Proof-Net Categories

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Preface

This study is the continuation of a project in categorial proof theory, which occupied us in the last few years and yielded the book [22]. This is a kind of appendix to that book, whose results are applied here. An acquaintance with that previous book is not absolutely necessary, provided the reader is prepared to trust the results on which we rely. It is, however, very desirable. Practically no other literature is presupposed, except for the sake of motivation. We rely only on very standard notions of logic and category theory, which, if by any chance they are not already known to the reader, may be found in [22].

The aim and context of our work are set forth in the introductory chapter. Our results should be of general interest to graduate students and researchers in general proof theory. They demonstrate how generality of proofs provides a criterion of identity for proofs. We believe these results bring something also to categorists interested in coherence questions, to whom they may illustrate the usefulness of syntactical methods in category theory. They should be of particular interest to investigators of linear logic, symmetric monoidal closed categories and star-autonomous categories. These or related matters seem to be interesting too in the borderline areas of theoretical computer science. This is, however, not a text belonging to that science. Our aims, our terminology and our style come from the related but, nevertheless, different and older field of logic.

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Chapter 1

Introduction

In this introductory chapter we state the aim of this work, and present the context, i.e. previous work related to the subject matter we treat. We also give a summary of the whole text.

§1.1. Aim and context

The aim of this work is to give a systematic account of the connection that exists between star-autonomous categories and the Kelly-Mac Lane graphs implicit in proof nets for the multiplicative fragment without propositional constants of linear logic. Star-autonomous categories are symmetric monoidal closed categories that have an object $\bot$ such that the canonical natural transformation from the identity functor to the functor $(\cdot \to \bot) \to \bot$ is a natural isomorphism (see §§3.1-2 and §3.8; here $\cdot \to \cdot$ is the internal hom-bifunctor).

For some results of this work it will perhaps be claimed that they are known—that they have already been established. We do not believe this claim is justified, and we have decided to present matters anew because we are not satisfied with the treatment they have received up to now. But even if it were true that these results are known, we think that they deserve a systematic and detailed presentation, following the canons of rigour that used to be the rule in logic. We feel there is a need for such a presentation, and we want to supply it.

There are in mathematics theorems that are more difficult to conjecture
CHAPTER 1. INTRODUCTION

than to prove. Such is, for example, the Theorem of Pythagoras, or the theorem that $\sqrt{2}$ is not rational. There are, on the other hand, theorems that are more difficult to prove than to conjecture (and there are no doubt theorems were the conjecture and the proof are of equal difficulty).

The results we are going to present are of a kind called in logic completeness theorems. Such theorems are often not difficult to conjecture, but their proofs can be quite demanding. The prime example of such a result is the completeness theorem for first-order predicate logic. The axiomatization of this logic existed long before a precise completeness proof was given by Gödel, and throughout this period it was assumed the axiomatization is complete, with a more or less precise notion of completeness being envisaged. Nearer to our topic, we have the coherence theorem for symmetric monoidal closed categories proved by Kelly and Mac Lane in [32] (see the end of §3.1), where it also seems it was easier to conjecture the theorem than to prove it.

From the inception of proof nets in the late 1980s (see [26] and [13]), it could have been realized that they are connected with the graphs one finds in Kelly’s and Mac Lane’s coherence theorem. The earliest explicit reference for that we know about is [4] (see also [5]). It was also soon suggested that the multiplicative fragment of classical linear logic, which has an involutive negation that satisfies De Morgan laws, is closely related to Barr’s star-autonomous categories, which stem from [1] (see [33], [42] and [2]). A number of results have been proposed since as completeness results connecting proof nets and particular categories (see the beginning of [28] for a recent survey). It seems to be an accepted opinion nowadays that, in the words of [28], “...the identifications [of proofs imposed by proof nets] correspond to coherences of free star-autonomous categories”. The purpose of our work is to examine this opinion, and find out how much truth there is in it.

The problem with this opinion is that on the side of proof nets we do not have in the standard treatment the multiplicative propositional constants, while on the side of star-autonomous categories we have the corresponding unit objects. In the presence of these units, an unrestricted coherence theorem with respect to graphs of the Kelly-Mac Lane kind is not forthcoming. Kelly and Mac Lane had for their coherence theorem for symmetric
monoidal closed categories of [32] a proviso concerning the unit object of
the monoidal structure (see the end of §3.1, and see [43] for further work
concerning this proviso), but we are not aware that a similar coherence
involving a proviso for the units of star-autonomous categories has been
proved up to now. (We provide such a result in Chapter 4 below.)

Two courses are open in this situation. The first course, which we will
follow, is to reject the units on the side of star-autonomous categories,
define precisely the resulting notion of category, and prove a standard, un-
restricted, coherence result for it, akin to Kelly’s and Mac Lane’s coherence.
(We do that in Chapter 2.) It is desirable to show here that the proposed
notion of star-autonomous category without units catches exactly the cor-
responding fragment of star-autonomous categories, in a sense to be made
precise in terms of category theory. (We do that in Chapters 3 and 4.) After
that only, one can establish a match between the equations assumed for the
categories and those imposed by the proof nets without the multiplicative
propositional constants, and so vindicate the established opinion.

Relying on the unrestricted coherence for star-autonomous categories
without units, one can obtain a restricted coherence theorem for standard
star-autonomous categories. This coherence theorem has a proviso concern-
ing the units: they are allowed to occur only in places such that the objects
in which they are involved are isomorphic either to objects not involving
the units or to one of the units. (This is the coherence result of Chapter 4
mentioned above, which will be phrased precisely in that chapter.) We will
show that this proviso is of the same kind as the proviso that Kelly and
Mac Lane had.

