A COMPLETENESS THEOREM FOR THE LAMBEK CALCULUS
OF SYNTACTIC CATEGORIES

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Introduction

In [5] LAMBEK has formulated a calculus of syntactic categories, related to a calculus of residuals, which is meant to underlie categorial grammars. In [1] BUSZKOWSKI mentions a natural semigroup modelling of this calculus, without claiming completeness. Recently, BUSZKOWSKI has published in [2] a completeness theorem for the product-free fragment of LAMBEK's calculus (for various axiomatizations of this fragment see [8] and [3]). According to BUSZKOWSKI it seems this is the first published result of this sort for LAMBEK's calculus. This completeness theorem of BUSZKOWSKI is given with respect to a new type of structures which he calls "category systems".

In this paper we shall prove a completeness theorem for the full Lambek calculus with respect to partially ordered semigroup models using an interpretation somewhat different from BUSZKOWSKI's interpretation of [1]. More precisely, we shall consider partially ordered semigroups (in the sense of [4], p. 153) and subsets in these semigroups which are closed from below (i.e., if $M$ is a subset, $e_2 \in M$ and $e_1 \leq e_2$, then $e_1 \in M$). Our proof will be entirely syntactical: it will consist in an embedding of LAMBEK's calculus in a first-order theory of partially ordered semigroups which has unary predicates interpreted by subsets closed from below. These unary predicates stand for syntactic categories. We shall suggest an intuitive interpretation of these semigroup models. After our main completeness result we shall consider completeness results of the same type for fragments of LAMBEK's calculus — in particular for the product-free fragment.

1. The system L

The Lambek calculus is the following system which we shall call $L$.

The language of $L$, called $\mathcal{L}(L)$, consists of

(I) denumerably many primitive category expressions: $P_1, P_2, \ldots$,

(II) three expressions for category operations: $\cdot, /, \\mid$,

(III) one expression for a category relation: $\rightarrow$.

Category expressions in $\mathcal{L}(L)$ behave like terms. We shall use $X_1, X_2, \ldots$ as schematic letters for category expressions. If $X_1$ and $X_2$ are category expressions, then $X_1 \cdot X_2$, $X_1 / X_2$ and $X_1 \\mid X_2$ are category expressions, and $X_1 \rightarrow X_2$ is a formula of $\mathcal{L}(L)$. There are no other formulae in $\mathcal{L}(L)$. 
The axioms and rules of $L$ are:

(a) $X_1 \rightarrow X_1$

(b) $(X_1 \cdot X_2) \cdot X_3 \rightarrow X_1 \cdot (X_2 \cdot X_3)$

(c) $X_1 \cdot X_2 \rightarrow X_3$

(d) $X_1 \rightarrow X_3/X_2$

(c') $X_2 \cdot X_1 \rightarrow X_3$

(e) $X_1 \rightarrow X_2 \cdot X_2 \rightarrow X_3$

X_1 \rightarrow X_3

The system $S$ called $S.$

The first-order theory of partially ordered semigroups we shall consider will be called $S.$

The language of $S,$ called $\mathcal{L}(S),$ consists of:

(i) denumerably many primitive term constants: $p_1, p_2, \ldots,$

(ii) denumerably many term variables: $x_1, x_2, \ldots$.

(iii) one expression for a term operation: $\cdot$,

(iv) denumerably many primitive category expressions: $P_1, P_2, \ldots$ (identical with (i)),

(v) two expressions for term relations: $\leq, =,$

(vi) the logical constants: $\rightarrow, \leftrightarrow, \&,$ $\vee, \neg, \forall x_i, \exists x_i$.

We shall use $a_1, a_2, \ldots$ as schematic letters for terms. If $a_1$ and $a_2$ are terms, then $a_1 + a_2$ is a term, and $a_1 \leq a_2$ and $a_1 = a_2$ are formulae. Category expressions in $\mathcal{L}(S)$ behave like unary predicates: so expressions of the form $P_i a_1$ are formulae. (Intuitively, $P_i a_1$ means "$a_1$ is of the category $P_i".)$ Other formulae of $\mathcal{L}(S),$ involving logical constants from (vi), are built as usual.

