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MODAL DUALITY THEORY

Kosta Došen

Abstract. This talk is about some results concerning the duality between modal algebras and frames. The presentation of these results is preceded by an introductory part, in which an assessment is made of modal logic in the light of contemporary research. There the generality of this research is stressed, and it is to illustrate this generality that attention is focused on modal duality theory, one of the most abstract areas of modal logic.

Introduction

Modal logic is the general theory of unary propositional operators. This is not a definition one is likely to find in textbooks. Neither is it a definition applicable to modal logic from the beginning of its history in the twentieth century. A change of subject occurred in modal logic in the sixties, with the advent of formidable model-theoretic tools. Before, modal logicians studied particular systems, which were meant to formalize the notions of *necessity* and *possibility*, and they produced a real jungle of such systems. After the sixties, modal logicians were increasingly less concerned with particular systems, and concentrated their attention on methods with which they could deal with whole classes of systems. These classes cover the more traditional systems of modal logic, but they include also many things whose connexion with *necessity* and *possibility*, in spite of some family resemblances, is at best remote. Nowadays, particular systems in works of modal logic often occur only as examples, to make

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this or that technical point, and for no other purpose.

Another change of direction of research occurred in the seventies. Now the point was not so much the development of tools to deal with this or that particular system, or even with whole classes of systems: the model-theoretic tools themselves became an object of study. Of course, the abstract study of models can have repercussions on their development for eventual application. However, as it happens often in mathematics, this application is not the main inspiration: the abstract study of models is motivated by independent mathematical interest.

It is because of this concern with whole classes of systems, and with the abstract study of models, that we claim that modal logic is the general theory of unary propositional operators.

What are the unary propositional operators modal logic deals with? We said these are not anymore only the traditional operators "it is necessary that", and its dual, "it is possible that". The search for a single system formalizing these operators has probably come to an end. There is no such single system. The two operators above can have a variety of meanings, depending on the context they are used in. What modal logic can give us are tools to deal with practically any of these particular meanings.

Because of this vagueness in the meaning of "it is necessary that", the study of this operator was replaced by the study of unary operators whose meaning is similar, but more precise - and also more interesting for mathematics. It seems safe to say that "it is necessary that" is not anymore the central unary operator of modal logic. If there is such a central operator, then that would be "it is provable that".

Again, this is not a claim one is likely to find in textbooks, but a number of facts could substantiate this claim.

First, the best known modal logics are S4, S5, and some logics in their vicinity. Already on an intuitive level, the connexion between these logics and provability is quite strong (see [Lemmon 1959]). On a more technical level, there is a famous translation of Heyting's logic into S4, or into logics in the vicinity of S4 (cf. [Došen 1986]), and this translation justifies reading S4-like necessity operators as "it is provable that". This translation connects modal logic with topology, and, in particular, with Tarski's Cn operator, which is of a topological inspiration (see references in [Czelakowski & Malinowski 1985]). Why this translation works could be realized from the Gentzen-style syntactical analysis of S4 and S5 in [Došen 1985] and [Došen 1986a] (cf. [Scott 1971]). Note that this analysis finds a connection with provability for S5, and S5-like logics, as well.

The central role of provability in modal logic could also be substantiated by the very great success of the modal analysis of Gödel's arithmetical provability predicate (see [Boolos 1979]), which also involves logics in the vicinity of S4.

But, there are many more unary operators besides "it is provable that" modal logic is able to deal with. Some of the most famous are: "it will always be the case that", "it is known that" and "it is obligatory that". These unary operators are studied in branches of modal logic which are called respectively: tense logic, epistemic logic and deontic logic. Recently, a new branch of modal logic, called dynamic logic, has developed around the study of operators drawn from computer science, like the operator "after every computation according to the programme P it is true that".

The binary propositional connectives studied by modal logic, like the connective of strict implication, are definable in terms of nonmodal connectives and unary modal operators. Nowadays, these binary connectives are very seldom taken as primitive, as modal logic has increasingly become conscious of its vocation to study unary operators.

