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THE FIRST AXIOMATIZATION OF RELEVANT LOGIC

ABSTRACT. This is a review, with historical and critical comments, of a paper by I. E. Orlov from 1928, which gives the oldest known axiomatization of the implication-negation fragment of the relevant logic R. Orlov's paper also foreshadows the modal translation of systems with an intuitionistic negation into S4-type extensions of systems with a classical, involutive, negation. Orlov introduces the modal postulates of S4 before Becker, Lewis and Gödel. Orlov's work, which seems to be nearly completely ignored, is related to the contemporaneous work on the axiomatization of intuitionistic logic.

In (1928), I. E. Orlov gave an axiomatization of the implication-negation fragment of the *relevant logic* R, and this may well be the very first axiomatization of relevant logic. It is perhaps not fortuitous that the first axiomatization of relevant logic should appear at the same time when the first axiomatizations of intuitionistic logic were being produced.

This is not the only achievement of Orlov's paper. He also foreshadows the *modal translation* of systems with an intuitionistic negation into S4-type extensions of systems with a classical, involutive, negation. By "modal translation" we mean a translation that prefixes the necessity operator to subformulae of a nonmodal formula. That there is such a modal translation from Heyting's logic into the modal logic S4, which is based on classical logic, has been claimed by Gödel (1933), and has become a very well known fact about Heyting's logic. In my paper (1992), it is shown that there are analogous modal translations from intuitionistic variants of linear logic, relevant logic and BCK logic into S4-type extensions of the respective systems with a negation like classical negation. Metaphorically speaking, the S4 necessity operator subdues classical negation and transforms it into an intuitionistic one. Such a modal translation for relevant logic is only foreshadowed in Orlov's paper. But he introduces quite explicitly the characteristic modal postulates of S4, in the form they have in

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Gödel (1933), and this is an older appearance in print of these postulates than any usually referred to. What makes Orlov fall short of S4 is only that he adds these postulates to relevant logic instead of classical logic.

Because of all that, we shall present here a review of Orlov's thoroughly and haplessly ignored paper. After this review, which will cover the next three sections, we shall compare, in the final section, Orlov's work with the contemporaneous work on the axiomatization of intuitionistic logic.

1. INTRODUCTION

(Orlov's calculus of compatibility: §1, p. 263)

I was set on the trail of Orlov (1928) by a reference to it in Chagrov and Zakharyashchev (1990). In two sentences of their survey, Chagrov and Zakharyashchev mention Orlov as a precursor of the axiomatization of S4 and of Gödel's modal translation of Heyting's logic into S4. They don't mention that Orlov is a precursor of relevant logic. Orlov's paper is listed in Church's bibliography (1936, p. 201), and it is given a short review in the *Jahrbuch über die Fortschritte der Mathematik* (vol. 54, 1928, p. 54). I know of no other reference to Orlov's paper. I was unable to find whether Orlov has written any other paper. I ignore even his full name. (The "I." of Orlov's initials "I. E." is transliterated in the French résumé after the paper by "J.", as if "Ivan" has become "Jean". The "M." preceding the French "J." is for "Monsieur", a custom *Matematicheskiĭ Sbornik* was to abandon some years later. I could not find Orlov's name in the roster of the Moscow Mathematical Society printed in *Matematicheskiĭ Sbornik* from 1922 until 1929. I only know, from the title page of (1928), that Orlov is from Moscow, as Kolmogorov and Glivenko were too.)[1]

The title of Orlov's paper is in Russian "Ischislenie sovmestnosti predlozhenii", and it has a short, and not too informative, résumé in French entitled "Sur la théorie de la compatibilité des propositions". By "compatibility", which is a translation of "sovmestnost'", Orlov understands exactly the same thing that Anderson and Belnap (1975, §27.1.4, pp. 345–346) have tried calling "co-tenability". In the relevant logic literature, the same notion has also been called "intensional conjunction" or "consistency", and has more recently acquired the name "fusion" (see Routley, Meyer *et al.*, 1982, pp. 365–367, and Dunn, 1986, p. 130; in Girard, 1987, this notion has been reintroduced as "times").

