On Passing from Singular to Plural Consequences

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Abstract. In the early 1970s, Dana Scott proved a theorem characterizing the minimal and maximal multiple-conclusion consequence relations (here called *plural*) that extend conservatively a consequence relation based on the usual (*singular*) notion. This paper analyzes and develops this result. First, it gives analogous, more symmetrical, results concerning extensions of preorders. Since the compactness property of the consequence relations is not presupposed, various infinitary forms of cut are envisaged. Next, a theorem generalizing Scott's is proved for consequence relations that need not be compact. Some variants of this theorem hold also for consequence relations where structural rules are restricted.

§0. Introduction. Let us call a consequence relation *singular* if, as usual, it holds between sets of formulae, understood as premises, and single formulae, understood as conclusions. The connection between $\Gamma \vdash A$, i.e. the assertion that the formula A is a consequence of the set of formulae Γ, and Tarski's consequence operation **Cn** is provided by the equivalence

$$(1) \qquad \Gamma \vdash A \;\Leftrightarrow\; A \in \mathbf{Cn}(\Gamma).$$

What is sometimes called *single-conclusion* consequence relations covers not only singular consequence relations, but also relations that may hold between a set of formulae and the empty set; i.e., we may have $\Gamma \vdash \varnothing$, too. This slightly more general notion, embodied in Gentzen's sequents for intuitionistic logic, may be needed in some contexts, but is not so important for us here: we eschew some not very essential technical epicycles if we require that we always have a formula on the right-hand side of $\vdash$ (cf. the remark after Theorem 3 at the end of §3). Anyway, in many usual logical systems we have a formula, like the constant absurd proposition, that may replace the empty set on the right-hand side of $\vdash$.

A *plural* consequence relation holds between two sets of formulae. Such consequence relations, sometimes called *multiple-conclusion*, underlie Gentzen's sequents for classical logic (but there are multiple-conclusion sequent formulations of intuitionistic logic, too). The interpretation of plural consequence relations in terms of natural deduction is somewhat problematic (though a whole book [Shoesmith & Smiley 1978] and several later papers are devoted to this matter). However, semantically, a plural consequence relation makes perfect sense when we understand it as holding between premises taken conjunctively and conclusions taken disjunctively. It embodies the Boolean dualities of classical logic, and underlies Gentzen's neat formalization of this logic. (We shall not deal here with plural consequence relations where both premises and conclusions are taken conjunctively. In principle, this is nothing but a handy notation for matters pertaining to singular consequence relations.)

To pass from a plural consequence relation $\Vdash$ to its singular mate $\vdash$, we define $\vdash$ by the obvious equivalence

$$(2) \qquad \Gamma \vdash A \Leftrightarrow \Gamma \Vdash \{A\}.$$

On the other hand, it is less obvious what plural consequence relations may correspond to a given singular consequence relation. In Theorem 1.2 of [1974, p. 415] Scott gave a characterization of the minimal and maximal plural consequence relations that extend conservatively a given singular consequence relation (i.e., they agree with the singular consequence relation in the sense of (2)). A version of this theorem with an alternative characterization of Scott's maximal plural consequence relation may be found in [Gabbay 1981, chapter 1.1, Theorem 13, p. 8]. Here we prove a few theorems extending Scott's result in more than one direction.

First, we consider passing from preorders, which are primitive consequence relations, singular on both sides, to singular or plural consequence relations. We prove in this context theorems analogous to Scott's, characterizing the minimal and maximal singular or plural consequence relations that extend conservatively a given preorder. This enables us to exhibit some symmetries hidden beneath Scott's characterization, which should make clearer its meaning.

The minimal plural consequence relation $\Vdash$ extending conservatively a singular consequence relation $\vdash$ is obtained by defining $\Gamma \Vdash \Delta$ as $\Gamma \vdash B$ for some B in Δ; for the minimal $\Vdash$ extending a preorder $\leq$, we define $\Gamma \Vdash \Delta$ as $A \leq B$ for some A in Γ and some B in Δ. The characterization of the maximal $\Vdash$ extending $\vdash$ boils down to understanding $\Gamma \Vdash \Delta$ as $\Gamma \vdash \vee\Delta$, where $\vee\Delta$ is a disjunction of the formulae in Δ; for the maximal $\Vdash$ extending $\leq$, we understand $\Gamma \Vdash \Delta$ as $\wedge\Gamma \leq \vee\Delta$, where $\wedge\Gamma$ is a conjunction of the formulae in Γ. We shall see that the difference between these characterizations of the minimal and maximal relations $\Vdash$ reduces essentially to the order of quantifiers.

Next, in our approach, the sets of formulae Γ and Δ in $\Gamma \vdash A$ and $\Gamma \Vdash \Delta$ (and hence also the conjunctions and disjunctions we have just mentioned) need not be finite. On the other hand, in Scott's approach one finds built-in the usual assumption that consequence relations, both singular and plural, are *compact*. This means that for singular consequence relations one has

$$\Gamma \vdash A \Rightarrow (\exists \Gamma' \subseteq \Gamma)(\Gamma' \textit{ is finite } \& \ \Gamma' \vdash A).$$

For plural consequence relations, Scott assumes compactness simply by defining them as relations between finite sets of formulae. Indeed, most authors seem to estimate that Gentzen's sequents make sense only with finite collections of formulae on the two sides of the turnstile. However, noncompact consequence relations arise in second-order or higher-order logic, in logic with infinitary connectives, in arithmetic with the ω-rule, or whenever finite axiomatizability fails, and we may use sequents to talk about such consequence relations.

We shall show that compactness is not essential for proving results of which Scott's theorem is a particular case, or analogous results involving preorders. This requires envisaging various infinitary forms of cut, involving infinitely many applications of ordinary cut rules. Nevertheless, our results apply also to compact consequence relations, where cut appears in its ordinary finite form. In the compact case, we only lose some of the distinctions we shall draw, which permit us to formulate the more general results.

Shoesmith and Smiley in [1978, chapters 1, 2, 5] treat systematically of many matters related to this paper, and, in particular, in [1978, section 5.1, pp. 73-75] they envisage generalizing Scott's result to noncompact consequence relations. However, for their notion of plural consequence relation, stronger than ours (see §2 below), there is not always a maximal plural consequence relation that extends conservatively a given singular consequence relation.

Finally, the style of our exposition will differ somewhat from Scott's in disregarding Tarski's **Cn** operation. Of course, the connection with **Cn** may be easily found via equivalence (1) (or, as in [Scott 1974, Proposition 1.1, p. 415], a variant of (1) where compactness is built-in). But it seems that sequents provide a more perspicuous language, not only for proving things Gentzen introduced them for.

We follow Scott in concentrating on notions of consequence relation appropriate for classical and intuitionistic logic. If similar results can be proved for consequence relations of other logics — for example, substructural logics, where the structural rules of thinning, contraction or permutation may be missing — we shall deal with them in more detail on another occasion. However, at the very end we have some results about extending conservatively singular consequence relations for which thinning, and also contraction and permutation, on the left need not hold.

§1. From preorders to singular consequence relations. Let L be a nonempty set of objects called *formulae*. We shall deal only with structural rules involving sequents built with the formulae of L. So, it does not matter what particular expressions we have in the language L: it can be a completely arbitrary nonempty set. We shall use the metavariables $A, B, C, \ldots$, possibly with indices, for members of L, and $\Gamma, \Delta, \Theta, \ldots$, possibly with indices, for arbitrary (empty, finite or infinite) subsets of L. By **P**(L) we denote the power set of L.