Modal logic deals mainly with propositional systems. This is to be expected of a theory concerned with unary propositional operators. To consider these operators together with quantifiers often complicates matters, and prevents results to be stated sharply. In a certain sense, quantifiers too are unary propositional operators, but, of course, the apparatus of binding of variables makes them fall out of the field of propositional logic (see, however, [Kuhn 1980] for an attempt to treat quantifiers as modal operators).

Is modal logic able to deal with arbitrary unary propositional operators? Without trying to answer this question with precision, it seems safe to say that the success of modal logic in dealing with particular operators, and the considerable sophistication of its tools, makes it very probable that practically any unary operator for which some axioms are offered could be dealt with. That means, modal logic could try to give models, and answer technical questions concerning completeness, decidability, and the like, with a reasonable chance of success.

This contrasts with the situation we find in the study of nonclassical propositional logics. There is as yet no logical theory able to claim the title of "general theory of binary propositional connectives". Such a theory should cover not only two-valued, or many-valued, or intuitionistic, or relevant connectives, but any connectives we might wish to consider.

The generality of modal logic, though great, is not such that it could not be greater. One limitation comes from a nearly exclusive concern with the interpretation of modal operators found in <u>Kripke semantics</u>:

$$x \models \Box A \iff \forall y (xRy \Rightarrow y \models A),$$
$$x \models \Diamond A \iff \exists y (xRy \;\&\; y \models A).$$

Although there were sporadic attempts to modify this interpretation, like, for example, the following, using an n+1-ary relation R [Jennings, Johnston & Schotch 1980]:

$$x \models \Box A \iff \forall y_1 \ldots \forall y_n (xRy_1 \ldots y_n \Rightarrow (y_1 \models A \;\underline{\text{or}} \ldots \underline{\text{or}}\; y_n \models A)),$$

and though there is a well-known more general interpretation called <u>neighbourhood semantics</u> (sometimes also called <u>Scott--Montague semantics</u>), the enormous majority of papers in the central areas of modal logic deals with Kripke semantics. The reasons for that are probably the connexions with relativized quantifiers, the possibility to deal effectively with the main logics around S4 and S5, and the already considerable generality of Kripke semantics itself. Another reason is probably that the great body of papers in the general study of model-theoretic tools deals with Kripke semantics: to change now the object of study would be like changing the rules of chess, and having to revise the bulk of existing chess theory.

Another limitation of contemporary modal logic is the fact that it studies unary operators added to a Boolean basis; i.e., the nonmodal context in which these operators are introduced is classical. This is quite understandable: this logical context is not only the simplest, but presumably the most important. However, a really general theory of unary operators should pay attention to unary operators in nonclassical logics too. A start in the investigation of intuitionistic modal logic (see [Božić & Došen 1984] and [Došen 1985a]), and relevant modal logic (see [Božić 1983]), seems to have been made. An output of these investigations is the analysis of negation as a modal operator, an analysis suggested by the greater fluidity of negation in nonclassical logics (see [Božić 1983], [Došen 1986b] and [Došen 1986c]). What has not even been started is the analysis of modal logic in some kind of minimal logic. However, the discovery of this minimal logic is probably tied to the creation of a general theory of binary propositional connectives.

Modal model theory may be divided into: completeness theory, which studies completeness problems involving modal systems and various types of models; correspondence theory, which studies the definability of conditions on models by modal formulae, and the other way round; and duality theory, which studies the interconnexions between types of models in a general algebraic setting. In the second part of this talk we shall present some rudiments of modal duality theory, in order to illustrate the abstract level of studies in modern modal logic. The results which we shall discuss are from [Došen 1986d], which develops ideas of [Goldblatt 1976] and [Thomason 1975].

An extensive survey of modal logic, including duality theory, and its interconnexions with other areas of research, can be found in the second volume of the Handbook of Philosophical Logic [Gabbay & Guenthner 1984], and in particular in the first chapter [Bull & Segerberg 1984], and in the fourth chapter [van Benthem 1984]. A useful guide to the modern literature is also [Bull 1982], [Bull 1983] and [Bull 1985]. Basic notions of category theory, which we need in the second part, can be found in the introductory parts of [Pareigis 1970].