The second course is to add the multiplicative propositional constants
without restriction on the side of proof nets, and claim that a completeness
result connecting the modified proof nets and star-autonomous categories
is the desired coherence result. This second course is more favoured in the
existing literature, cited below and in Chapter 7. We should immediately
notice that with this course coherence cannot be understood in the sense of
Kelly and Mac Lane. Also, no precise notion of star-autonomous category
without units arises.

As far as we know, the only coherence result in the style of Kelly and
Mac Lane proved up to now for star-autonomous categories is still Kelly’s
and Mac Lane’s own result of [32], which is, as we said above, about symmetric monoidal closed categories with a proviso concerning the unit object of the monoidal structure. Richard Blute in [5] purports to prove a general coherence result, which should yield coherence for star-autonomous categories without units with respect to Kelly-Mac Lane graphs. We find, however, this proof excessively wanting. The notion of star-autonomous category without units is not precisely defined. We do not know what is the “usual theory without units” of star-autonomous categories (mentioned in [5], pp. 9, 15), and one of our purposes in this work is to supply a language of arrow terms for that theory and the appropriate equations between these arrow terms. We could not find either of these in [5], or anywhere else. Recently, attempts have been made in [35] and [27] to define a notion of star-autonomous category without units, but the approach of these papers, different from ours, is not equational (at least not in our sense).

It is not, however, the case that once the notion of star-autonomous category without units is made precise, one obtains from the sketch in [5] (p. 23, right-to-left direction of Theorem 10.2) a recipe for proving coherence for this notion. A substantial part of the proof is covered by the sentence: “This amounts to a straightforward case analysis.” This sentence occurs in a context where no specific equations are stated, and it is claimed that these equations cover a cut-elimination procedure. This is usually the most arduous part of a proof of coherence (see, for example, [32]).

Robert Seely and Robin Cockett in [12] (p. 104) consider coherence for star-autonomous categories without units to be “fairly straightforward, even trivial”, and they refer to [5] and [6] for an exposition. We have already discussed [5], while in [6] we find neither a definition of star-autonomous category without units, nor a coherence result for them in the sense of Kelly and Mac Lane. Instead, the latter paper is about coherence for star-autonomous categories with units (and it is presumed that these are the weakly distributive categories of [11] with negation added) with respect to an extension of proof nets with multiplicative propositional constants. The subject of our work is to a great extent this matter previously dismissed as straightforward, or even trivial.

We will give an equational formulation of the notion of star-autonomous category without units, which we call proof-net category, and we will prove
coherence for this notion with respect to Kelly-Mac Lane graphs, which means that there is a faithful functor from the proof-net category freely generated by a set of objects into the category whose arrows are these graphs. Another possibility would be to define the notion of proof-net category by coherence, i.e. by the existence of the faithful functor into the category whose arrows are graphs. This way, however, we would have no information about the axioms, which are the combinatorial building blocks of our notion.

The notion of monoidal category was introduced in such a nonaxiomatic way, via coherence, by Bénabou in [3], and in the axiomatic way, such as we favour, by Mac Lane in [37]. For Bénabou, coherence is built into the definition, and for Mac Lane it is a theorem. One could analogously define the theorems of classical propositional logic as being the tautologies (this is done, for example, in [9], Sections 1.2-3), in which case completeness would not be a theorem, but would be built into the definition.

To take another example, we could easily define nonaxiomatically a notion of Boolean category with respect to graphs of the Kelly-Mac Lane kind. (In this notion, conjunction would not be a product, because the diagonal arrows and the projections would not make natural transformations, and, analogously, disjunction would not be a coproduct; see [22], Section 14.3.) The resulting notion would not be trivial—the resulting freely generated categories would not be preorders—but its nonaxiomatic definition would be trivial. We are looking for nontrivial axiomatic definitions. Such definitions give information about the combinatorial building blocks of our notions, as Reidemeister moves give information about the combinatorial building blocks of knot equivalence (see [8], Chapter 1). Our axiomatic equational definition of proof-net category is of this nontrivial, combinatorially informative, kind. Coherence of proof-net categories is for us a theorem, whose proof requires considerable effort.

§1.2. Summary

This study is a continuation of [22], whose ideas and style we have followed in general. Many notions we need are exposed more systematically in that book, which the reader may consult for more detailed explanations and
definitions, and also for motivation from the perspective of general proof theory or categorial proof theory. At some key points, we rely on results proved before. In Chapter 2 we rely on matters proved in [20], [21] and [22], and in particular on a coherence result from [22] (Symmetric Net Coherence of Section 7.6). In Chapter 3 we rely on the coherence result of Kelly and Mac Lane for symmetric monoidal closed categories of [32]. We rely also on some well-known elementary notions of category theory, which may all be found in [38] or [22], and for the sake of motivation we rely on some acquaintance with linear logic and the proof nets of [26]. Except for that, we have strived to make our exposition self-contained to a great extent.

First, we give in Chapter 2 a precise definition of a notion that may be considered to correspond to star-autonomous categories without units. This notion, which we call proof-net category, is obtained by extending with an operation that corresponds to negation the notion of symmetric net category of [22] (Section 7.6); the notion of symmetric net category corresponds to the notion of linear (alias weakly) distributive category of [11] without units. For proof-net categories we prove in Chapter 2 a coherence result with respect to Kelly-Mac Lane graphs.

In Chapter 3, we prove precisely in categorial terms the equivalence of the notion of star-autonomous category with a notion amounting to the notion of linearly distributive category with negation of [11]. The latter notion is obtained by extending with the units our notion of proof net category. This categorial equivalence result was foreshadowed in [11] (Section 4, Theorem 4.5), but there its “straightforward” proof was said to depend on “pretty horrid” diagrams, and practically the whole of it was left “to the faith of the reader”. The proof we supply is indeed pretty lengthy, though we have shortened it considerably by relying on Kelly’s and Mac Lane’s coherence for symmetric monoidal closed categories and on our coherence from Chapter 2 for proof-net categories. We can only imagine how “horrid” it would be without these tools, which are not mentioned in [11].