A *preorder* on L is, as usual, a subset $\leq$ of $L \times L$ that satisfies

(≤ 1) $\qquad C \leq C,$

(≤ 2) $\qquad A \leq C \Rightarrow (C \leq B \Rightarrow A \leq B).$

Note that (≤ 1) and (≤ 2) could be replaced by either of the equivalences

$$A \leq C \Leftrightarrow \forall B(C \leq B \Rightarrow A \leq B),$$
$$C \leq B \Leftrightarrow \forall A(A \leq C \Rightarrow A \leq B),$$

whose left-to-right directions are (≤ 2). The other directions amount to (≤ 1). (Assuming the right-to-left implication of the first equivalence we obtain (≤ 1) by putting C for A; we obtain this implication from (≤ 1) by instantiating B by C. We proceed quite analogously with the second equivalence.) Hence reflexivity (≤ 1) and transitivity (≤ 2), which is a primitive form of cut, may be conceived as converse to each other.

A *singular consequence relation* on L is a subset $\vdash$ of $\mathbf{P}(\mathrm{L}) \times \mathrm{L}$ that satisfies

($\vdash 1$) $\qquad \{C\} \vdash C,$

($\vdash 2$) $\qquad (\forall C \in \Theta)\, \Gamma_1 \vdash C \Rightarrow (\Theta \cup \Gamma_2 \vdash B \Rightarrow \Gamma_1 \cup \Gamma_2 \vdash B).$

The reflexivity postulate ($\vdash 1$) is familiar from Gentzen's sequent systems, whereas ($\vdash 2$) embodies cut and thinning on the left. The cut of ($\vdash 2$) does not cover only the ordinary form of cut, where Θ in ($\vdash 2$) is a singleton (this singleton cut yields all instances of ($\vdash 2$) where Θ is nonempty and finite): it covers also cases where Θ is infinite. For Θ empty, $(\forall C \in \Theta)\, \Gamma_1 \vdash C$ is true for every Γ_1, and hence ($\vdash 2$) yields thinning on the left:

($\vdash 3$) $\qquad \Gamma_2 \vdash B \Rightarrow \Gamma_1 \cup \Gamma_2 \vdash B.$

Our notion of singular consequence relation is equivalent to the notion of single-conclusion consequence relation of [Shoesmith & Smiley 1978, section 1.1, p. 15].

We can replace ($\vdash 1$) and ($\vdash 2$) by either of the equivalences

$$(\forall C \in \Theta)\, \Gamma_1 \vdash C \Leftrightarrow \forall \Gamma_2 \forall B(\Theta \cup \Gamma_2 \vdash B \Rightarrow \Gamma_1 \cup \Gamma_2 \vdash B),$$
$$\Theta \cup \Gamma_2 \vdash B \Leftrightarrow \forall \Gamma_1((\forall C \in \Theta)\, \Gamma_1 \vdash C \Rightarrow \Gamma_1 \cup \Gamma_2 \vdash B),$$

whose left-to-right directions are ($\vdash 2$). The other directions amount to ($\vdash 1$) in the presence of ($\vdash 3$). So, the reflexivity postulate and cut may be conceived as converse to each other.

Another alternative is to replace ($\vdash 1$) and ($\vdash 2$) by

($\vdash 1'$) $\qquad (\forall C \in \Theta)\, \Theta \vdash C,$

($\vdash 2'$) $\qquad (\forall C \in \Theta)\, \Gamma \vdash C \Rightarrow (\Theta \vdash B \Rightarrow \Gamma \vdash B).$

It is clear that ($\vdash 1$) yields ($\vdash 1'$) with the help of ($\vdash 3$), and that ($\vdash 2$) yields ($\vdash 2'$) by taking Γ_2 empty. For the converse, we obtain ($\vdash 1$) from ($\vdash 1'$) by taking Θ a singleton. To obtain ($\vdash 2$), let us first deduce ($\vdash 3$) from ($\vdash 1'$) and ($\vdash 2'$). As an instance of ($\vdash 1'$) we have $(\forall C \in \Gamma_1 \cup \Gamma_2)\, \Gamma_1 \cup \Gamma_2 \vdash C$, and hence $(\forall C \in \Gamma_2)\, \Gamma_1 \cup \Gamma_2 \vdash C$; then from $\Gamma_2 \vdash B$, with the help of ($\vdash 2'$), we obtain $\Gamma_1 \cup \Gamma_2 \vdash B$. Now for ($\vdash 2$), suppose $(\forall C \in \Theta)\, \Gamma_1 \vdash C$. With ($\vdash 3$) we derive $(\forall C \in \Theta \cup \Gamma_2)\, \Gamma_1 \cup \Gamma_2 \vdash C$, which with $\Theta \cup \Gamma_2 \vdash B$ and ($\vdash 2'$) yields $\Gamma_1 \cup \Gamma_2 \vdash B$.

Still another alternative is to replace ($\vdash$ 1) and ($\vdash$ 2) by the equivalence

$$(\forall C \in \Theta)\, \Gamma \vdash C \;\Leftrightarrow\; \forall B(\Theta \vdash B \;\Rightarrow\; \Gamma \vdash B),$$

whose left-to-right direction is ($\vdash$ 2'). The other direction amounts to ($\vdash$ 1').

Starting from a given singular consequence relation $\vdash$, we may define $A \leq B$ as $\{A\} \vdash B$, and check in a straightforward way that the relation $\leq$ so defined is a preorder. Conversely, starting from a given preorder $\leq$, we define the relations

$$\Gamma \vdash_{\min} B =_{\mathrm{df}} (\exists A \in \Gamma)\, A \leq B,$$
$$\Gamma \vdash_{\max} B =_{\mathrm{df}} \forall E((\forall A \in \Gamma)\, E \leq A \;\Rightarrow\; E \leq B).$$

We are primarily interested in these definitions for preorders $\leq$, but we may envisage them for any relation $\leq \subseteq \mathrm{L} \times \mathrm{L}$. (For example, in Theorem 1(ii) below, we assume that $\leq$ is only reflexive.)

If for our preorder $\leq$ it holds that for some formula denoted by $\wedge\Gamma$, for every E,

$$(\wedge) \qquad (\forall A \in \Gamma)\, E \leq A \;\Leftrightarrow\; E \leq \wedge\Gamma,$$

then the definition of $\vdash_{\max}$ yields the equivalence

$$\Gamma \vdash_{\max} B \;\Leftrightarrow\; \wedge\Gamma \leq B.$$

We usually have $\wedge\Gamma$ for every finite Γ: it is the conjunction of the formulae in Γ if Γ is nonempty, and the constant true proposition if Γ is empty. For infinite Γ, we may use universal quantification or infinite conjunctions. Algebraically, $\wedge\Gamma$ is like the infimum of the set Γ (it is unique if $\leq$ is antisymmetric).

Let us note that for $\leq$ a preorder we can derive

$$\Gamma \vdash_{\min} B \;\Leftrightarrow\; (\exists A \in \Gamma)\forall E(E \leq A \;\Rightarrow\; E \leq B),$$

whereas for Γ nonempty we derive (classically, but not intuitionistically)

$$\Gamma \vdash_{\max} B \;\Leftrightarrow\; \forall E(\exists A \in \Gamma)(E \leq A \;\Rightarrow\; E \leq B).$$

We have the following theorem:

Theorem 1. (i) If $\leq$ is a preorder, then $\vdash_{\min}$ and $\vdash_{\max}$ are singular consequence relations.