Modal algebras and frames

A frame F is a nonempty set C, the carrier of F, together with some associated relations or functions defined over C. We can imagine that a modal model is made out of a frame in two steps: first we spread over the frame a modal algebra, and then we define a valuation on this algebra.

A modal algebra A is a Boolean algebra with an additional unary operation L. If we consider modal logic in a nonclassical setting, the underlying algebra need not be Boolean: it can be a Heyting algebra, or something else. A valuation is a homomorphism v from a propositional language with a modal operator $\Box$ into A so that for formulae φ:

$$v(\Box\varphi) = L\,v(\varphi).$$

In modal duality theory valuations don't play an essential role: once we have spread a modal algebra over a frame, valuations are obtained automatically. It is this business of spreading which becomes the main subject.

To spread a modal algebra over a frame means to define it in terms of the frame. The power set $\mathcal{P}C$ of C is of course a Boolean algebra, but we can also consider subalgebras of this power set algebra. The problem is to define in these set Boolean algebras $\mathcal{A}F$ a unary operation L in terms of the relations or functions of the frame F. There are two ways of doing this, which give rise to two distinct types of frames.

First, we have relational frames, where we are given a binary relation $R \subseteq C^2$. In terms of R we can define a successor function $S: C \to \mathcal{P}C$ by $S(x) = \{y : xRy\}$ (the members of S(x) are the successors of x). Conversely, in terms of S we can define R by $xRy \Leftrightarrow y \in S(x)$. So, relational frames and successor frames amount to the same thing. With such frames F, in the power set algebra, or its subalgebra, $\mathcal{A}F$, for $B \subseteq C$ in $\mathcal{A}F$, we define L by:

$$LB = \{x\colon S(x) \subseteq B\}.$$

Second, we have neighbourhood frames, where we are given a neighbourhood function $N: C \to \mathcal{P}(\mathcal{A}F)$, the set $\mathcal{A}F$ being a subset of $\mathcal{P}C$ (the set $N(x)$ is the set of neighbourhoods of x). In terms of N we define L by:

$$LB = \{x\colon B \in N(x)\}.$$

Now, it is clear that, conversely, N can be defined in terms of L by:

$$N(x) = \{B\colon x \in LB\}.$$

So, a neighbourhood frame is essentially a set modal algebra spread over a carrier. The function N may be taken as defined in terms of L.

Is the same thing true for relational frames, viz. is R always definable in terms of L? The answer is: no. If $\mathcal{A}F$ is the whole power set $\mathcal{P}C$, then, indeed, we have:

$$(1)\quad S(x) = \bigcap \{B\colon x \in LB\},$$

or equivalently:

$$xRy \Leftrightarrow \forall B(x \in LB \Rightarrow y \in B).$$

However, if $\mathcal{A}F$ is not the whole power set $\mathcal{P}C$, but a proper subalgebra of $\mathcal{P}C$, then we may have:

$$S(x) \subsetneq \bigcap \{B\colon x \in LB\}.$$

The problem is $S(x)$ need not be an element of $\mathcal{A}F$: if it were, we would have (1). So, we distinguish a subtype of relational frames where (1) holds: we call these frames reducible frames.

The question from which modal duality theory starts is:

for an arbitrary modal algebra A, can we find an isomorphic algebra $\mathcal{A}F$ spread over some frame F? The answer is: yes, there is always such a neighbourhood frame. If the algebra A is normal, i.e. if L1 = 1 and $L(b_1 \cap b_2) = Lb_1 \cap Lb_2$, there is always an isomorphic $\mathcal{A}F$ where F is a relational frame.

These answers are usually couched as results in category theory: one establishes duality (categorial equivalence with contravariant functors) between categories of modal algebras and categories of frames. These results of category theory yield much more than an answer to our original question. They induce us to try to translate algebraic theorems (of which we know much more) into theorems about frames. For example, we might try to answer the following: for what constructions on frames are closed classes of frames which correspond to modal algebras which make a variety? What on frames corresponds to homomorphic images, subalgebras, direct products?