In Chapter 4, we prove that with the notion of proof-net category we have not only caught the notion of star-autonomous category without units, but with its help we can also obtain a coherence result for star-autonomous categories with respect to Kelly-Mac Lane graphs—a result of the same kind as Kelly’s and Mac Lane’s coherence result for symmetric monoidal closed
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categories. This result involves a proviso concerning the units, but does not exclude them completely (as we announced in the preceding section). This coherence of star-autonomous categories is a powerful tool for verifying whether a diagram of arrows commutes in star-autonomous categories.

After all that, the established opinion on the connection between proof nets and star-autonomous categories, which we mentioned in the preceding section, may be rephrased as the statement that the identifications of proofs imposed by proof nets correspond well to the equations of proof-net categories, and since proof-net categories catch the unit-free portion of star-autonomous categories, the opinion seems vindicated. We find that before it was accepted just on faith.

It is not true, however, that the identifications of proofs imposed by proof nets stem only from the cut-elimination procedure. There are also equations that serve to equate different cut-free proofs. These are equations similar to the so-called *permutative* reductions of natural deduction, which permute the order of rules in cut-free proofs, and also equations that atomize the identity axiomatic sequents. Such equations are indeed incorporated in the usual notion of proof net, and are there invisible, but in modifications of this notion they may reappear (cf. [24]). It is, in general, tricky to justify equations just by reference to cut elimination, because cut elimination tends to be sensitive to a particular syntax, and also to a particular procedure (cf. [14], Section 0.3.1 and passim). A justification independent of the vagaries of syntax is obtained by coherence theorems in the style of Mac Lane.

In Chapter 5, we consider how the assumptions concerning the involutive unary operation corresponding to negation, which we have in proof-net categories and star-autonomous categories, are tied to a particular kind of adjunction where an endofunctor is adjoint to itself.

In Chapter 6, we consider proof-net categories that have arrows corresponding to the mix principle of linear logic, and we prove coherence for the resulting notion by adapting the coherence proof for proof-net categories of Chapter 2.

In Chapter 7, the final chapter, we discuss the relationship between the Kelly-Mac Lane graphs and proof nets, which justifies the name we have given to proof-net categories. In general proof theory, one of the main
problems is the investigation of identity of proofs (see [15] or [22], Sections 1.3-4), and it is desirable to find efficient means to check this identity. We approach coherence questions in that spirit, and we expect coherence theorems to yield a decision procedure (preferably easy) to answer the question whether a diagram of arrows commutes. From that standpoint, Kelly-Mac Lane graphs are the relevant core of proof nets, which we can use to answer efficiently the question whether two proofs are equal in the multiplicative fragment without propositional constants of linear logic, and also, according to the coherence theorem of Chapter 4, in a larger fragment of linear logic, where the multiplicative propositional constants occur at particular places. At the very end, we discuss further papers related to our work, and express some opinions on proof nets in the context of general proof theory.
Chapter 2

Coherence of Proof-Net Categories

In this chapter we define our notion of proof-net category. This notion is based on the notion of symmetric net category of [22] (Section 7.6); these are categories with two multiplications, $\wedge$ and $\vee$, associative and commutative up to isomorphism, which have moreover arrows of the *dissociativity* type $A \wedge (B \vee C) \to (A \wedge B) \vee C$ (called *linear or weak* distribution by the authors of [11]). The symmetric net category freely generated by a set of objects is called $\text{DS}$. To symmetric net categories we add an operation on objects corresponding to negation, which is involutive up to isomorphism. With these operations come appropriate arrows. A number of equations between arrows, of the kind called *coherence conditions* in category theory, are satisfied in proof-net categories.

We introduce a category $\text{Br}$ whose arrows are called *Brauerian split equivalences* of finite ordinals. These equivalence relations, which stem from results in representation theory from the 1930s, amount to the graphs used by Kelly and Mac Lane for their coherence theorem of symmetric monoidal categories. Brauerian split equivalences express generality of proofs in linear logic (see [20] and [21]).

The coherence theorem for proof-net categories says that there is a faithful functor from the proof-net category $\text{PN}^\neg$ freely generated by a set of objects into $\text{Br}$. We call theorems of this kind *coherence theorems*. The coherence theorem for $\text{PN}^\neg$ yields an elementary decision procedure for
verifying whether a diagram of arrows commutes in $\mathbf{PN}^\rightarrow$, and hence also in every proof-net category. This is a very useful tool, which will facilitate calculations later on.

The coherence theorem for $\mathbf{PN}^\rightarrow$ is proved by finding a category $\mathbf{PN}$, equivalent to $\mathbf{PN}^\rightarrow$, in which negation can be applied only to the generating objects, and coherence is first established for $\mathbf{PN}$ by relying on coherence for symmetric net categories, previously established in [22] (Chapter 7), and on an additional normalization procedure involving negation.

§2.1. The category $\mathbf{DS}$

The objects of the category $\mathbf{DS}$ are the formulae of the propositional language $\mathcal{L}_{\wedge,\vee}$, generated from a set $\mathcal{P}$ of propositional letters, which we call simply letters, with the binary connectives $\wedge$ and $\vee$. We use $p, q, r, \ldots$, sometimes with indices, for letters, and $A, B, C, \ldots$, sometimes with indices, for formulae. As usual, we omit the outermost parentheses of formulae and other expressions later on.

To define the arrows of $\mathbf{DS}$, we define first inductively a set of expressions called the arrow terms of $\mathbf{DS}$. Every arrow term of $\mathbf{DS}$ will have a type, which is an ordered pair of formulae of $\mathcal{L}_{\wedge,\vee}$. We write $f : A \vdash B$ when the arrow term $f$ is of type $(A, B)$. (We use the turnstile $\vdash$ instead of the more usual $\rightarrow$, which we reserve for a connective and a bifunctor.) We use $f, g, h, \ldots$, sometimes with indices, for arrow terms.