The system $S$ is axiomatized by the following

(1) logical postulates:

(1.1) the axioms and rules of the first-order predicate calculus involving the constants of (vi),

(1.2) $a_i = a_i,$

(1.3) $(a_1 = a_2 \& A[a_1]) \Rightarrow A[a_2],$ where the formula $A[a_2]$ is obtained from the formula $A[a_1]$ by replacing zero or more occurrences of $a_1$ by $a_2$, with the usual provisos for variables;

(2) axiom-schemata for partial order:

(2.1) $a_1 \leq a_1,$

(2.2) $(a_1 \leq a_2 \& a_2 \leq a_3) \Rightarrow a_1 \leq a_3,$

(2.3) $(a_1 \leq a_2 \& a_2 \leq a_3) \Rightarrow a_1 = a_2.$
(3) **axiom-schemata connecting** \( \leq \) with \(+\) and **category expressions**:

- **(3.1)** \( a_1 \leq a_2 \Rightarrow a_1 + a_3 \leq a_2 + a_3 \).
- **(3.2)** \( a_1 \leq a_2 \Rightarrow a_3 + a_1 \leq a_3 + a_2 \).
- **(3.3)** \( (a_1 \leq a_2 \& P_1 a_2) \Rightarrow P_1 a_1 \):

(4) **semigroup axiom-schema**: \((a_1 + a_2) + a_3 = a_1 + (a_2 + a_3)\).

Next we shall introduce a system \( S' \) closely related to \( S \). Let \( \mathcal{L}(S') \) differ from \( \mathcal{L}(S) \) only by having the expressions of (II) and (III) in addition to what we have in \( \mathcal{L}(S) \). We have in \( \mathcal{L}(S') \) complex unary predicates of the form \( X_1 \cdot X_2 \), \( X_1/X_2 \), and \( X_1 \setminus X_2 \), and also formulae of the form \( X_1 a_1 \) and \( X_1 \rightarrow X_2 \). It is clear that \( \mathcal{L}(L) \) is a proper sublanguage of \( \mathcal{L}(S') \).

The system \( S' \) differs from \( S \) by having in addition to (1)–(4) the following axiom-schemata

- **(5)** \((X_1 \cdot X_2) a_1 \leftrightarrow \exists x_2 \exists x_2 (X_1 x_1 \& X_2 x_2 \& a_1 \leq x_1 + x_2)\);
- **(6.1)** \((X_1/X_2) a_1 \leftrightarrow \forall x_2 (X_2 x_2 \Rightarrow X_1 (a_1 + x_2))\);
- **(6.2)** \((X_1 \setminus X_2) a_1 \leftrightarrow \forall x_2 (X_2 x_2 \Rightarrow X_1 (x_2 + a_1))\);
- **(7)** \( X_1 \rightarrow X_2 \leftrightarrow \forall x_1 (X_1 x_1 \Rightarrow X_2 x_1) \).

It is easily shown by induction on the complexity of \( X_1 \) that \((a_1 \leq a_2 \& X_1 a_2) \Rightarrow X_1 a_1 \) is provable in \( S' \) for every \( X_1 \).

It is clear that (5)–(7) could be understood as specifying definitions which introduce some abbreviations in \( \mathcal{L}(S) \), and we shall assume that \( \cdot, /, \setminus \) and \( \rightarrow \) have been introduced in \( \mathcal{L}(S) \) by these definitions. Then it is trivial to show that the theorems of \( S \) and \( S' \) coincide. We have introduced the system \( S' \) only for convenience.

3. **Embedding of \( L \) into \( S' \)**

We can prove the following lemma.

**Lemma 1.** \( \vdash_L X_1 \rightarrow X_2 \), then \( \vdash_S X_1 \rightarrow X_2 \).

This is done by a straightforward induction on the length of proof of \( X_1 \rightarrow X_2 \) in \( L \), and we shall omit this proof.