(ii) If $\leq \subseteq \mathrm{L} \times \mathrm{L}$ is reflexive and $\vdash \subseteq \mathbf{P}(\mathrm{L}) \times \mathrm{L}$ satisfies ($\vdash$ 2), then

$$\forall A, B(A \leq B \;\Leftrightarrow\; \{A\} \vdash B) \text{ if and only if } \vdash_{\min} \subseteq \vdash \subseteq \vdash_{\max}.$$

Proof. (i) That $\vdash_{\min}$ and $\vdash_{\max}$ satisfy ($\vdash$ 1) is trivial. For ($\vdash$ 2), suppose ($\mathrm{I}_{\min}$) $(\forall C \in \Theta)$ $\Gamma_1 \vdash_{\min} C$ and ($\mathrm{II}_{\min}$) $\Theta \cup \Gamma_2 \vdash_{\min} B$. From ($\mathrm{II}_{\min}$) we obtain that for some $D \in \Theta \cup \Gamma_2$ we have $D \leq B$. If $D \in \Gamma_2$, we obtain immediately $\Gamma_1 \cup \Gamma_2 \vdash_{\min} B$. If $D \in \Theta$, then from ($\mathrm{I}_{\min}$)

we obtain that for some $A \in \Gamma_1$ we have $A \leq D$. Then, by ($\leq$ 2), we get $A \leq B$, from which $\Gamma_1 \cup \Gamma_2 \vdash_{min} B$ follows. Hence ($\vdash$ 2) holds for $\vdash_{min}$.

Suppose now (I_{max}) $(\forall C \in \Theta)\, \Gamma_1 \vdash_{max} C$ and (II_{max}) $\Theta \cup \Gamma_2 \vdash_{max} B$. If $(\forall A \in \Gamma_1 \cup \Gamma_2)$ $E \leq A$, then from (I_{max}) we obtain $(\forall C \in \Theta)\ E \leq C$, and from ($II_{max}$) we obtain $(\forall C \in \Theta)\, E \leq C \Rightarrow E \leq B$, which yields $E \leq B$. Hence ($\vdash$ 2) holds for $\vdash_{max}$.

(ii — only if part) If $\Gamma \vdash_{min} B$, then for some $A \in \Gamma$ we have $A \leq B$, and hence $\{A\} \vdash B$. By ($\vdash$ 3) (which is an instance of ($\vdash$ 2)), we obtain $\Gamma \vdash B$. So, $\vdash_{min} \subseteq \vdash$. Suppose now $\Gamma \vdash B$. If $(\forall A \in \Gamma)\ E \leq A$, then $(\forall A \in \Gamma)\ \{E\} \vdash A$, and by ($\vdash$ 2), we obtain $\{E\} \vdash B$. This yields $E \leq B$, and so, $\vdash \subseteq \vdash_{max}$.

(ii — if part) If $A \leq B$, then we get immediately $\{A\} \vdash_{min} B$, and hence $\{A\} \vdash B$. If $\{A\} \vdash B$, then $\{A\} \vdash_{max} B$, which yields $A \leq B$ with the help of ($\leq$ 1) (this is the only place in the proof of (ii) where we need the reflexivity of $\leq$). *q.e.d.*

From this theorem it follows that the set of all singular consequence relations that extend conservatively a given preorder coincides with the set of all singular consequence relations in the interval between $\vdash_{min}$ and $\vdash_{max}$. This set is nonempty, because $\vdash_{min} \subseteq \vdash_{max}$ (which follows from the transitivity of $\leq$), and we can substitute $\vdash_{min}$ and $\vdash_{max}$ for $\vdash$ in (ii).

§2. From preorders to plural consequence relations. A *plural consequence relation* on L is a subset $\Vdash$ of $\mathbf{P}(L) \times \mathbf{P}(L)$ that satisfies

($\Vdash$ 1) $\quad \{C\} \Vdash \{C\}$,

($\Vdash$ 2.1 #) $\quad (\forall C \in \Theta)\, \Gamma_1 \Vdash \{C\} \Rightarrow (\Theta \cup \Gamma_2 \Vdash \Delta_2 \Rightarrow \Gamma_1 \cup \Gamma_2 \Vdash \Delta_2)$,

($\Vdash$ 2.2 #) $\quad (\forall C \in \Theta)\, \{C\} \Vdash \Delta_2 \Rightarrow (\Gamma_1 \Vdash \Delta_1 \cup \Theta \Rightarrow \Gamma_1 \Vdash \Delta_1 \cup \Delta_2)$.

For Θ empty, ($\Vdash$ 2.1 #) and ($\Vdash$ 2.2 #) yield respectively thinning on the left and on the right:

($\Vdash$ 3.1) $\quad \Gamma_2 \Vdash \Delta_2 \Rightarrow \Gamma_1 \cup \Gamma_2 \Vdash \Delta_2$,

($\Vdash$ 3.2) $\quad \Gamma_1 \Vdash \Delta_1 \Rightarrow \Gamma_1 \Vdash \Delta_1 \cup \Delta_2$.

We can replace ($\Vdash$ 1), ($\Vdash$ 2.1 #) and ($\Vdash$ 2.2 #) either by the two equivalences

$$(\forall C \in \Theta)\, \Gamma_1 \Vdash \{C\} \Leftrightarrow \forall \Gamma_2, \Delta_2 (\Theta \cup \Gamma_2 \Vdash \Delta_2 \Rightarrow \Gamma_1 \cup \Gamma_2 \Vdash \Delta_2),$$
$$(\forall C \in \Theta)\, \{C\} \Vdash \Delta_2 \Leftrightarrow \forall \Gamma_1, \Delta_1 (\Gamma_1 \Vdash \Delta_1 \cup \Theta \Rightarrow \Gamma_1 \Vdash \Delta_1 \cup \Delta_2),$$

or by the two equivalences

$$\Theta \cup \Gamma_2 \Vdash \Delta_2 \Leftrightarrow \forall \Gamma_1 ((\forall C \in \Theta)\, \Gamma_1 \Vdash \{C\} \Rightarrow \Gamma_1 \cup \Gamma_2 \Vdash \Delta_2),$$
$$\Gamma_1 \Vdash \Delta_1 \cup \Theta \Leftrightarrow \forall \Delta_2 ((\forall C \in \Theta)\, \{C\} \Vdash \Delta_2 \Rightarrow \Gamma_1 \Vdash \Delta_1 \cup \Delta_2).$$

Other alternatives are to use either the following four assumptions

$$(\forall C \in \Theta)\, \Theta \Vdash \{C\},$$
$$(\forall C \in \Theta)\, \{C\} \Vdash \Theta,$$

$$(\forall C\in\Theta)\,\Gamma\Vdash\{C\}\;\Rightarrow\;(\Theta\Vdash\Delta\;\Rightarrow\;\Gamma\Vdash\Delta),$$
$$(\forall C\in\Theta)\,\{C\}\Vdash\Delta\;\Rightarrow\;(\Gamma\Vdash\Theta\;\Rightarrow\;\Gamma\Vdash\Delta),$$

or the two equivalences

$$(\forall C\in\Theta)\,\Gamma\Vdash\{C\}\;\Leftrightarrow\;\forall\Delta(\Theta\Vdash\Delta\;\Rightarrow\;\Gamma\Vdash\Delta),$$
$$(\forall C\in\Theta)\,\{C\}\Vdash\Delta\;\Leftrightarrow\;\forall\Gamma(\Gamma\Vdash\Theta\;\Rightarrow\;\Gamma\Vdash\Delta).$$