Let us sketch how our duality results look like. On the algebraic side let us take the category MA of modal algebras defined by:

objects: modal algebras,
morphisms: homomorphisms,

and the category NMA of normal modal algebras which differs from MA by requiring that its objects be normal modal algebras.

On the frame side we have first the category DF of descriptive neighbourhood frames defined by:

objects: descriptive neighbourhood frames,
morphisms: frame morphisms.

We shall define descriptive neighbourhood frames together with a functor $\mathcal{F}$ which will associate with every modal algebra A a frame $\mathcal{F}A$ spread over A, and with every homomorphism h between modal algebras a frame morphism $\mathcal{F}h$. If A is a modal algebra, $\mathcal{F}A$ will have a carrier C^A made of all ultrafilters of A, and if for an element b of A we have $q(b) = \{X \in C^A: b \in X\}$, then $N^A(X) = \{q(b): Lb \in X\}$. The mapping q is an isomorphism from A to $\mathcal{A}(\mathcal{F}A)$, as in Stone's Representation Theorem. Now, dually, we define a mapping $p: C \to \mathcal{P}(\mathcal{P}C)$ by $p(x) = \{B: x \in B\}$, where B is in $\mathcal{A}F$. A frame is <u>descriptive</u> iff p is <u>one-one</u> and <u>onto</u>.

The frame morphisms of DF are defined as follows. If $f: C_1 \to C_2$ and $(\mathcal{A}f)(B_2) = \{x_1: f(x_1) \in B_2\}$, then f is a <u>frame morphism</u> iff for every B_2 in $\mathcal{A}F_2$:

(i) $(\mathcal{A}f)(B_2) \in \mathcal{A}F_1$,

(ii) $(\mathcal{A}f)(B_2) \in N_1(x_1) \Leftrightarrow B_2 \in N_2(f(x_1))$.

A frame morphism is a <u>frame isomorphism</u> iff f is <u>one-one</u> and <u>onto</u>, and f^{-1} is also a frame morphism (which is not automatically satisfied). The mapping p defined above is a frame isomorphism from a descriptive F to $\mathcal{F}(\mathcal{A}F)$. Now, if $h: A_1 \to A_2$ is a homomorphism, we define the frame morphism $\mathcal{F}h: C^{A_2} \to C^{A_1}$ by $(\mathcal{F}h)(X_2) = \{b_1: h(b_1) \in X_2\}$. So, we have completely defined the functor $\mathcal{F}$ from categories of modal algebras into categories of frames. In the same way, $\mathcal{A}$ is a functor from categories of frames into categories of modal algebras. The functors $\mathcal{A}$ and $\mathcal{F}$ are contravariant.

It is possible to establish the following theorem:

THEOREM 1. <u>The categories</u> MA <u>and</u> DF <u>are dual by the functors</u> $\mathcal{A}$ <u>and</u> $\mathcal{F}$.

This theorem means that the following diagrams commute:

$$\begin{array}{ccc} A_1 & \xrightarrow{\;h\;} & A_2 \\ {\scriptstyle q_1}\downarrow & & \downarrow{\scriptstyle q_2} \\ \mathcal{A}(\mathcal{F}A_1) & \xrightarrow{\;\mathcal{A}(\mathcal{F}h)\;} & \mathcal{A}(\mathcal{F}A_2) \end{array} \qquad \begin{array}{ccc} F_1 & \xrightarrow{\;f\;} & F_2 \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_2} \\ \mathcal{F}(\mathcal{A}F_1) & \xrightarrow{\;\mathcal{F}(\mathcal{A}f)\;} & \mathcal{F}(\mathcal{A}F_2) \end{array}$$

A neighbourhood frame F is a <u>filter frame</u> iff for every $x \in C$ we have that N(x) is a filter (not necessarily proper) of $\mathcal{A}F$. The category DFF is defined by the following:

<u>objects</u>: descriptive filter frames,

<u>morphisms</u>: frame morphisms.