Next we introduce the mapping \( t \) from \( \mathcal{L}(S') \) to the metalanguage of \( L \) satisfying

- \( t(p_i) = P_i \).
- \( t(x_i) = \) the \( i \)-th schematic letter for category expressions,
- \( t(a_1 + a_2) = t(a_1) \cdot t(a_2) \),
- \( t(X_1) = X_1 \).
- \( t(X_1 a_1) = \vdash_L t(a_1) \rightarrow t(X_1) \) (i.e. \( \vdash_L t(a_1) \rightarrow X_1 \)).
- \( t(a_1 \leq a_2) = \vdash_L t(a_1) \rightarrow t(a_2) \),
- \( t(a_1 = a_2) = \vdash_L t(a_1) \rightarrow t(a_2) \) and \( \vdash_L t(a_2) \rightarrow t(a_1) \),
- \( t(X_1 \rightarrow X_2) = \vdash_L t(X_1) \rightarrow t(X_2) \) (i.e. \( \vdash_L X_1 \rightarrow X_2 ) \),
\[ t(A \Rightarrow B) = \text{iff } t(A), \text{ then } t(B), \]
\[ t(A \Leftrightarrow B) = t(A) \text{ iff } t(B), \]
\[ t(A \& B) = t(A) \text{ and } t(B), \]
\[ t(A \lor B) = t(A) \text{ or } t(B), \]
\[ t(\neg A) = \text{not } t(A), \]
\[ t(\forall x, A) = \text{for every } X_i, t(A), \]
\[ t(\exists x, A) = \text{for some } X_i, t(A), \]

where \( A \) and \( B \) are formulae of \( \mathcal{L}(S') \).

Remark. The translation of \( x \) is the schematic letter "\( X_i \)" whereas the translation of the category expression \( X_i \) is the category expression \( X_i \). However, since in the proof of Lemma 2 below we shall be translating schemata for formulae of \( \mathcal{L}(S') \) rather than these formulae, and since in these schemata the schematic letters \( X_1, X_2, \ldots \) might occur, in order to avoid confusion we shall translate the variables \( x_1, x_2, \ldots \) by new schematic letters for category expressions: \( Y_1, Y_2, \ldots \)

Then we can prove the following lemma.

Lemma 2. If \( \vdash_{S^*} A \), then \( t(A) \).

Proof. By induction on the length of proof of \( A \) in \( S' \). It is trivial to show that if \( A \) is an axiom from (1.1), then \( t(A) \) is true. For the other axioms we have the following.

(1.2) This case follows immediately from (a).

(1.3) We show that if \( \vdash_L X_1 \rightarrow X_2 \) and \( \vdash_L X_2 \rightarrow X_1 \) and \( C[X_1] \), then \( C[X_2] \), where \( C[X_2] \) is obtained from the formula \( C[X_1] \) of the metalanguage of \( L \) by replacing zero or more occurrences of \( X_1 \) by \( X_2 \). For that it is enough to show that if \( \vdash_L X_1 \rightarrow X_2 \) and \( \vdash_L X_2 \rightarrow X_1 \), then \( \vdash_L X_3[X_1] \rightarrow X_3[X_2] \) and \( \vdash_L X_3[X_2] \rightarrow X_3[X_1] \), where \( X_3[X_2] \) is obtained from the category expression \( X_3[X_1] \) by replacing zero or more occurrences of \( X_1 \) by \( X_2 \). This is done using the fact that \( L \) is closed under the rules

\[
\begin{align*}
(m) & \quad \frac{X_1 \rightarrow X_2 \quad X_3 \rightarrow X_4}{X_1 \cdot X_3 \rightarrow X_2 \cdot X_4} \\
(n) & \quad \frac{X_1 \rightarrow X_2 \quad X_3 \rightarrow X_4}{X_1 / X_4 \rightarrow X_2 / X_3} \quad (n') \quad \frac{X_1 \rightarrow X_2 \quad X_3 \rightarrow X_4}{X_4 \cdot X_1 \rightarrow X_3 \cdot X_2}
\end{align*}
\]

(see [5], p. 164).

(What we have just shown, together with (a) and (e), guarantees that \( \rightarrow \) represents a partial-ordering relation in \( L \).)

(2) The cases (2.1) and (2.2) follow immediately from (a) and (e), and (2.3) is trivial.

(3) The cases (3.1) and (3.2) follow immediately from (m), and (3.3) from (e).

(4) This case follows immediately from (b) and (b').