Orlov says at the beginning of his paper that, by rejecting some axioms from the propositional calculus, he will obtain a system that, unlike classical logic, which is based on material implication, will be able to represent relevance between propositions in a symbolic form. He also wants to connect this new system with intuitionism. He says that, in intuitionism, notions are defined in terms of certain unary operations on propositions, like "it is provable that", for which he refers to Brouwer (1925). Orlov thinks that such propositional operations could not be introduced consistently into classical logic, but can be so introduced into his new logic. With hindsight, we are able to say that they can be introduced consistently into both.

2. RELEVANT LOGIC

(Orlov's calculus of compatibility: §§1–5, pp. 263–279)

Orlov says that classical logic deals with the truth and falsity of premises in a deduction, by which he may also mean that in a classical deduction premises are joined by a truth-functional conjunction. This concern with truth and falsity is foreign to Orlov's system, where instead premises in a deduction are joined by a "logical product", i.e. conjunction, that he calls "compatibility". This is why Orlov calls his system "the calculus of compatibility of propositions". These remarks of Orlov should be compared with the fact that, in a Gentzen formulation of relevant logic, we can characterize fusion $\bullet$ by the rules:

$$\frac{\Gamma_1, A, B, \Gamma_2 \vdash \Delta}{\Gamma_1, A \bullet B, \Gamma_2 \vdash \Delta}$$

where the double line means that we have two rules, one going downwards and the other upwards. In Gentzen systems with structural rules adequate for classical or intuitionistic logic, the same double-line rule would characterize extensional conjunction (cf. Došen, 1988, §§1.2, 1.7).

Orlov defines "p is compatible with q" as "p does not imply the negation of q". The notions of implication and negation, in terms of which this definition of compatibility is given, are the primitives of his system. In an intuitive comment on his implication, Orlov stresses that it is a relevant implication, based on "a connexion of meaning", and should not be confused with material implication, which is necessary but not sufficient for his implication. As a consequence of that, we shall not have "if p is compatible with q, then p", and we must also reject "if p, then (if q, then p)". However, Orlov finds it remarkable that many important logical principles, whose proof as given by Peano and Russell depended on the banned logical principles, can still be proved in his system.

We don't follow Orlov's notation, which he derived from Hilbert, for the formal presentation of his system. Instead, we use what looks like more standard notation, noting in parentheses where this notation differs from Orlov's. Orlov's system is based on a propositional language with the binary connective $\rightarrow$ of implication and the unary connective $\neg$ of negation (Orlov uses Hilbert's "overlined bar" for negation, as Anderson and Belnap do too in 1975). As schematic letters for propositional formulae, we use A, B, C, . . . (Orlov uses a, b, c, . . .). Orlov's system has the following axiom-schemata:

(Ax. 1) $A \rightarrow \neg\neg A$

(Ax. 2) $\neg\neg A \rightarrow A$

(Ax. 3) $A \rightarrow \neg(A \rightarrow \neg A)$

(Ax. 4) $(A \rightarrow B) \rightarrow (\neg B \rightarrow \neg A)$

(Ax. 5) $(A \rightarrow (B \rightarrow C)) \rightarrow (B \rightarrow (A \rightarrow C))$

(Ax. 6) $(B \rightarrow C) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$

and *modus ponens*, which Orlov calls (Ax. 7):

$$\frac{A \qquad A \rightarrow B}{B}$$

as a primitive rule. (I ought to mention that, in my copy of Orlov, 1928, which comes from an offset reprint of *Matematicheskiĭ Sbornik* published by Kraus Reprint Ltd (Vaduz, 1965), both (Ax. 1) and (Ax. 2) are given as $A \to A$, the double overlined bar that stands for $\neg\neg$ missing in both cases. However, to make sense of the proof of $A \to A$ on p. 267, which should proceed with the help of (Ax. 1), (Ax. 2) and (Ax. 6), and of other things later on, we must take (Ax. 1) and (Ax. 2) as above. This obvious misprint occurs perhaps only in the offset text, which is a little bit pale and has other similar misprints. Orlov's bad luck with Hilbert's overlined bar is one more reason for not using it here.)