Starting from a given plural consequence relation $\Vdash$, we may define $A\leq B$ as $\{A\}\Vdash\{B\}$, and check in a straightforward way that the relation $\leq$ so defined is a preorder. Conversely, starting from a given preorder $\leq$, we define the relations

$$\Gamma\Vdash_{\min}\Delta\;=_{\mathrm{df}}\;(\exists A\in\Gamma)(\exists B\in\Delta)\,A\leq B,$$
$$\Gamma\Vdash_{\max}\Delta\;=_{\mathrm{df}}\;\forall E,F(((\forall A\in\Gamma)\,E\leq A\;\;\&\;\;(\forall B\in\Delta)\,B\leq F)\;\Rightarrow\;E\leq F).$$

These definitions may also be envisaged for any relation $\leq\,\subseteq L\times L$, and not only preorders $\leq$.

If for our preorder $\leq$ it holds that for some formula $\wedge\Gamma$ we have for every E the equivalence ($\wedge$), mentioned above, and for some formula $\vee\Delta$, for every F,

($\vee$) $$(\forall B\in\Delta)\,B\leq F\;\Leftrightarrow\;\vee\Delta\leq F,$$

then the definition of $\Vdash_{\max}$ yields the equivalence

$$\Gamma\Vdash_{\max}\Delta\;\Leftrightarrow\;\wedge\Gamma\leq\vee\Delta.$$

We usually have $\vee\Delta$ for every finite Δ: it is the disjunction of the formulae in Δ if Δ is nonempty, and the constant absurd proposition if Δ is empty. For infinite Δ, we may use existential quantification or infinite disjunctions. Algebraically, $\vee\Delta$ is like the supremum of the set Δ (it is unique if $\leq$ is antisymmetric).

Let us note that for $\leq$ a preorder we can derive

$$\Gamma\Vdash_{\min}\Delta\;\Leftrightarrow\;(\exists A\in\Gamma)(\exists B\in\Delta)\forall E,F((E\leq A\;\;\&\;\;B\leq F)\;\Rightarrow\;E\leq F),$$

whereas for Γ and Δ nonempty we derive (classically, but not intuitionistically)

$$\Gamma\Vdash_{\max}\Delta\;\Leftrightarrow\;\forall E,F(\exists A\in\Gamma)(\exists B\in\Delta)((E\leq A\;\;\&\;\;B\leq F)\;\Rightarrow\;E\leq F).$$

Let us also note that for Δ a singleton we obtain the equivalences

$$\Gamma\Vdash_{\min}\{B\}\;\Leftrightarrow\;(\exists A\in\Gamma)\,A\leq B,$$
$$\Gamma\Vdash_{\max}\{B\}\;\Leftrightarrow\;\forall E((\forall A\in\Gamma)\,E\leq A\;\Rightarrow\;E\leq B),$$

which correspond exactly to the definitions of $\vdash_{\min}$ and $\vdash_{\max}$ in §1.

Consider now the following generalizations of ($\Vdash$ 2.1 #) and ($\Vdash$ 2.2 #):

($\Vdash$ 2.1) $$(\forall C\in\Theta)\,\Gamma_1\Vdash\Delta_1\cup\{C\}\;\Rightarrow\;(\Theta\cup\Gamma_2\Vdash\Delta_2\;\Rightarrow\;\Gamma_1\cup\Gamma_2\Vdash\Delta_1\cup\Delta_2),$$

($\Vdash$ 2.2) $$(\forall C\in\Theta)\,\{C\}\cup\Gamma_2\Vdash\Delta_2\;\Rightarrow\;(\Gamma_1\Vdash\Delta_1\cup\Theta\;\Rightarrow\;\Gamma_1\cup\Gamma_2\Vdash\Delta_1\cup\Delta_2).$$

We obtain ($\Vdash$ 2.1 #) from ($\Vdash$ 2.1) by taking Δ_1 empty, and ($\Vdash$ 2.2 #) from ($\Vdash$ 2.2) by taking Γ_2 empty. Note that for Θ a singleton ($\Vdash$ 2.1) and ($\Vdash$ 2.2) boil down to the same plural form of cut. This singleton cut implies all the instances of ($\Vdash$ 2.1) and ($\Vdash$ 2.2) where Θ is nonempty and finite. Hence these two principles are equivalent when Θ is nonempty and finite. They also boil down to the same thing when Θ is empty: in this case, either of ($\Vdash$ 2.1) and ($\Vdash$ 2.2) yields thinning on both the left and right, i.e. both ($\Vdash$ 3.1) and ($\Vdash$ 3.2). So, ($\Vdash$ 2.1) and ($\Vdash$ 2.2) can be distinguished only when Θ is infinite.

The principles ($\Vdash$ 2.1) and ($\Vdash$ 2.2) may be found in [Shoesmith & Smiley 1978, section 2.1, p. 32]. However, the cut principle embodied in the notion of multiple-conclusion consequence relation of that work is still more general. It says that if for every partition $\langle\Theta_1, \Theta_2\rangle$ of Θ we have $\Theta_1\cup\Gamma \Vdash \Delta\cup\Theta_2$, then $\Gamma \Vdash \Delta$.

Note that ($\Vdash$ 1), ($\Vdash$ 2.1) and ($\Vdash$ 2.2) amount to the two equivalences

$$(\forall C\in\Theta)\,\Gamma_1 \Vdash \Delta_1\cup\{C\} \;\Leftrightarrow\; \forall\Gamma_2,\Delta_2(\Theta\cup\Gamma_2 \Vdash \Delta_2 \;\Rightarrow\; \Gamma_1\cup\Gamma_2 \Vdash \Delta_1\cup\Delta_2),$$
$$(\forall C\in\Theta)\,\{C\}\cup\Gamma_2 \Vdash \Delta_2 \;\Leftrightarrow\; \forall\Gamma_1,\Delta_1(\Gamma_1 \Vdash \Delta_1\cup\Theta \;\Rightarrow\; \Gamma_1\cup\Gamma_2 \Vdash \Delta_1\cup\Delta_2),$$

or to the two equivalences

$$\Theta\cup\Gamma_2 \Vdash \Delta_2 \;\Leftrightarrow\; \forall\Gamma_1,\Delta_1((\forall C\in\Theta)\,\Gamma_1 \Vdash \Delta_1\cup\{C\} \;\Rightarrow\; \Gamma_1\cup\Gamma_2 \Vdash \Delta_1\cup\Delta_2),$$
$$\Gamma_1 \Vdash \Delta_1\cup\Theta \;\Leftrightarrow\; \forall\Gamma_2,\Delta_2((\forall C\in\Theta)\,\{C\}\cup\Gamma_2 \Vdash \Delta_2 \;\Rightarrow\; \Gamma_1\cup\Gamma_2 \Vdash \Delta_1\cup\Delta_2).$$

Note also that ($\Vdash$ 2.1) and ($\Vdash$ 2.2) may follow from ($\Vdash$ 2.1 #) and ($\Vdash$ 2.2 #) if in L we have connectives that, like classical negation, enable us to transport formulae from one side of a sequent to the other, yielding an equivalent sequent.