We can prove the following:

THEOREM 2. <u>The categories</u> NMA <u>and</u> DFF <u>are dual by the functors</u> $\mathcal{A}$ <u>and</u> $\mathcal{F}$.

Reducible relational frames are intertranslatable not with filter frames, but with a slightly more restrictive type of neighbourhood frames, which we call <u>hyperfilter frames</u>; for every B in $\mathcal{A}F$ these frames satisfy:

$$\bigcap N(x) \subseteq B \;\Rightarrow\; B \in N(x).$$

(Note that this does not entail that for every x the set N(x) is a complete filter, i.e. it does not entail that $\bigcap N(x) \in N(x)$, since $\bigcap N(x)$ need not belong to $\mathcal{A}F$.) A hyperfilter frame is always a filter frame, but for infinite frames we don't necessarily have the converse. A reducible frame becomes a hyperfilter frame with the following definition of N:

$$N(x) = \{B : S(x) \subseteq B\},$$

where B is in $\mathcal{A}F$. Conversely, a hyperfilter frame becomes a reducible frame with the following definition of S:

$$S(x) = \bigcap N(x).$$

Frame morphisms on descriptive hyperfilter frames can equivalently be defined by replacing (ii) by:

$$\text{(ii}'\text{)}\ S_2(f(x_1)) = \{f(x_2)\colon x_2 \in S_1(x_1)\}.$$

Although not every filter frame is a hyperfilter frame, every descriptive filter frame is a hyperfilter frame. So, descriptive filter frames are intertranslatable with descriptive reducible frames. Our Theorem 2 then amounts to a result of [Goldblatt 1976] which establishes duality between NMA and the category of descriptive reducible frames with frame morphisms.

Let us now consider the frames F of a more usual kind, where $\mathcal{A}F$ is the whole power set algebra. We shall call such frames <u>full</u>. What categories of modal algebras are dual with categories of full frames? An answer is provided by the following.

Let CAA be the category:

<u>objects</u>: complete atomic modal algebras,

<u>morphisms</u>: complete homomorphisms,

and let FNF be the category:

<u>objects</u>: full neighbourhood frames,

<u>morphisms</u>: frame morphisms.

The functor $\mathcal{G}$ from CAA to FNF is defined by the following. If A is an object of CAA, then for the frame $\mathcal{G}A$ we have:

$$C^A = \{a\colon a \text{ is an atom of } A\},$$

$$N^A(a) = \{Z \subseteq C^A\colon a \leqslant L \cup Z\}.$$

If $h:A_1 \to A_2$ is a complete homomorphism, then the frame morphism $\mathcal{G}h:C^{A_2} \to C^{A_1}$ is defined by $(\mathcal{G}h)(a_2)=a_1 \Leftrightarrow a_2 \leqslant h(a_1)$. Then we can prove the following:

THEOREM 3. _The categories_ CAA _and_ FNF _are dual by the functors_ $\mathcal{A}$ _and_ $\mathcal{G}$.

If NCAA differs from CAA by requiring moreover that its objects be normal modal algebras, and FFNF differs from FNF by requiring moreover that its objects be filter frames, we can prove:

THEOREM 4. _The categories_ NCAA _and_ FFNF _are dual by the functors_ $\mathcal{A}$ _and_ $\mathcal{G}$.

Full relational frames, which are always reducible, are intertranslatable with full hyperfilter frames, as above. Full relational frames are the usual Kripke frames for modal logic.

Let now CCAA be the category which differs from CAA by requiring moreover that its objects be modal algebras which satisfy:

$$L \bigcap \{b_i : i \in I\} = \bigcap \{L b_i : i \in I\}.$$

These modal algebras are normal. Next, let HFNF be the category which differs from FNF by requiring moreover that its objects be hyperfilter frames (in these frames for every x we now have that N(x) is a complete filter). The following theorem, with which we conclude this lecture, can be derived from [Thomason 1975]:

THEOREM 5. _The categories_ CCAA _and_ HFNF _are dual by the functors_ $\mathcal{A}$ _and_ $\mathcal{G}$.

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Matematički Institut
Knez Mihailova 35
11000 Beograd
Yugoslavia