For all formulae $A, B$ and $C$ of $\mathcal{L}_{\wedge,\vee}$ the following primitive arrow terms:

$1_A : A \vdash A$,

$\hat{b}_{A,B,C} : A \wedge (B \wedge C) \vdash (A \wedge B) \wedge C$, $\hat{v}_{A,B,C} : A \vee (B \vee C) \vdash (A \vee B) \vee C$,

$\hat{b}_{A,B,C} : (A \wedge B) \wedge C \vdash A \wedge (B \wedge C)$, $\hat{v}_{A,B,C} : (A \vee B) \vee C \vdash A \vee (B \vee C)$,

$\hat{e}_{A,B} : A \wedge B \vdash B \wedge A$, $\hat{c}_{A,B} : B \vee A \vdash A \vee B$,

$d_{A,B,C} : A \wedge (B \vee C) \vdash (A \wedge B) \vee C$

are arrow terms of $\mathbf{DS}$. If $g : A \vdash B$ and $f : B \vdash C$ are arrow terms of $\mathbf{DS}$, then $f \circ g : A \vdash C$ is an arrow term of $\mathbf{DS}$; and if $f : A \vdash D$ and $g : B \vdash E$ are arrow terms of $\mathbf{DS}$, then $f \xi g : A \xi B \vdash D \xi E$, for $\xi \in \{\wedge, \vee\}$, is an arrow term of $\mathbf{DS}$. This concludes the definition of the arrow terms of $\mathbf{DS}$. 
§2.1. The category $\text{DS}$

Next we define inductively the set of equations of $\text{DS}$, which are expressions of the form $f = g$, where $f$ and $g$ are arrow terms of $\text{DS}$ of the same type. We stipulate first that all instances of $f = f$ and of the following equations are equations of $\text{DS}$:

\[
\begin{align*}
(\text{cat } 1) & \quad f \circ 1_A = 1_B \circ f = f : A \vdash B, \\
(\text{cat } 2) & \quad h \circ (g \circ f) = (h \circ g) \circ f,
\end{align*}
\]

for $\xi \in \{\wedge, \vee\}$,

\[
\begin{align*}
(\xi 1) & \quad 1_A \xi 1_B = 1_{A \land B}, \\
(\xi 2) & \quad (g_1 \circ f_1) \xi (g_2 \circ f_2) = (g_1 \xi g_2) \circ (f_1 \xi f_2),
\end{align*}
\]

for $f : A \vdash D$, $g : B \vdash E$ and $h : C \vdash F$,

\[
\begin{align*}
\langle \beta \rangle & \quad (f \xi g) h = (f \xi h) \circ (g \xi h), \\
\langle \gamma \rangle & \quad \gamma A \beta B = \text{nat}(f \land g), \\
\langle \delta \rangle & \quad \delta A \beta B = \text{nat}(f \lor g), \\
\langle d \rangle & \quad (f \land g) h = d_{A,B,C} = d_{D,E,F} \circ (f \land g).
\end{align*}
\]

\[
\begin{align*}
\langle \eta \rangle & \quad \eta A \beta B = \text{nat}(f \circ g), \\
\langle \epsilon \rangle & \quad \epsilon A \beta B = \text{nat}(f \circ g), \\
\langle \theta \rangle & \quad (f \circ g) h = d_{A,B,C} = d_{D,E,F} \circ (f \circ g).
\end{align*}
\]

For $d^R_{A,B,C}$, $d^R_{A,B,C,D}$, and $d^R_{A,B,C,D,E}$:

\[
\begin{align*}
\langle \theta \rangle & \quad \theta A \beta B = \text{nat}(f \circ g), \\
\langle \rho \rangle & \quad \rho A \beta B = \text{nat}(f \circ g), \\
\langle \sigma \rangle & \quad (f \circ g) h = d_{A,B,C} = d_{D,E,F} \circ (f \circ g).
\end{align*}
\]

for $f : A \vdash D$, $g : B \vdash E$ and $h : C \vdash F$,
The set of equations of $\mathbf{DS}$ is closed under symmetry and transitivity of equality and under the rules

\[(\text{cong } \xi) \quad \frac{f = f_1 \quad g = g_1}{f \xi g = f_1 \xi g_1}\]

where $\xi \in \{\circ, \wedge, \vee\}$; if $\xi$ is $\circ$, then $f \circ g$ is defined (namely, $f$ and $g$ have appropriate, composable, types), and analogously for $f_1 \circ g_1$.

On the arrow terms of $\mathbf{DS}$ we impose the equations of $\mathbf{DS}$. This means that an arrow of $\mathbf{DS}$ is an equivalence class of arrow terms of $\mathbf{DS}$ defined with respect to the smallest equivalence relation such that the equations of $\mathbf{DS}$ are satisfied (see [22], Section 2.3, for details).

The equations $(\xi 1)$ and $(\xi 2)$ are called bifunctorial equations. They say that $\wedge$ and $\vee$ are biendofunctors (i.e. 2-endofunctors in the terminology of [22], Section 2.4).

It is easy to show that for $\mathbf{DS}$ we have the equations

\[
\begin{align*}
(b^- \text{ nat}) & \quad (f \xi (g \xi h)) = b^-_{A,B,C} = b^-_{D,E,F} \circ ((f \xi g) \xi h), \\
(d^R \text{ nat}) & \quad (h \vee (g \wedge f)) = d^R_{C,B,A} = d^R_{F,E,D} \circ ((h \vee g) \wedge f).
\end{align*}
\]

We call these equations and other equations with “nat” in their names, like those in the list above, naturality equations. Such equations say that $b^-$, $b^-$, $\hat{\circ}$, etc. are natural transformations.