Consider now the axiomatization of the implication-negation fragment of R given in Anderson and Belnap (1975, §14.2.4, p. 144). It has, in addition to *modus ponens*, the following axiom-schemata, whose names in Anderson and Belnap (1975) we write on the right-hand side:

(Ax. 2)	$\neg\neg A \to A$	(7)
(Ax. 5)	$(A \to (B \to C)) \to (B \to (A \to C))$	(3)
(Ax. 6)	$(B \to C) \to ((A \to B) \to (A \to C))$	(2)
	$A \to A$	(4)
	$(A \to \neg B) \to (B \to \neg A)$	(5)
	$(A \to \neg A) \to \neg A$	(6)

Let us now indulge in a small and easy exercise in axiom-chopping. First, we sketch how we prove the additional schemata (4), (5) and (6) in Orlov's system. For (4), we use:

$$(\neg\neg A \to A) \to ((A \to \neg\neg A) \to (A \to A)) \quad \text{(Ax. 6)}$$

together with (Ax. 2), (Ax. 1) and *modus ponens* (as Orlov does on p. 267). For (5), we have:

$(A \to \neg B) \to (\neg\neg B \to \neg A)$	(Ax. 4)
$\neg\neg B \to ((A \to \neg B) \to \neg A)$	(Ax. 5), *modus ponens*
$B \to ((A \to \neg B) \to \neg A)$	(Ax. 6), (Ax. 1), *modus ponens*
$(A \to \neg B) \to (B \to \neg A)$	(Ax. 5), *modus ponens*.

Finally, for **(6)**, we have:

$$A \to \neg(A \to \neg A) \qquad \text{(Ax. 3)}$$

$$(A \to \neg A) \to \neg A \qquad \text{(5), \textit{modus ponens}.}$$

(Orlov notes on p. 269 that **(5)** is provable in his system, and on p. 268 that **(6)** is provable.)

Now we sketch how we prove Orlov's (Ax. 1), (Ax. 3) and (Ax. 4) in the implication-negation fragment of R. For (Ax. 1), we have:

$$(\neg A \to \neg A) \to (A \to \neg\neg A) \qquad \text{(5)}$$

together with **(4)** and *modus ponens*. For (Ax. 3), we use **(6)**, **(5)** and *modus ponens*. Finally, for (Ax. 4), we have:

$$B \to \neg\neg B \qquad \text{(Ax. 1)}$$

$$(A \to B) \to (A \to \neg\neg B) \qquad \text{(2), \textit{modus ponens};}$$

this together with:

$$(A \to \neg\neg B) \to (\neg B \to \neg A) \qquad \text{(5)}$$

and **(2)** and *modus ponens* gives (Ax. 4).

So, Orlov's system is the implication-negation fragment of R. To do justice to Orlov, we shall call this system from now on OR.

Orlov's axiomatization of OR is not separative, in the sense that we cannot prove the schemata:

$$A \to A$$

$$(A \to (A \to B)) \to (A \to B)$$

which we need for the implicational fragment of R (see Church's axiomatization $R_{\to I}$ in Anderson and Belnap, 1975, §8.3.4, p. 79), without proceeding via axioms involving negation. The implicational fragment of R had to wait Moh's (1950) and Church's (1951) for its first axiomatization. (Though Church has introduced the implicational fragment of R only in 1951, the idea on which the implication of R is based is analogous to what is in his λ-I calculus of the thirties, where the term $\lambda x.t$ is not well-formed if x does not occur free in the term t.) However, to use a phrase from Anderson and Belnap (1975, §28.1,

p. 349), where it is incorrectly claimed that Moh's and Church's system is the oldest relevant logic, Orlov's achievement "deserves at least respect".