We can easily prove the following lemma:

Lemma 1. If $\leq\,\subseteq L\times L$ is transitive, then $\Vdash_{min}$ satisfies ($\Vdash$ 2.1) and ($\Vdash$ 2.2).

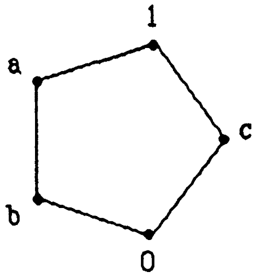
With the help of this lemma, and by imitating the proof of Theorem 1, we obtain the following theorem:

Theorem 2. (i) If $\leq$ is a preorder, then $\Vdash_{min}$ and $\Vdash_{max}$ are plural consequence relations.

(ii) If $\leq\,\subseteq L\times L$ is reflexive and $\Vdash\,\subseteq \mathbf{P}(L)\times\mathbf{P}(L)$ satisfies ($\Vdash$ 2.1 #) and ($\Vdash$ 2.2 #), then

$$\forall A,B(A\leq B \;\Leftrightarrow\; \{A\}\Vdash\{B\}) \text{ if and only if } \Vdash_{min}\subseteq\;\Vdash\;\subseteq\;\Vdash_{max}.$$

Note that to prove ($\Vdash$ 1) for $\Vdash_{max}$ in (i) we use the transitivity of $\leq$. To show that $\Vdash_{max}$ in (i) need not satisfy ($\Vdash$ 2.1) and ($\Vdash$ 2.2), let L be $\{1, a, b, c, 0\}$ with $\leq$ the lattice ordering of the nondistributive lattice



and take $\Gamma_1 = \Gamma_2 = \{a\}$, $\Delta_1 = \Delta_2 = \{b\}$ and $\Theta = \{c\}$.

It follows from Theorem 2 that the set of all plural consequence relations that extend conservatively a given preorder coincides with the set of all plural consequence relations in the interval between $\Vdash_{min}$ and $\Vdash_{max}$. This set is nonempty, because $\Vdash_{min} \subseteq \Vdash_{max}$ (which follows from the transitivity of $\leq$).

§3. From singular to plural consequence relations. Starting from a given plural consequence relation $\Vdash$, we may define $\Gamma \vdash B$ as $\Gamma \Vdash \{B\}$, and check in a straightforward way that the relation $\vdash$ so defined is a singular consequence relation (for that we use only ($\Vdash$ 1) and ($\Vdash$ 2.1 #)). Conversely, starting from a given singular consequence relation $\vdash$, we define the relations

$$\Gamma \Vdash_{MIN} \Delta =_{df} (\exists B \in \Delta)\, \Gamma \vdash B,$$
$$\Gamma \Vdash_{MAX} \Delta =_{df} \forall \Xi \forall F((\forall B \in \Delta)\, \{B\} \cup \Xi \vdash F \Rightarrow \Gamma \cup \Xi \vdash F).$$

These definitions may also be envisaged for any relation $\vdash\, \subseteq \mathbf{P}(\mathrm{L}) \times \mathrm{L}$, and not only singular consequence relations $\vdash$. They match exactly the characterization of $\vdash_{\cap}$ and $\vdash_{\cup}$ in [Shoesmith & Smiley 1978, section 5.1, p. 74].

If for our singular consequence $\vdash$ relation it holds that for some formula $\vee\Delta$, for every Ξ and every F,

$$(\forall B \in \Delta)\, \{B\} \cup \Xi \vdash F \Leftrightarrow \{\vee\Delta\} \cup \Xi \vdash F,$$

then our definition of $\Vdash_{MAX}$ yields the equivalence

$$\Gamma \Vdash_{MAX} \Delta \Leftrightarrow \Gamma \vdash \vee\Delta.$$

We usually have $\vee\Delta$ for every finite Δ: it is the disjunction of the formulae in Δ if Δ is nonempty, and the constant absurd proposition if Δ is empty. For infinite Δ, we may use existential quantification or infinite disjunctions.

For $\Vdash_{MAX}$ it is easy to derive, with the help of ($\vdash$ 3),

$$\Gamma \Vdash_{MAX} \Delta \Leftrightarrow (\forall \Xi \supseteq \Gamma) \forall F((\forall B \in \Delta)\, \{B\} \cup \Xi \vdash F \Rightarrow \Xi \vdash F).$$

This last equivalence matches exactly Scott's definition of the corresponding notion, and the definition of $\Vdash_{MIN}$ above is also Scott's [1974, Theorem 1.2, clauses (i) and (ii), p. 415].

Let us also note that, with the help of ($\vdash$ 1) and ($\vdash$ 2) (actually, ($\vdash$ 2 cut) of §4 below would do instead of ($\vdash$ 2)), we can derive

$$\Gamma \Vdash_{\mathrm{MIN}} \Delta \Leftrightarrow (\exists B \in \Delta)\forall \Xi \forall F(\{B\} \cup \Xi \vdash F \Rightarrow \Gamma \cup \Xi \vdash F),$$

whereas for Δ nonempty we derive (classically, but not intuitionistically)

$$\Gamma \Vdash_{\mathrm{MAX}} \Delta \Leftrightarrow \forall \Xi \forall F(\exists B \in \Delta)(\{B\} \cup \Xi \vdash F \Rightarrow \Gamma \cup \Xi \vdash F).$$

We can also derive, with the help of ($\vdash$ 1') and ($\vdash$ 2),

$$\Gamma \Vdash_{\mathrm{MIN}} \Delta \Leftrightarrow (\exists B \in \Delta)\forall \Xi_1, \Xi_2 \forall F(((\forall A \in \Gamma)\, \Xi_1 \vdash A \;\&\; \{B\} \cup \Xi_2 \vdash F) \Rightarrow \Xi_1 \cup \Xi_2 \vdash F),$$
$$\Gamma \Vdash_{\mathrm{MAX}} \Delta \Leftrightarrow \forall \Xi_1, \Xi_2 \forall F(((\forall A \in \Gamma)\, \Xi_1 \vdash A \;\&\; (\forall B \in \Delta)\, \{B\} \cup \Xi_2 \vdash F) \Rightarrow \Xi_1 \cup \Xi_2 \vdash F).$$

From these last two equivalences, for Γ nonempty we derive

$$\Gamma \Vdash_{\mathrm{MIN}} \Delta \Leftrightarrow (\exists B \in \Delta)\forall \Xi_1, \Xi_2 \forall F(\exists A \in \Gamma)((\Xi_1 \vdash A \;\&\; \{B\} \cup \Xi_2 \vdash F) \Rightarrow \Xi_1 \cup \Xi_2 \vdash F),$$

while for Γ and Δ nonempty we derive

$$\Gamma \Vdash_{\mathrm{MAX}} \Delta \Leftrightarrow \forall \Xi_1, \Xi_2 \forall F(\exists A \in \Gamma)(\exists B \in \Delta)((\Xi_1 \vdash A \;\&\; \{B\} \cup \Xi_2 \vdash F) \Rightarrow \Xi_1 \cup \Xi_2 \vdash F)$$

(in both cases we proceed classically, but not intuitionistically). The last six equivalences should be compared with the definitions of $\vdash_{\mathrm{min}}$, $\vdash_{\mathrm{max}}$, $\Vdash_{\mathrm{min}}$ and $\Vdash_{\mathrm{max}}$ in §§1-2, and the equivalences following these definitions, which are all based on the same underlying pattern.