The equations $(d\wedge)$, $(d\vee)$, $(d\hat{\circ})$ and $(d\hat{\circ})$ stem from [11] (Section 2.1; see [10], Section 2.1, for an announcement). The equation $(d\hat{\circ})$ of [22] (Section 7.2) amounts with $(\hat{\circ})$ to the present one.

§2.2. The category $\mathbf{PN}^-$

The category $\mathbf{PN}^-$ is defined as $\mathbf{DS}$ save that we make the following changes and additions. Instead of $\mathcal{L}_{\wedge, \vee}$, we have the propositional language $\mathcal{L}_{\neg, \wedge, \vee}$, which has in addition to what we have for $\mathcal{L}_{\wedge, \vee}$ the unary connective $\neg$.

To define the arrow terms of $\mathbf{PN}^-$, in the inductive definition we had for the arrow terms of $\mathbf{DS}$ we assume in addition that for all formulae $A$ and $B$ of $\mathcal{L}_{\neg, \wedge, \vee}$ the following primitive arrow terms:
\[\begin{align*}
\hat{\Delta}_{B,A}: & \ A \vdash A \land (\neg B \lor B), \\
\check{\Sigma}_{B,A}: & \ (B \land \neg B) \lor A \vdash A,
\end{align*}\]

are arrow terms of \(\mathbf{PN}^{-}\). We call the index \(B\) of \(\hat{\Delta}_{B,A}\) and \(\check{\Sigma}_{B,A}\) the crown index, and \(A\) the stem index. The right conjunct \(\neg B \lor B\) in the target of \(\hat{\Delta}_{B,A}\): \(A \vdash A \land (\neg B \lor B)\) is the crown of \(\hat{\Delta}_{B,A}\), and the left disjunct \(B \land \neg B\) in the source of \(\check{\Sigma}_{B,A}\): \((B \land \neg B) \lor A \vdash A\) is the crown of \(\check{\Sigma}_{B,A}\). We have analogous definitions of crown and stem indices, and crowns, for \(\check{\Sigma}, \hat{\Delta}', \check{\Sigma}'\), \(\hat{\Delta}'\) and \(\check{\Sigma}'\), which will be defined below. (The symbol \(\Delta\) should be associated with the Latin \textit{dexter}, because in \(\hat{\Delta}_{B,A}, \hat{\Delta}'_{B,A}, \hat{\Delta}_{B,A}\) and \(\hat{\Delta}'_{B,A}\) the crown is on the right-hand side of the stem; analogously, \(\Sigma\) should be associated with \textit{sinister}.)

To define the arrows of \(\mathbf{PN}^{-}\), we assume in the inductive definition we had for the equations of \(\mathbf{DS}\) the following additional equations, which we call the \(\mathbf{PN}\) \textit{equations} (and not \(\mathbf{PN}^{-}\) equations):

\[\begin{align*}
(\hat{\Delta} \text{ nat}) & \quad (f \land 1_{B \lor B}) \circ \hat{\Delta}_{B,A} = \hat{\Delta}_{B,D} \circ f, \\
(\check{\Sigma} \text{ nat}) & \quad f \circ \check{\Sigma}_{B,A} = \check{\Sigma}_{B,D} \circ (1_{B \land \neg B} \lor f), \\
(\check{b} \hat{\Delta}) & \quad \check{b}_{A,B,-C \lor C} \circ \hat{\Delta}_{C,A \land B} = 1_{A} \land \hat{\Delta}_{C,B}, \\
(\check{b} \check{\Sigma}) & \quad \check{\Sigma}_{C,B \lor A} \circ \check{b}_{A,-C \land B} = \check{\Sigma}_{C,B} \lor 1_{A},
\end{align*}\]

for \(\check{\Sigma}_{B,A} \overset{df}{=} \check{c}_{A,-B \lor B} \circ \check{\Delta}_{B,A}: A \vdash (B \lor B) \land A\),

\[\begin{align*}
(d \check{\Sigma}) & \quad d_{-A \lor A,B,C} \circ \check{\Sigma}_{A,B \lor C} = \check{\Sigma}_{A,B} \lor 1_{C},
\end{align*}\]

for \(\check{\Delta}_{B,A} \overset{df}{=} \check{\Sigma}_{B,A} \circ \check{c}_{B \land -B,A}: A \lor (B \land -B) \vdash A\),

\[\begin{align*}
(d \check{\Delta}) & \quad \check{\Delta}_{A,C \land B} \circ d_{C,B, A \land -A} = 1_{C} \land \check{\Delta}_{A,B}, \\
(\check{\Sigma} \check{\Delta}) & \quad \check{\Sigma}_{A,A} \circ d_{-A \land -A,A} \circ \check{\Delta}_{A,A} = 1_{A}.
\end{align*}\]

for \(\check{\Delta}'_{B,A} \overset{df}{=} (1_{A} \land \check{c}_{B,-B}) \circ \check{\Delta}_{B,A} : A \vdash A \land (B \lor -B)\) and

\[\begin{align*}
(\check{\Sigma} \check{\Delta}') & \quad \check{\Sigma}_{A,-A} \circ d_{-A,A, A \land -A} \circ \check{\Delta}'_{A,-A} = 1_{A}.
\end{align*}\]

It is easy to show that for \(\mathbf{PN}^{-}\) we have the equations...
\( (\Sigma \text{ nat}) \quad (1_{\neg B \lor B} \land f) \ast \hat{\Sigma}_{B,A} = \hat{\Sigma}_{B,D} \ast f, \)

\( (\Delta \text{ nat}) \quad f \ast \hat{\Delta}_{B,A} = \hat{\Delta}_{B,D} \ast (f \lor 1_{B \land \neg B}). \)

The naturality equations \((\Delta \text{ nat})\) and \((\Sigma \text{ nat})\) together with these say that
\( \Delta, \Sigma, \check{\Sigma} \) and \( \check{\Delta} \) are natural transformations in the stem index only, i.e. in
the second index.