Orlov does not consider extending OR with extensional conjunction and disjunction. He sticks only to implication and negation, and the intensional connectives defined from them (in the terminology of Girard, 1987, he remains in the multiplicative fragment). In terms of his primitives, Orlov formulates the following definitions:

$$A \bullet B =_{\text{df}} \neg(A \to \neg B) \qquad \text{(p. 265)}$$

$$A + B =_{\text{df}} \neg(\neg A \bullet \neg B) \qquad \text{(p. 275)}$$

Orlov also uses extensively a Sheffer-like stroke defined by:

$$A|B =_{\text{df}} \neg(A \bullet B) \qquad \text{(p. 265)}$$

which, however, we shall eschew using here. The connective $\bullet$ is the compatibility, or fusion, connective, written as $\circ$ in the relevant logic tradition (in Girard, 1987, the small circle has acquired a cross to become the tensor product sign $\otimes$; Orlov's dot is somewhat smaller and lower than ours). The connective $+$, which could equivalently be defined by:

$$A + B =_{\text{df}} \neg A \to B$$

is the intensional disjunction of Anderson and Belnap (1975, §27.1.4, p. 344) (which, in Girard, 1987, has taken the garb of an inverted ampersand and is called "par"; Orlov's sign for $+$ is the ordinary wedge $\vee$). On p. 276, Orlov comments briefly on his intensional disjunction and says that it involves "a connexion of meaning".

The bulk of Orlov's paper is devoted to proving various theorems in OR and making intuitive comments on them. He shows on p. 267 that $\to$ could be defined in terms of $\bullet$ and $\neg$, since in OR we can prove:

$$(A \to B) \to \neg(A \bullet \neg B)$$

and the converse implication. Next, he finds that $\bullet$ is associative and commutative; i.e., we can prove in OR:

$$((A \bullet B) \bullet C) \to (A \bullet (B \bullet C)) \qquad \text{(p. 270)}$$

$$(A \bullet (B \bullet C)) \to ((A \bullet B) \bullet C) \qquad \text{(p. 276)}$$

$$(A \bullet B) \to (B \bullet A) \qquad \text{(p. 269)}$$

(the last schema corresponds to Gentzen's structural rule of Permutation, or Interchange, on the left of a sequent). Orlov's (Ax. 3) is:

$$A \to (A \bullet A)$$

(which corresponds to the structural rule of Contraction on the left), but he notes, without proof, on p. 268 that in OR we don't always have the *mingle* principle:

$$(A \bullet A) \to A$$

(this schema corresponds to the structural rule converse to Contraction on the left, which is a restricted form of Thinning, or Weakening, and which we may call Expansion on the left). Next, Orlov proves in OR:

$$(A \to (A \to B)) \to (A \to B) \qquad \text{(p. 273)}$$

$$(A \to B) \to ((\neg A \to B) \to B) \qquad \text{(p. 274)}$$

$$(A \to (B \to C)) \to ((A \bullet B) \to C) \qquad \text{(p. 271)}$$

$$((A \bullet B) \to C) \to (A \to (B \to C)) \qquad \text{(p. 271)}$$

$$((A \bullet B) \to C) \to ((A \bullet \neg C) \to \neg B) \qquad \text{(p. 272)}$$

$$((A \to B) \bullet (B \to C)) \to (A \to C) \qquad \text{(p. 272)}$$

$$((A \to B) \bullet A) \to B \qquad \text{(p. 273)}$$

$$(A \to (B \bullet \neg B)) \to \neg A \qquad \text{(p. 273)}$$

$$((A \to B) \bullet (C \to D)) \to ((A \bullet C) \to (B \bullet D)) \qquad \text{(p. 277)}$$

$$(A \to B) \to ((A \bullet C) \to (B \bullet C)) \qquad \text{(p. 274)}$$

$$((A \to B) \bullet (A \to C)) \to (A \to (B \bullet C)) \qquad \text{(p. 278)}$$