(5) We show that

\[ \vdash_L X_3 \rightarrow X_1 \cdot X_2 \text{ iff for some } Y_1 \text{ and for some } Y_2, \vdash_L Y_1 \rightarrow X_1 \text{ and } \vdash_L Y_2 \rightarrow X_2 \text{ and } \vdash_L X_3 \rightarrow Y_1 \cdot Y_2. \]
From left to right we simply take $X_1$ to be $Y_1$ and $X_2$ to be $Y_2$. For the other direction we have

$$
\frac{Y_1 \rightarrow X_1 \quad Y_2 \rightarrow X_2}{X_3 \rightarrow X_1 \cdot X_2}. \quad (m)
$$

$$
\frac{Y_1 \cdot Y_2 \rightarrow X_1 \cdot X_2}{X_3 \rightarrow X_1 \cdot X_2}. \quad (e)
$$

(6.1) We show that

$$
\vdash_L X_3 \rightarrow X_1/X_2 \text{ iff for every } Y_2, \text{ if } \vdash_L Y_2 \rightarrow X_2, \text{ then } \vdash_L X_3 \cdot Y_2 \rightarrow X_1.
$$

From left to right we have

$$
\frac{X_3 \rightarrow X_1/X_2 \quad Y_2 \rightarrow X_2}{X_3 \cdot Y_2 \rightarrow (X_1/X_2) \cdot X_2}. \quad (m)
$$

$$
\frac{X_1/X_2 \rightarrow X_1/X_2 \quad (X_1/X_2) \cdot X_2 \rightarrow X_1}{X_3 \cdot Y_2 \rightarrow X_1}. \quad (d)
$$

$$
\frac{(X_1/X_2) \cdot X_2 \rightarrow X_1}{X_3 \cdot Y_2 \rightarrow X_1}. \quad (e)
$$

For the other direction we have as an instance of the right-hand side that if $\vdash_L X_2 \rightarrow X_2$, then $\vdash_L X_3 \cdot X_2 \rightarrow X_1$. Using (a) we obtain $\vdash_L X_3 \rightarrow X_1$, from which $\vdash_L X_3 \rightarrow X_1/X_2$ follows by (c).

(6.2) This case is analogous to case (6.1).

(7) We show that

$$
\vdash_L X_1 \rightarrow X_2 \text{ iff for every } Y_1, \text{ if } \vdash_L Y_1 \rightarrow X_1, \text{ then } \vdash_L Y_1 \rightarrow X_2.
$$

From left to right we use (e). For the other direction we have as an instance of the right-hand side that if $\vdash_L X_1 \rightarrow X_1$, then $\vdash_L X_1 \rightarrow X_2$, from which $\vdash_L X_1 \rightarrow X_2$ follows using (a).

It remains only to remark that the rules of the predicate calculus hold in the metatheory of $L$. □

As a corollary of this lemma we obtain that if $\vdash_S X_1 \rightarrow X_2$, then $\vdash_L X_1 \rightarrow X_2$, which together with Lemma 1 and the connection between $S$ and $S'$ gives the following embedding theorem for $L$.

Theorem 1. $\vdash_L X_1 \rightarrow X_2$ iff $\vdash_S X_1 \rightarrow X_2$.

From this theorem we obtain our completeness theorem for $L$.

Theorem 2. $\vdash_L X_1 \rightarrow X_2$ iff $X_1 \rightarrow X_2$ is true in every model of $S$.

Let $S_1$ be the system which differs from $S$ only by lacking (2.3); i.e., $\leq$ in $S_1$ stands for a preordering relation. Then by going over our proof it can easily be shown that $\vdash_L X_1 \rightarrow X_2$ iff $\vdash_{S_1} X_1 \rightarrow X_2$, i.e., iff $X_1 \rightarrow X_2$ is true in every model of $S_1$.