We also have the following abbreviations:
\[
\hat{\Sigma}^\prime_{B,A} = df \land \Sigma \quad \hat{\Delta}^\prime_{B,A} = df \lor \Delta
\]

If \( \Xi \) stands for either \( \Delta \) or \( \Sigma \) and \( \xi \in \{\land, \lor\} \), then for every \((\hat{\xi} \text{ nat})\) equation
we have in \( \text{PN}^{-}\) the equation \((\hat{\xi}' \text{ nat})\), which differs from \((\hat{\xi} \text{ nat})\) by re-
placing \( \hat{\xi} \) by \( \hat{\xi}' \), and the index of \( 1 \) by the appropriate index. For example,
we have
\[
(\hat{\Delta}^\prime \text{ nat}) \quad (f \land 1_{B \lor \neg B}) \ast \hat{\Delta}^\prime_{B,A} = \hat{\Delta}^\prime_{B,D} \ast f.
\]

As alternative primitive arrow terms for defining \( \text{PN}^{-}\) we could take one
of \( \land \Xi \) or \( \land \Xi' \) and one of \( \lor \Xi \) or \( \lor \Xi' \).

We can also derive for \( \text{PN}^{-}\) the following equations:
\[
(\hat{b} \Delta \Sigma) \quad \hat{b}_{\neg B \lor B, C} \ast (\hat{\Delta}_{B,A} \land 1_C) = 1_A \land \hat{\Sigma}_{B,C},
\]
\[
(\hat{b} \Sigma) \quad \hat{b}_{C \lor B, A} \ast \hat{\Sigma}_{C,B \land A} = \hat{\Sigma}_{C,B} \land 1_A.
\]

For the first equation, with indices omitted, we have
\[
\hat{b}^{-} \cdot (\hat{\Delta} \land 1) = \hat{b}^{-} \cdot \hat{\Delta} \ast (1 \land \hat{\Delta}) = \hat{\Delta}, \quad \text{by } (\hat{\Delta} \text{ nat}) \text{ and } (\hat{\Delta} \text{ nat}),
\]
\[
= \hat{b}^{-} \cdot \hat{\Delta} \ast \hat{\Delta} \ast \hat{\Delta}, \quad \text{by } (\hat{\Delta} \text{ nat}),
\]
\[
= (1 \land \hat{\Delta}) \ast \hat{\Delta}, \quad \text{with } (\hat{\Delta} \text{ nat}) \text{ and } (\hat{\Delta} \text{ nat}),
\]
\[
= 1 \land \hat{\Sigma}, \quad \text{by } (\hat{b} \Delta),
\]
and for the second equation we have
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\[
\hat{b}^{-} \circ \hat{\Sigma} = \hat{b}^{-} \circ \hat{c} \circ \hat{b}^{-} \circ (1 \wedge \hat{\Delta}), \quad \text{with } (\hat{b} \hat{\Delta}),
\]
\[
= (\hat{c} \wedge 1) \circ \hat{b}^{-} \circ (1 \wedge \hat{c}) \circ (1 \wedge \hat{\Delta}), \quad \text{by } (\hat{b} \hat{c}),
\]
\[
= \hat{\Sigma} \wedge 1, \quad \text{with } (\hat{b} \hat{\Delta} \hat{\Sigma}).
\]

We derive analogously with the help of $(\hat{b} \hat{\Sigma})$ the equations

\[
(\hat{b} \hat{\Delta} \hat{\Sigma}) \quad (\hat{\Delta}_{\text{B,A}} \vee 1_C) \circ \hat{b}^{-}_{\text{A,B\neg B,C}} = 1_A \vee \hat{\Sigma}_{\text{B,C}},
\]
\[
(\hat{b} \hat{\Delta}) \quad \hat{\Delta}_{\text{C,AVB}} \circ \hat{b}^{-}_{\text{A,B,CA\neg C}} = 1_A \vee \hat{\Delta}_{\text{C,B}}.
\]

The arrows $\hat{\Delta}_{\text{B,A}}: A \vdash A \wedge (\neg B \vee B)$ and $\hat{\Sigma}_{\text{B,A}}: A \vdash (\neg B \vee B) \wedge A$ are analogous to the arrows of types $A \vdash A \wedge \top$ and $A \vdash \top \wedge A$ that one finds in monoidal categories. However, $\hat{\Delta}_{\text{B,A}}$ and $\hat{\Sigma}_{\text{B,A}}$ do not have inverses in $\mathbf{PN}^-$. The equations $(\hat{b} \hat{\Delta})$, $(\hat{b} \hat{\Delta} \hat{\Sigma})$, $(\hat{b} \hat{\Sigma})$ are analogous to equations that hold in monoidal categories (see [38], Section VII.1, [22], Section 4.6, and §3.1 below). An analogous remark can be made for $\hat{\Sigma}_{\text{B,A}}$ and $\hat{\Delta}_{\text{B,A}}$.