Then comes on p. 276 a series of schemata involving $+$ that indicate various possibilities of definitions; namely, Orlov finds that, in OR, we can prove:

$$(A + B) \to (\neg A \to B)$$

$$(A \to B) \to (\neg A + B)$$

$$(A \bullet B) \to \neg(\neg A + \neg B)$$

and the converse implications. Orlov also finds that + is associative and commutative; i.e., he proves in OR:

$$((A + B) + C) \to (A + (B + C)) \qquad \text{(p. 277)}$$

$$(A + (B + C)) \to ((A + B) + C) \qquad \text{(p. 277)}$$

$$(A + B) \to (B + A) \qquad \text{(p. 276)}$$

(the last schema corresponds to the structural rule of Permutation on the right). We also have in OR:

$$((A \to B) \bullet (C \to D)) \to ((A + C) \to (B + D)) \qquad \text{(p. 278)}$$

$$(A \to B) \to ((A + C) \to (B + C))$$

(in Orlov's formulation of the last schema, (13c) on p. 277, the clause $b \vee c$ is a misprint for $b \to c$). Orlov also notes on p. 265 (footnote 2), when he compares OR with the axioms of classical propositional logic given in Ackermann's dissertation (1925), that, in OR, we can prove:

$$(A \to C) \to ((B \to C) \to ((A + B) \to C))$$

whereas we don't always have:

$$A \to (B \to A)$$

$$(A \bullet B) \to A, \quad (A \bullet B) \to B$$

$$A \to (A + B), \quad B \to (A + B)$$

$$A \to (\neg A \to B).$$

(The first three principles correspond to the structural rule of Thinning on the left, and the last three to Thinning on the right. To complete our comments on structural rules, we should add something Orlov does not say explicitly; namely, in OR, we have $(A + A) \to A$, which corresponds to Contraction on the right, and we don't always have $A \to (A + A)$, which corresponds to Expansion on the right.) In the same footnote, concerning $A \to (B \to (A \bullet B))$, Orlov seems to have made an oversight (or perhaps it is another misprint); this schema is provable in OR.

Orlov also notes briefly on p. 277 that the distributivity of $\bullet$ over + fails. What he presumably has in mind is that, in OR, we cannot prove the schema:

$$((A \bullet B) + (A \bullet C)) \rightarrow (A \bullet (B + C)).$$

The converse implication, as well as $(A \bullet (B + C)) \rightarrow ((A \bullet B) + C)$, is provable in OR (the converse of this last schema, of course, fails even when $\rightarrow$ is classical implication, $\bullet$ is classical conjunction and $+$ is classical disjunction).

On p. 275, Orlov introduces the definition:

$$A \rightleftarrows B \ =_{\text{df}} \ (A \rightarrow B) \bullet (B \rightarrow A)$$

and proves in OR:

$$(A \rightleftarrows B) \rightarrow (A \rightarrow B)$$

$$(A \rightleftarrows B) \rightarrow (B \rightarrow A)$$

(cf. Anderson and Belnap, 1975, §29.7, p. 434).

This concludes our list of theorems of OR proved by Orlov. What is missing from this list are only variations, which seem unessential, and some obvious generalizations of associativity and replacement principles.

3. INTUITIONISTIC AND MODAL LOGIC

(Orlov's calculus of compatibility: §§6–7, pp. 279–286)

In the remainder of his paper, Orlov deals with intuitionism. He introduces in his propositional language a unary propositional operator, i.e. unary connective, $\Box$ (which he writes Φ) such that $\Box A$ should mean intuitively "it is provable that A". Orlov knows that, from an intuitionistic point of view, only propositions that amount to propositions of the form $\Box A$ can be meaningfully asserted in mathematics. However, his is a broader view, where a proposition A may be considered as being true or false independently of its provability, and where, after introducing the operator $\Box$, we may distinguish A from $\Box A$.

Orlov also defines in terms of $\Box$ and $\neg$ the unary propositional operator X by:

$$XA \ =_{\text{df}} \ \Box\neg A \qquad \text{(p. 281)}$$

With XA, we should capture the intuitionistic "it is absurd that A", i.e. intuitionistic negation. Orlov needs X to represent Brouwer's pronouncements about absurdity.