We prove the following lemmata:

Lemma 2. If $\vdash \subseteq \mathbf{P}(\mathrm{L}) \times \mathrm{L}$ satisfies ($\vdash$ 2), then $\Vdash_{\mathrm{MIN}}$ satisfies ($\Vdash$ 2.1) and ($\Vdash$ 2.2).

Proof. Suppose for ($\Vdash$ 2.1) that (I_1) $(\forall C \in \Theta)\ \Gamma_1 \Vdash_{\mathrm{MIN}} \Delta_1 \cup \{C\}$ and (II_1) $\Theta \cup \Gamma_2 \Vdash_{\mathrm{MIN}} \Delta_2$. If $C \in \Theta$, then from (I_1) we obtain either $\Gamma_1 \vdash B_1$, for some $B_1 \in \Delta_1$, or $\Gamma_1 \vdash C$. From the first disjunct we derive $\Gamma_1 \cup \Gamma_2 \vdash B_1$ with the help of ($\vdash$ 3) (which is an instance of ($\vdash$ 2)). If, however, there is no $C \in \Theta$ such that this first disjunct is true, then $(\forall C \in \Theta)\ \Gamma_1 \vdash C$. From this and ($\mathrm{II}_1$), with the help of ($\vdash$ 2), we derive $\Gamma_1 \cup \Gamma_2 \vdash B_2$ for some $B_2 \in \Delta_2$. This proves that $\Vdash_{\mathrm{MIN}}$ satisfies ($\Vdash$ 2.1).

Suppose now for ($\Vdash$ 2.2) that (I_2) $(\forall C \in \Theta)$ $\{C\} \cup \Gamma_2 \Vdash_{\mathrm{MIN}} \Delta_2$ and (II_2) $\Gamma_1 \Vdash_{\mathrm{MIN}} \Delta_1 \cup \Theta$. Then from ($\mathrm{II}_2$) we know that for some $B_1 \in \Delta_1 \cup \Theta$ we have $\Gamma_1 \vdash B_1$. If $B_1 \in \Delta_1$, we use ($\vdash$ 3). If $B_1 \in \Theta$, then, by (I_2), we have for some $B_2 \in \Delta_2$ that $\{B_1\} \cup \Gamma_2 \vdash B_2$. This and $\Gamma_1 \vdash B_1$ yield $\Gamma_1 \cup \Gamma_2 \vdash B_2$ with the help of ($\vdash$ 2). This proves that $\Vdash_{\mathrm{MIN}}$ satisfies ($\Vdash$ 2.2). *q.e.d.*

Lemma 3. If $\vdash \subseteq \mathbf{P}(\mathrm{L}) \times \mathrm{L}$ satisfies ($\vdash$ 2), then $\Vdash_{\mathrm{MAX}}$ satisfies ($\Vdash$ 2.1 #) and ($\Vdash$ 2.2).

Proof. Suppose for ($\Vdash$ 2.1 #) that (I_1) $(\forall C \in \Theta)\ \Gamma_1 \Vdash_{\mathrm{MAX}} \{C\}$ and (II_1) $\Theta \cup \Gamma_2 \Vdash_{\mathrm{MAX}} \Delta_2$. If $(\forall B \in \Delta_2)$ $\{B\} \cup \Xi \vdash F$, then from (II_1) we obtain $\Theta \cup \Gamma_2 \cup \Xi \vdash F$. With $(\forall C \in \Theta)\ \Gamma_1 \vdash C$, which follows from ($I_1$), this yields $\Gamma_1 \cup \Gamma_2 \cup \Xi \vdash F$ by using ($\vdash$ 2). We have proved that $\Vdash_{MAX}$ satisfies ($\Vdash$ 2.1 #). (Note that this way we don't obtain ($\Vdash$ 2.1) for $\Vdash_{MAX}$.)

Suppose now for ($\Vdash$ 2.2) that (I_2) $(\forall C \in \Theta)$ $\{C\} \cup \Gamma_2 \Vdash_{MAX} \Delta_2$ and (II_2) $\Gamma_1 \Vdash_{MAX} \Delta_1 \cup \Theta$. If $(\forall B \in \Delta_1 \cup \Delta_2)$ $\{B\} \cup \Xi \vdash F$, then from (I_2) we obtain $(\forall C \in \Theta)$ $\{C\} \cup \Gamma_2 \cup \Xi \vdash F$. With

$$(\forall B \in \Theta)\ \{B\} \cup \Gamma_2 \cup \Xi \vdash F \ \Rightarrow\ \Gamma_1 \cup \Gamma_2 \cup \Xi \vdash F,$$

which follows from (II_2), we obtain $\Gamma_1 \cup \Gamma_2 \cup \Xi \vdash F$. (To derive from ($II_2$) the implication we have displayed, we use ($\vdash$ 3) in order to obtain $(\forall B \in \Delta_1)$ $\{B\} \cup \Gamma_2 \cup \Xi \vdash F$ from $(\forall B \in \Delta_1)$ $\{B\} \cup \Xi \vdash F$.) This proves that $\Vdash_{MAX}$ satisfies ($\Vdash$ 2.2). *q.e.d.*

That ($\Vdash$ 2.1) cannot replace ($\Vdash$ 2.1 #) in Lemma 3 is shown by the following counterexample (adapted from [Shoesmith & Smiley 1978, section 5.2, Theorem 5.7, p. 75]). Let L be infinite and let $\Gamma \vdash B$ hold if and only if either Γ is infinite or $B \in \Gamma$. This relation $\vdash$ is a singular consequence relation, but with $\Gamma_1 = \Gamma_2 = \varnothing$, $\Delta_1 = \Delta_2 = \{A\}$, Θ infinite and $A \notin \Theta$, the principle ($\Vdash$ 2.1) fails for $\Vdash_{MAX}$.

This asymmetry of Lemma 3, which consists in having ($\Vdash$ 2.1 #) rather than ($\Vdash$ 2.1), has perhaps something to do with the fact that ($\Vdash$ 2.1) is involved in the following deduction. If Θ is the set of all instances of a formula A that are obtained by substituting a constant term for the variable x, and B is a formula in which x does not occur free, then from $(\forall C \in \Theta)$ $\{\forall x(B \vee A)\} \Vdash \{B, C\}$ and $\Theta \Vdash \{\forall x A\}$ we may infer by using ($\Vdash$ 2.1) the intuitionistically unacceptable sequent $\{\forall x(B \vee A)\} \Vdash \{B, \forall x A\}$.