We can also derive for $\mathbf{PN}^-$ the following equations by using essentially $(d \hat{\Sigma})$ and $(d \hat{\Delta})$:

\[
(d^R \hat{\Delta}) \quad d^R_{\text{C,B,\neg A\neg A}} \circ \hat{\Delta}_{\text{A,C\vee B}} = 1_C \vee \hat{\Delta}_{\text{A,B}},
\]
\[
(d^R \hat{\Sigma}) \quad \hat{\Sigma}_{\text{A,B\wedge C}} \circ d^R_{\text{A\neg A\neg A,B,C}} = \hat{\Sigma}_{\text{A,B}} \wedge 1_C.
\]

These two equations could replace $(d \hat{\Sigma})$ and $(d \hat{\Delta})$ for defining $\mathbf{PN}^-$. The analogues of the equations $(d \hat{\Sigma})$, $(d \hat{\Delta})$, $(d^R \hat{\Sigma})$ and $(d^R \hat{\Delta})$ may be found in [11] (Section 2.1), where they are assumed for linearly (alias weakly) distributive categories with negation (cf. [22], Section 7.9).

It is easy to infer that in $\mathbf{PN}^-$ we have analogues of the equations $(\hat{b} \hat{\Delta})$, $(\hat{b} \hat{\Delta} \hat{\Sigma})$, $(\hat{b} \hat{\Sigma})$, $(\hat{b} \hat{\Delta} \hat{\Sigma})$, $(\hat{b} \hat{\Delta})$, $(d \hat{\Sigma})$, $(d \hat{\Delta})$, $(d^R \hat{\Sigma})$ and $(d^R \hat{\Delta})$ obtained by replacing $\hat{\Xi}$ by $\hat{\Xi}'$, and the indices of the form $\neg B \vee B$ and $B \wedge \neg B$ by $B \vee \neg B$ and $\neg B \wedge B$ respectively. For example, we have

\[
(\hat{b} \hat{\Delta}) \quad \hat{b}^{-}_{\text{A,B,C\vee C}} \circ \hat{\Delta}'_{\text{C,A\wedge B}} = 1_A \wedge \hat{\Delta}'_{\text{C,B}}.
\]

We can also derive for $\mathbf{PN}^-$ the following equations by using essentially $(\hat{\Sigma} \hat{\Delta})$ and $(\hat{\Sigma}' \hat{\Delta}')$:
(\tilde{\mathcal{Y}} \tilde{\mathcal{S}})\quad \tilde{\Delta}_{A,A} \ast d^R_{A,\neg A,A} \ast \tilde{\mathcal{S}}_{A,A}^{\vee} = 1_A,
(\tilde{\mathcal{S}} \tilde{\Delta})\quad \tilde{\Delta}_{\neg A,\neg A} \ast d^R_{\neg A,\neg A,\neg A} \ast \tilde{\mathcal{S}}_{\neg A,\neg A} = 1_{\neg A}.

These two equations could replace (\tilde{\mathcal{Y}} \tilde{\mathcal{S}}) and (\tilde{\mathcal{S}} \tilde{\Delta}) for defining PN\textsuperscript{\neg}. The equations (\tilde{\mathcal{Y}} \tilde{\mathcal{S}}), (\tilde{\mathcal{S}} \tilde{\Delta}), (\tilde{\mathcal{S}} \tilde{\mathcal{Y}}), and (\tilde{\mathcal{S}} \tilde{\Delta}) are related to the triangular equations of an adjunction (see [38], Section IV.1, and §5.1 below; see also the next section). The analogues of these equations may be found in [11] (Section 4).

A proof-net category is a category with two biendofunctors \wedge and \vee, a unary operation \neg on objects, and the natural transformations \wedge b \rightarrow^\xi b, \wedge b \leftarrow^\xi b, \vee c \rightarrow^\xi c, \vee c \leftarrow^\xi c, \wedge d \rightarrow^\xi d, and \wedge d \leftarrow^\xi d that satisfy the equations (\xi b), (\xi d), . . . , (\xi c) of PN\textsuperscript{\neg}. The category PN\textsuperscript{\neg} is up to isomorphism the free proof-net category generated by the set of letters P (the set P may be understood as a discrete category).

If \beta is a primitive arrow term of PN\textsuperscript{\neg} except \textbf{1}_B, then we call \beta-terms of PN\textsuperscript{\neg} the set of arrow terms defined inductively as follows: \beta is a \beta-term; if f is a \beta-term, then for every A in L\wedge,\vee we have that \textbf{1}_A \in f and f \in f, where \xi \in \{\wedge, \vee\}, are \beta-terms.

In a \beta-term the subterm \beta is called the head of this \beta-term. For example, the head of the \textbf{b}_{\neg C,\neg D} \wedge (\textbf{b}_{\neg C,\neg D} \vee \textbf{1}_E) is \textbf{b}_{\neg C,\neg D}.

We define \textbf{1}-terms as \beta-terms by replacing \beta in the definition above by \textbf{1}_B. So \textbf{1}-terms are headless.

An arrow term of the form \textit{f}_n \circ \ldots \circ \textit{f}_1, where \textit{n} \geq 1, with parentheses tied to \circ associated arbitrarily, such that for every \textit{i} \in \{1, \ldots, \textit{n}\} we have that \textit{f}_i is composition-free is called factorized. In a factorized arrow term \textit{f}_n \circ \ldots \circ \textit{f}_1 the arrow terms \textit{f}_i are called factors. A factor that is a \beta-term for some \beta is called a headed factor. A factorized arrow term is called headed when each of its factors is either headed or a \textbf{1}-term. A factorized arrow term \textit{f}_n \circ \ldots \circ \textit{f}_1 is called developed when \textit{f}_1 is a \textbf{1}-term and if \textit{n} > 1, then every factor of \textit{f}_n \circ \ldots \circ \textit{f}_2 is headed. It is sometimes useful to write the factors of a headed arrow term one above the other, as it is done for example in Figure 1 at the end of §2.5.

By using the categorial equations (\textit{cat} 1) and (\textit{cat} 2) and bifunctorial equations we can easily prove by induction on the length of \textit{f} the following
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lemma.

**Development Lemma.** For every arrow term $f$ there is a developed arrow term $f'$ such that $f = f'$ in $PN^\neg$.

Analogous definitions of $\beta$-term and developed arrow term can be given for $DS$, and an analogous Development Lemma can be proved for $DS$.