In a competent discussion of the right understanding of the intuitionistic "it is provable that A", Orlov says that this refers to provability in principle, which of course covers also cases when A has actually been proved or is obvious. In the realm of the finite, "it is provable that A or it is absurd that A" holds, even though this may seem to be in conflict with practical limitations. However, according to intuitionistic lights, this disjunction does not hold in the transfinite realm. Orlov states that intuitionists have *de facto* introduced into mathematics the propositional operation corresponding to $\Box$, and propositional operations defined in terms of it, though they don't represent these operations symbolically.

Then Orlov switches to more formal matters. In the propositional language enlarged with $\Box$, he formulates the following axiom-schemata (p. 281):

(Ax. 8) $\Box A \to A$

(Ax. 9) $\Box A \to \Box\Box A$

(Ax. 10) $\Box(A \to B) \to (\Box A \to \Box B)$

and the rule of *necessitation* (see the first paragraph on p. 282):

$$\frac{A}{\Box A}.$$

Let us first note that these are exactly the modal postulates of S4, in the standard form they have acquired after Gödel (1933). Orlov does not only anticipate Gödel; he also anticipates Becker (1930), to whom the first formulation of S4 posulates is usually credited (see Lewis and Langford, 1932, pp. 497, 501, and Hughes and Cresswell, 1968, p. 236, footnote 215; Becker, 1930, pp. 526–531 also foreshadows Gödel's modal translation of 1933). The modal postulates of S4 are, of course, analogous to Kuratowski's axioms for closure algebras from (1922) and, with a more latitudinarian attitude, one could credit Kuratowski with the axiomatization of S4. However, though the connexion between modal logic and topology did not wait very long to be discovered, it doesn't seem that logicians took their posulates for S4 directly from topology. At least, there is no reason to think that Orlov did.

The modal postulates of S4 should in Orlov's opinion be added to his system OR. When the axiomatization of OR is extended with these modal posulates, we call the resulting system ORS4. (Note that the axiomatization of the system $R^{\Box}$ in Anderson and Belnap, 1975, §27.1.3, pp. 343–344, involves also the axiom-schema:

$$(\Box A \wedge \Box B) \rightarrow \Box(A \wedge B)$$

where $\wedge$ is extensional conjunction.) Orlov then proceeds to prove a number of theorems in ORS4 and to comment on them in the light of Brouwer (1925a). For example, he finds that in ORS4 we can prove (p. 283):

$$\Box A \rightarrow \Box\neg\Box\neg\Box A$$

which shows that from A follows the absurdity of the absurdity of A. Similarly, we can prove in ORS4 (p. 285):

$$\Box\neg\Box\neg\Box\neg\Box A \rightarrow \Box\neg\Box A$$

and the converse implication, which justifies Brouwer's theorem that "the absurdity of the absurdity is equivalent to the absurdity". Orlov also notes (p. 285) that:

$$\Box\neg\Box\neg\Box\neg A \rightarrow \Box\neg A$$

is not always provable in ORS4 (indeed, it fails in S4). Orlov finds that the right way to understand intuitionistic notions in ORS4 is to consider connectives whose arguments are of the form $\Box A$ and not simply A.

In the last paragraph of his paper, Orlov comes to a rather odd conclusion. He thinks that the modal postulates of S4 can be added to OR, but cannot be added to classical logic. Orlov's opinion is that his modal theorems become senseless if we interpret $\rightarrow$ as material implication, and he somehow infers from the classical principle "all true propositions are equivalent" that we cannot add the modal postulates of S4 to classical logic without also adding $A \rightarrow \Box A$, which would, of course, make our modal system trivial and useless.