To formulate the next theorem we need the following instance of ($\Vdash$ 2.2), where Δ_1 is empty:

($\Vdash$ 2.2 $\flat$) $\quad (\forall C \in \Theta)\{C\} \cup \Gamma_2 \Vdash \Delta_2 \ \Rightarrow\ (\Gamma_1 \Vdash \Theta \ \Rightarrow\ \Gamma_1 \cup \Gamma_2 \Vdash \Delta_2).$

If we have ($\Vdash$ 1) and ($\Vdash$ 3.2), then we can derive ($\Vdash$ 2.2 #) from ($\Vdash$ 2.2 $\flat$); if moreover we have ($\Vdash$ 3.1), we can derive ($\Vdash$ 2.2) from ($\Vdash$ 2.2 $\flat$). Since ($\Vdash$ 2.2 #) gives ($\Vdash$ 3.2) as an instance, ($\Vdash$ 2.2 $\flat$) and ($\Vdash$ 3.2) amount to ($\Vdash$ 2.2 $\flat$) and ($\Vdash$ 2.2 #) in the presence of ($\Vdash$ 1). For Γ_2 empty, ($\Vdash$ 2.2 $\flat$) coincides with ($\Vdash$ 2.2 #) where Δ_1 is empty. For Θ a singleton, ($\Vdash$ 2.2 $\flat$) coincides with an instance of ($\Vdash$ 2.1 #); it yields all instances of ($\Vdash$ 2.1 #) where Θ is nonempty and finite. Now we can give our theorem:

Theorem 3. (i) If $\vdash$ is a singular consequence relation, then $\Vdash_{\mathrm{MIN}}$ and $\Vdash_{\mathrm{MAX}}$ are plural consequence relations that satisfy ($\Vdash$ 2.2).

(ii) If $\vdash \subseteq \mathbf{P}(\mathrm{L}) \times \mathrm{L}$ satisfies ($\vdash$ 1) and $\Vdash \subseteq \mathbf{P}(\mathrm{L}) \times \mathbf{P}(\mathrm{L})$ satisfies ($\Vdash$ 2.2 $\flat$) and ($\Vdash$ 3.2), then

$$\forall\Gamma\forall B(\Gamma \vdash B \Leftrightarrow \Gamma \Vdash \{B\}) \text{ if and only if } \Vdash_{\mathrm{MIN}} \subseteq \Vdash \subseteq \Vdash_{\mathrm{MAX}}.$$

Proof. (i) It is trivial to prove that $\Vdash_{\mathrm{MIN}}$ and $\Vdash_{\mathrm{MAX}}$ satisfy ($\Vdash$ 1). For the rest we apply Lemmata 2 and 3.

(ii — only if part) If $\Gamma \Vdash_{\mathrm{MIN}} \Delta$, then for some $B \in \Delta$ we have $\Gamma \vdash B$, and hence $\Gamma \Vdash \{B\}$. By ($\Vdash$ 3.2), we obtain $\Gamma \Vdash \Delta$. So, $\Vdash_{\mathrm{MIN}} \subseteq \Vdash$. Suppose now $\Gamma \Vdash \Delta$. If $(\forall B \in \Delta)$ $\{B\} \cup \Xi \vdash F$, then $(\forall B \in \Delta)$ $\{B\} \cup \Xi \Vdash \{F\}$, and by ($\Vdash$ 2.2 $\flat$), we obtain $\Gamma \cup \Xi \Vdash \{F\}$. This yields $\Gamma \cup \Xi \vdash F$, and so $\Vdash \subseteq \Vdash_{\mathrm{MAX}}$.

(ii — if part) If $\Gamma \vdash B$, then we get immediately $\Gamma \Vdash_{\mathrm{MIN}} \{B\}$, and hence $\Gamma \Vdash \{B\}$. If $\Gamma \Vdash \{B\}$, then $\Gamma \Vdash_{\mathrm{MAX}} \{B\}$, which yields $\Gamma \vdash B$ with the help of ($\vdash$ 1) (this is the only place in the proof of (ii) where we need ($\vdash$ 1)). *q.e.d.*

From this theorem it follows that the set of all plural consequence relations satisfying ($\Vdash$ 2.2) that extend conservatively a given singular consequence relation coincides with the set of all plural consequence relations satisfying ($\Vdash$ 2.2) in the interval between $\Vdash_{\mathrm{MIN}}$ and $\Vdash_{\mathrm{MAX}}$. This set is nonempty, because $\Vdash_{\mathrm{MIN}} \subseteq \Vdash_{\mathrm{MAX}}$ (which follows from ($\vdash$ 2)).

Had we dealt with single-conclusion consequence relations $\vdash$ mentioned in the introduction, for which we may also have $\Gamma \vdash \varnothing$, then $\Gamma \Vdash_{\mathrm{MIN}} \Delta$ could be defined as

$$(\exists B \in \Delta)\, \Gamma \vdash B \;\; or \;\; (\Delta = \varnothing \;\&\; \Gamma \vdash \Delta).$$

In the definition of $\Vdash_{\mathrm{MAX}}$, the variable F could range also over the empty set. All this because in Theorem 3 we might wish to have on the left-hand side of the equivalence of (ii)

$$\forall\Gamma\forall B(\Gamma \vdash B \Leftrightarrow \Gamma \Vdash \{B\}) \;\;\&\;\; \forall\Gamma(\Gamma \vdash \varnothing \Leftrightarrow \Gamma \Vdash \varnothing).$$

It is to eschew these things that we have preferred to stick to our notion of singular consequence relation, which always has a formula on the right-hand side of $\vdash$. However, modulo such adjustments, our results would hold in the more general context, too (cf. the parenthetical remark in the penultimate paragraph, below).

§4. Variants of Theorem 3. Let us now consider cases where $\vdash \subseteq \mathbf{P}(\mathrm{L}) \times \mathrm{L}$ is not necessarily a singular consequence relation in our sense, but satisfies the weaker version of ($\vdash$ 2) where Θ is a singleton, i.e. the ordinary cut principle

($\vdash$ 2 cut) $\qquad \Gamma_1 \vdash C \Rightarrow (\{C\} \cup \Gamma_2 \vdash B \Rightarrow \Gamma_1 \cup \Gamma_2 \vdash B).$

Either of the equivalences

$$\Gamma_1 \vdash C \Leftrightarrow \forall\Gamma_2\forall B(\{C\}\cup\Gamma_2 \vdash B \Rightarrow \Gamma_1\cup\Gamma_2 \vdash B),$$
$$\{C\}\cup\Gamma_2 \vdash B \Leftrightarrow \forall\Gamma_1(\Gamma_1 \vdash C \Rightarrow \Gamma_1\cup\Gamma_2 \vdash B)$$

amounts to ($\vdash$ 1) and ($\vdash$ 2 cut). It is clear that ($\vdash$ 2 cut) yields ($\vdash$ 2) for every nonempty finite Θ. Note that ($\vdash$ 2 cut) suffices to show that $\Vdash_{MIN} \subseteq \Vdash_{MAX}$.

Then, by going once again through the relevant parts of the proofs of Lemmata 2 and 3, we can prove a variant of Theorem 3 where (i) is replaced by

(i') If $\vdash \subseteq \mathbf{P}(L)\times L$ satisfies ($\vdash$ 1), ($\vdash$ 2 cut) and ($\vdash$ 3), then $\Vdash_{MIN}$ and $\Vdash_{MAX}$ satisfy ($\Vdash$ 1) and ($\Vdash$ 2.2).