4. An intuitive interpretation of $S$

The first-order theory of semigroups with unary predicates has a natural linguistic interpretation. Terms stand for words, $+$ stands for concatenation, and unary predicates stand for syntactic categories. How can we interpret linguistically the ordering relation $\leq$ of $S$ (or $S_1$)? There is probably more than one way to do that. One tentative suggestion is to read "$a_1 \leq a_2$" as "$a_1$ abbreviates $a_2$ or $a_1$ is equal to $a_2$". (The relation "$-$ abbreviates $-$" is neither reflexive nor symmetric.) It is easy to
check that with this interpretation all the axioms of $S$ are satisfied. The definition given by (5) is also justified: it enables $a_3$ to be of the category $X_1 \cdot X_2$ not only when $a_3$ is equal to the concatenation of the words $a_1$ of the category $X_1$ and $a_2$ of the category $X_2$, but also when $a_3$ only abbreviates a word of the form $a_1 + a_2$.

5. Completeness of fragments of $L$

It can easily be seen that our embedding theorem did not depend essentially on the infinity of the sets of primitive category expressions and primitive term constants. It could easily be adapted to cover finite cases.

It can also easily be seen that our proof can be adapted to show that the system obtained from $L$ by omitting (b) and (b') (see [6]) can be embedded in $S$ (or $S_1$) without (4).

Let us now consider fragments of $L$ obtained by making reductions in $\mathcal{L}(L)$. It can easily be shown that if from $\mathcal{L}(L)$ we omit / or \ (or both), and reject from $L$ the rules involving what we have omitted, the remaining calculus can still be embedded in $S$ (or $S_1$), without the definitions of (6.1) or (6.2), and is hence complete with respect to models of $S$ (or $S_1$).

Next we shall consider the product-free fragment of $L$ (i.e. the fragment without $\cdot$). Let $S_2$ be the system obtained from $S$ by omitting $\leq$ from $\mathcal{L}(S)$ and rejecting all the axioms involving $\leq$ (i.e. the axioms of (2) and (3)). In other words, $S_2$ is the first-order theory of semigroups with unary predicates. We show in the following lemma that $S$ is a conservative extension of $S_2$.

Lemma 3. If $A$ is a formula of $\mathcal{L}(S_2)$, then $\vdash_{S_2} A$ iff $\vdash_{S} A$.

Proof. From left to right this is trivial. For the other direction it is enough to check that the axioms of (2) and (3) hold in $S_2$ when we replace $\leq$ by $\leq$.

Then it is possible to show the following theorem.

Theorem 3. If $\cdot$ does not occur in $X_1 \rightarrow X_2$, then $\vdash_{L} X_1 \rightarrow X_2$ iff $\vdash_{S_2} X_1 \rightarrow X_2$.

Proof. From Theorem 1 we know that $\vdash_{L} X_1 \rightarrow X_2$ iff $\vdash_{S} X_1 \rightarrow X_2$. The expression $\leq$ does not occur in $X_1 \rightarrow X_2$ considered as a formula of $\mathcal{L}(S)$. Hence, by Lemma 3, $\vdash_{S} X_1 \rightarrow X_2$ iff $\vdash_{S_2} X_1 \rightarrow X_2$.

So, the product-free fragment of $L$ is complete with respect to models of $S_2$. This should be connected with Buszkowski’s completeness theorem for this fragment (see [2]).

6. Concluding remarks

A survey of our proof will show that we have embedded $L$ into the fragment of $S$ (or $S_1$) without $\vee$ and $\neg$. It will also show that we could have assumed the underlying first-order logic of $S$ (or $S_1$) to be intuitionistic rather than classical.

The interpretation of $L$ which Buszkowski envisaged in [1] differs essentially from ours only in replacing $a_1 \leq x_1 + x_2$ in (5) by $a_1 = x_1 + x_2$. With (5) so modified it can easily be shown that if $\vdash_{L} X_1 \rightarrow X_2$, then $\vdash_{S_2} X_1 \rightarrow X_2$. We leave open the question
whether the converse also holds, i.e., whether $L$ is not only sound but also complete with respect to partially ordered semigroup models where the partial-ordering relation is identity.

It is known that the first-order theory of semigroups is undecidable (see [7], p. 86). The system $S$ extends this theory only by adding the ordering relation and unary predicates to the language, and the axioms of (2) and (3). It is easy to show that $S$, $S_1$, and $S_2$ are conservative extensions of this theory (cf. the proof of Lemma 3). Hence, all these systems are also undecidable. According to our embedding theorems, the system $L$, which was shown decidable in [5], gives decidable fragments of these systems.

References


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