Excluded middle, whose traces can be found in:

$$(A \to B) \to ((\neg A \to B) \to B)$$

and other theorems of OR, did not make problems for understanding intuitionism in ORS4, but is deemed fatal if we want to reach intuitionism via the addition of $\Box$ to classical logic. It is as if Orlov thought that, before our propositions have come within the scope of $\Box$, it could make sense to have a truth-functional logic, but, once we have prefixed the operators $\Box$, we must look for a logic that is not truth-functional. This seems to be the least clear, and, in the light of latter developments, least fortunate, point of a paper otherwise graced by acumen and foresight.

The idea that intuitionistic logic could be axiomatized and then embedded by a modal translation in ORS4 is not made explicit by Orlov, but it is in the air when one reads him (or so it seems with hindsight). Of course, as he has put matters, the intuitionistic logic so axiomatized would not be Heyting's logic, but an intuitionistic variant of relevant logic (like the system considered in my paper, 1992). What would separate it from Heyting's logic is $A \to (B \to A)$ and $\neg A \to (A \to B)$ (which correspond to Gentzen's structural rule of Thinning). This intuitionistic relevant logic is interesting in its own right, and there is not necessity to consider relevant logic always with a classical negation that satisfies Orlov's (Ax. 2). However, such a classical negation, which was nearly exclusively considered in the Anderson and Belnap school (and was to reappear in the relevant logic without contraction of Girard, 1987), may already be found in what is presumably the first axiomatization of relevant logic.

4. THE AXIOMATIZATION OF INTUITIONISTIC LOGIC

Orlov's paper appeared at the time when the axiomatization of intuitionistic logic was in gestation, and one can find in his paper the spirit of that time. What Orlov did for relevant logic in (1928) is reminiscent of what Kolmogorov did for intuitionistic logic in (1925), in the same journal, and in the same idiom. In this paper, Kolmogorov gave the first axiomatization of the implication-negation fragment of so-called *minimal* logic (often named after Johansson), which is Heyting's logic without $\neg A \to (A \to B)$. (This logic is essentially negationless Heyting's logic, since minimal negation is purely implicational. When $\neg A$ is defined as $A \to \bot$, as in Gentzen, 1935, II, § 5.2, nothing in particular is assumed about $\bot$, so that all the properties of negation come from implication. In Heyting's logic, we assume $\bot \to B$. In relevant logic, we don't assume that, but, if we follow Orlov in accepting his (Ax. 2), we assume $((A \to \bot) \to \bot) \to A$.) Though Kolmogorov's paper has a footnote on disjunction and deals also with quantifiers, and though Orlov's paper deals with intensional conjunction and intensional disjunction, both papers are essentially about implication and negation. But isn't implication the heart of intuitionistic and relevant logic, and presumably of logic in general? And isn't negation, due to its at least partly implicational nature, just next to it? However, Orlov's axioms don't axiomatize separatively relevant implication, whereas the implicational axioms in Kolmogorov's paper, which he found in Hilbert (1923), give with *modus ponens* a complete axiomatization of the implicational fragment of Heyting's logic, the heart of Heyting's logic.

Kolmogorov's rejection of $\neg A \to (A \to B)$ is not of the same kind as Orlov's. Orlov was motivated by relevance and Kolmogorov not. However, it is interesting that, in his attempt to capture intuitionistic logic, which at that time was still pretty "intuitive", Kolmogorov accepted $A \to (B \to A)$ and did not accept $\neg A \to (A \to B)$. He says that both of these principles made their appearance only with the rise of symbolic logic, but, with an intuitionistically correct notion of implication, which Kolmogorov says explicitly is not relevant, the former principle should be accepted and the latter not (see 1925, II, §§ 2,4).

Kolmogorov's (1925) resembles Orlov's paper in one more respect. In his paper, Kolmogorov discovered the embedding via the double-negation translation of classical logic into intuitionistic logic, whereas Orlov foreshadowed the embedding via a modal translation of a logic with an intuitionistic negation into an S4-type extension of a logic with a classical negation. If intuitionistic double negation is conceived as a modal operator, these two translations are analogous (cf. Došen, 1986 and 1990).