Scott's Theorem 1.2 of [1974, p. 415] is obtained from this variant of Theorem 3 by taking that $\vdash$ is compact and that the relations $\Vdash$ (i.e. those included in $\mathbf{P}(L)\times\mathbf{P}(L)$) hold only between finite sets of formulae. This is why he can replace ($\Vdash$ 2.2) by the following plural form of ordinary singleton cut:

$$\Gamma \Vdash \Delta\cup\{C\} \Rightarrow (\{C\}\cup\Gamma \Vdash \Delta \Rightarrow \Gamma \Vdash \Delta),$$

together with thinning on the left and right (in the presence of thinning on the left and right, this cut amounts to ($\Vdash$ 2.2) with Θ a singleton, or a finite set). As we have remarked above, ($\Vdash$ 2.1) and ($\Vdash$ 2.2) cannot be distinguished for $\Vdash$ holding between finite sets of formulae. So, for Scott, $\Vdash_{MIN}$ and $\Vdash_{MAX}$ in (i') are plural consequence relations in our sense. But, in general, without the finiteness restriction, we need not have ($\Vdash$ 2.1 #), and hence neither ($\Vdash$ 2.1), for $\Vdash_{MIN}$ and $\Vdash_{MAX}$ in (i'). (This is shown by the following counterexample, adapted from [Shoesmith & Smiley 1978, section 1.1, Theorem 1.3, p. 18]): let L be infinite and let $\Gamma \vdash B$ hold if and only if either Γ is infinite or $B\in\Gamma$ or B is not A, for a chosen formula A; this relation $\vdash$ satisfies ($\vdash$ 1), ($\vdash$ 2 cut) and ($\vdash$ 3), but with $\Gamma_1 = \Gamma_2 = \varnothing$, $\Delta_2 = \{A\}$, Θ infinite and $A\notin\Theta$, the principle ($\Vdash$ 2.1 #) fails for $\Vdash_{MIN}$ and $\Vdash_{MAX}$.) Of course, compact relations $\vdash \subseteq \mathbf{P}(L)\times L$ that satisfy ($\vdash$ 2 cut) and ($\vdash$ 3) satisfy also ($\vdash$ 2), and vice versa. So, for Scott, $\vdash$ in (i') is a singular consequence relation in our sense.

Note that to derive some of the equivalences giving alternative definitions of $\Vdash_{MIN}$ and $\Vdash_{MAX}$ in §3 we have used the full power of ($\vdash$ 2), and not only its instances such as ($\vdash$ 2 cut) or ($\vdash$ 3). In the absence of ($\vdash$ 2), these equivalences need not hold. However, ($\vdash$ 2 cut) and ($\vdash$ 3) amount to ($\vdash$ 2) for compact relations $\vdash$. Moreover, for the equivalence of our definition of $\Vdash_{MAX}$ with Scott's we need only ($\vdash$ 3).

In [1978, section 5.2, pp. 75-79] Shoesmith and Smiley investigate how one may extend conservatively compact singular consequence relations to plural consequence relations that need not be compact, i.e. that need not satisfy

$$\Gamma \Vdash \Delta \Rightarrow (\exists \Gamma' \subseteq \Gamma)(\exists \Delta' \subseteq \Delta)(\Gamma' \textit{ is finite } \& \ \Delta' \textit{ is finite } \& \ \Gamma' \Vdash \Delta').$$

In particular, they give examples of such maximal plural couterparts that need not be compact. (The minimal plural counterparts of compact singular consequence relations are always compact.) Scott's $\Vdash_{MAX}$, which is compact, need not coincide with the possibly noncompact $\Vdash_{MAX}$ of the variant of Theorem 3 with (i'), but it can be characterized completely analogously, the only difference being that it holds between finite sets. And our proof of this variant of Theorem 3, where we never assume that the sets that enter into the relations $\Vdash$ *must* be infinite (they only *can* be such), yields a proof of Scott's theorem.

Let us mention still another variant of Theorem 3, where thinning on the left is not assumed for $\vdash$. In this variant, (i) of Theorem 3 is replaced by

(i") If $\vdash \subseteq \mathbf{P}(L) \times L$ satisfies ($\vdash$ 1) and ($\vdash$ 2 cut), then $\Vdash_{MIN}$ and $\Vdash_{MAX}$ satisfy ($\Vdash$ 1), ($\Vdash$ 2.2 $\flat$) and ($\Vdash$ 2.2 #).

The proof consists in another rehearsal of the relevant parts of the proofs of Lemmata 2 and 3. Actually, for the part involving $\Vdash_{MAX}$, the relation $\vdash$ can be any relation included in $\mathbf{P}(L) \times L$, not necessarily satisfying ($\vdash$ 1) and ($\vdash$ 2 cut). For $\Vdash_{MIN}$ in (i") we can show that it also satisfies ($\Vdash$ 2.1) where Θ is a singleton and $\Gamma_2 \subseteq \Gamma_1$, and ($\Vdash$ 2.2) where $\Gamma_2 \subseteq \Gamma_1$. Of course, ($\Vdash$ 2.2 $\flat$) and ($\Vdash$ 2.2 #) yield the principles ($\Vdash$ 2.2 $\flat$) and ($\Vdash$ 3.2) from (ii) of Theorem 3. This variant of Theorem 3, with (i) replaced by (i"), can be adapted to sequents $\Gamma \vdash B$ and $\Gamma \Vdash \Delta$ where Γ and Δ are sequences of formulae (possibly transfinite) rather than sets, and where thinning, contraction and permutation on the left are not presupposed for $\vdash$.

Let us note that if in (i") we replace ($\vdash$ 2 cut) by ($\vdash$ 2) where $\Theta \neq \varnothing$ (so, Θ can be finite or infinite, but ($\vdash$ 3) is not guaranteed), then $\Vdash_{MIN}$ and $\Vdash_{MAX}$ will satisfy, of course, what they satisfy in (i"), but they will satisfy moreover ($\Vdash$ 2.1 #) where $\Theta \neq \varnothing$ (the principle ($\Vdash$ 2.1 #) for Θ nonempty and finite follows from ($\Vdash$ 2.2 $\flat$)). The relation $\Vdash_{MIN}$ will satisfy also ($\Vdash$ 2.1) where $\Theta \neq \varnothing$ and $\Gamma_2 \subseteq \Gamma_1$, and as before, ($\Vdash$ 2.2) where $\Gamma_2 \subseteq \Gamma_1$.

For the relations $\Vdash$ of these last variants of Theorem 3 we have thinning on the right in ($\Vdash$ 3.2). We also have the following special form of thinning on both the left and right, obtained by taking Θ empty in ($\Vdash$ 2.2 $\flat$):

($\Vdash$ 3.2 $\flat$) $\qquad \Gamma_1 \Vdash \varnothing \Rightarrow \Gamma_1 \cup \Gamma_2 \Vdash \Delta_2.$

Otherwise, we don't have thinning on the left for $\Vdash$, as we don't have it for $\vdash$. (If we deal with relations $\vdash$ that admit $\Gamma \vdash \varnothing$, then for $\vdash$ in analogues of Theorem 3 we might need to assume a single-conclusion form of ($\Vdash$ 3.2 ♭), with Δ_2 a singleton or empty.) Of course, if in $\Gamma \Vdash \Delta$ we require that Δ be always nonempty, we shall not have ($\Vdash$ 3.2 ♭).

One may envisage other variants of our theorems, and in some of them one would need definitions of minimal or maximal consequence relations different from those we have given. For example, in a variant of Theorem 2, one may try to characterize the maximal plural consequence relation that extends conservatively a given preorder and satisfies ($\Vdash$ 2.1) and ($\Vdash$ 2.2), rather than only ($\Vdash$ 2.1 #) and ($\Vdash$ 2.2 #). Provided that it exists. The minimal such consequence relation is $\Vdash_{min}$, as stated in Lemma 1.

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