, New York 1963.

[4] K.Došen, Models for stronger normal intuitionistic modal logics, Studia Logica 44(1985).

[5] K.Došen, Negative modal operators in intuitionistic logic, Publications de l'Institut Mathématique n.s. 35(49)(1984).

[6] K.Došen, Negation as a modal operator, submitted to the Reports on Mathematical Logic.

[7] J.C.C.McKinsey and A.Tarski, On closed elements in closure algebras, Annals of Mathematics 47(1946),pp.122-162.

[8] C.Rauszer, An algebraic and Kripke-style approach to a certain extension of intuitionistic logic, Dissertationes Mathematicae 167(1980),pp.5-62.

[9] V.H.Sotirov, Modal theories with intuitionistic logic, in Mathematical Logic, Proceedings of the Conference on Mathematical Logic dedicated to A.A.Markov, Bulgarian Academy of Sciences, Sofia 1984.