Following a paper by Barzin and Errera, a polemic started in 1927 in the *Bulletin de la Classe des Sciences de l'Académie Royale de*

Belgique on whether Brouwer's ideas about logic make sense (Barzin's and Errera's paper, a paper by Lévy and a paper by Avsitidysky are in vol. 13, 1927, whereas Khinchin's, Glivenko's and Heyting's notes, mentioned below, are in succeeding volumes; papers on intuitionistic logical conceptions in the *Revue de Métaphysique et de Morale*, which preceded this polemic, are cited in van Stigt, 1990, § 5.12.3, pp. 288–294). Following another note by Barzin and Errera in *l'Enseignement Mathématique* (vol. 30, 1931), Heyting engaged in a polemical correspondence with these authors in this other journal (vol. 31, 1932). The reading of the polemic in *l'Enseignement Mathématique* is philosophically quite rewarding, but the first part of the polemic, in the *Bulletin*, is more interesting to us here.

It is first a testimony on other attempts, contemporaneous with Orlov's, to introduce operators related to "it is provable that" in order to understand Brouwer. Heyting's (1930a) is a comment on such attempts, which concludes with a diffident expectation that somebody will develop the logic of "it is provable that". Orlov's and Gödel's S4 postulates may be taken as a response to this expectation.

The note by Khinchin (1928) showed that Barzin's and Errera's critique of Brouwer rests on the intuitionistically, as well as relevantly, unpalatable principle $(A \to B) \vee (B \to A)$, which follows from $(C \to (A \vee B)) \to ((C \to A) \vee (C \to B))$.

The deservedly best remembered pieces of this polemic are Glivenko's (1928 and 1929). It is noteworthy that Glivenko hesitated in accepting $A \to (B \to A)$ and $\neg A \to (A \to B)$ before being convinced by Heyting that they should be assumed in intuitionistic logic (a discussion of Kolmogorov's and Glivenko's role in the axiomatization of Heyting's logic, including some of their letters to Heyting, may be found in Troelstra, 1990; there are no references to either Kolmogorov or Glivenko in Orlov's paper). Glivenko's axiomatization of the intuitionistic propositional calculus of (1929), which he reached by knowing Kolmogorov (1925) and in correspondence with Heyting, but also through his own efforts (as Heyting acknowledged in 1930a, pp. 957, 960), is more elegant and perspicuous than Heyting's axiomatization of (1930), which was produced in 1928 (Heyting said in 1978, p. 15, that his 1930 was imperfect, but it is not clear whether this refers to the form of his axioms, which he reproduced unchanged in 1971, pp. 105–106). Glivenko's axioms are truly in the spirit of Hilbert's school, and it is probably no accident that, in Gentzen's thesis (1935, V, §2), Glivenko's (1929) is given more prominence than Heyting's (1930). Gentzen's thesis, which came just a few years later, is unqualifiedly the most important work on the proof theory of logical calculuses, where the articulation of Heyting's logic became perfectly clear, and where the main principle governing relevant logic could be glimpsed: just imagine there is no Thinning.

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NOTE

[1] I am grateful to Aleksandr Chagrov for informing me, after this paper was accepted for publication, that Orlov's discovery of relevant logic is not so completely unknown in Russia as was my impression. In 1978, V. M. Popov presented a paper at a conference in Russia where he claimed that Orlov anticipated Church. The abstract of this paper of Popov, which I was unable to see, is mentioned in the preface of a book of readings on the semantics of modal and intensional logics edited by V. A. Smirnov in 1981. Chagrov informed me that Popov has written about Orlov in another paper, which appeared in 1986, but I was unable to see this paper too. I have found some other references to Orlov's paper in the Russian logical literature, but these mention neither Orlov's discovery of relevant logic nor his anticipation of S4.

The bibliography *Forty Years of Mathematics in the USSR: 1917–1957* (in Russian; A. G. Kurosh *et al.* eds, Gosudarstvennoe izdatel'stvo fiziko-matematicheskoĭ literatury, Moscow, 1959, vol. 2) lists (1928) as the only paper by I. E. Orlov.

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