§2.3. The category $Br$

We are now going to introduce a category called $Br$, which will serve to prove our main coherence result for proof-net categories. We will show that there is a faithful functor from $PN^\neg$ to $Br$. The name of the category $Br$ comes from “Brauerian”. The arrows of this category correspond to graphs, or diagrams, that were introduced in [7] in connection with Brauer algebras (see [45]). Analogous graphs were investigated in [23], and in [32] Kelly and Mac Lane relied on them to prove their coherence result for symmetric monoidal closed categories (see §3.1).

Let $M$ be a set whose subsets are denoted by $X, Y, Z, \ldots$. For $i \in \{s, t\}$ (where $s$ stands for “source” and $t$ for “target”), let $M^i$ be a set in one-to-one correspondence with $M$, and let $i: M \to M^i$ be a bijection. Let $X^i$ be the subset of $M^i$ that is the image of the subset $X$ of $M$ under $i$. If $u \in M$, then we use $u_i$ as an abbreviation for $i(u)$. We assume also that $M, M^s$ and $M^t$ are mutually disjoint.

For $X, Y \subseteq M$, let a *split relation* of $M$ be a triple $\langle R, X, Y \rangle$ such that $R \subseteq (X^s \cup Y^t)^2$. The set $X^s \cup Y^t$ may be conceived as the disjoint union of $X$ and $Y$. We denote a split relation $\langle R, X, Y \rangle$ more suggestively by $R: X \vdash Y$.

A split relation $R: X \vdash Y$ is a *split equivalence* when $R$ is an equivalence relation. We denote by $\text{part}(R)$ the partition of $X^s \cup Y^t$ corresponding to the split equivalence $R: X \vdash Y$.

We say that a split equivalence $R: X \vdash Y$ is *Brauerian* when every member of part($R$) is a two-element set. For $R: X \vdash Y$ a Brauerian split equivalence, every member of part($R$) is either of the form $\{u_s, v_t\}$, in which case it is called a *transversal*, or of the form $\{u_s, v_s\}$, in which case it is called a *cup*, or, finally, of the form $\{u_t, v_t\}$, in which case it is called a *cap*. 
For $X, Y, Z \subseteq M$, we want to define the composition $P \ast R: X \vdash Z$ of the split relations $R: X \vdash Y$ and $P: Y \vdash Z$ of $M$. For that we need some auxiliary notions.

For $X, Y \subseteq M$, let the function $\varphi^s: X \cup Y^t \to X^s \cup Y^t$ be defined by

$$\varphi^s(u) = \begin{cases} u_s & \text{if } u \in X \\ u & \text{if } u \in Y^t, \end{cases}$$

and let the function $\varphi^t: X^s \cup Y \to X^s \cup Y^t$ be defined by

$$\varphi^t(u) = \begin{cases} u & \text{if } u \in X^s \\ u_t & \text{if } u \in Y. \end{cases}$$

For a split relation $R: X \vdash Y$, let the relations $R^{-s} \subseteq (X \cup Y^t)^2$ and $R^{-t} \subseteq (X^s \cup Y)^2$ be defined by

$$(u, v) \in R^{-i} \iff (\varphi^i(u), \varphi^i(v)) \in R$$

for $i \in \{s, t\}$. Finally, for an arbitrary binary relation $R$, let $\text{Tr}(R)$ be the transitive closure of $R$.

Then we define $P \ast R$ by

$$P \ast R =_{df} \text{Tr}(R^{-t} \cup P^{-s}) \cap (X^s \cup Z^t)^2.$$ 

It is easy to conclude that $P \ast R: X \vdash Z$ is a split relation of $M$, and that if $R: X \vdash Y$ and $P: Y \vdash Z$ are (Brauerian) split equivalences, then $P \ast R$ is a (Brauerian) split equivalence.

We now define the category $Br$. The objects of $Br$ are the members of the set of finite ordinals $N$. (We have $0 = \emptyset$ and $n + 1 = n \cup \{n\}$, while $N$ is the ordinal $\omega$.) The arrows of $Br$ are the Brauerian split equivalences $R: m \vdash n$ of $N$. The identity arrow $1_n: n \vdash n$ of $Br$ is the Brauerian split equivalence such that

$$\text{part}(1_n) = \{\{m_s, m_t\} \mid m < n\}.$$ 

Composition in $Br$ is the operation $\ast$ defined above.

That $Br$ is indeed a category (i.e. that $\ast$ is associative and that $1_n$ is an identity arrow) is proved in [20] and [21]. This proof is obtained via an isomorphic representation of $Br$ in the category $Rel$, whose objects are
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the finite ordinals and whose arrows are all the relations between these objects. Composition in $Rel$ is the ordinary composition of relations. A direct formal proof would be more involved, though what we have to prove is rather clear if we represent Brauerian split equivalences geometrically (as this is done in [7], [23], and also in categories of tangles; see [31], Chapter 12, and references therein).

For example, for $R \subseteq (3^s \cup 9^t)^2$ and $P \subseteq (9^s \cup 1^t)^2$ such that

$$\text{part}(R) = \{\{0_s, 0_t\}, \{1_s, 3_t\}, \{2_s, 6_t\}\} \cup \{\{n_s, (n+1)_t\} \mid n \in \{1, 4, 7\}\},$$

$$\text{part}(P) = \{\{2_s, 0_t\}\} \cup \{\{n_s, (n+1)_t\} \mid n \in \{0, 3, 5, 7\}\},$$

the composition $P \ast R \subseteq (3^s \cup 1^t)^2$, for which we have

$$\text{part}(P \ast R) = \{\{0_s, 0_t\}, \{1_s, 2_s\}\},$$

is obtained from the following diagram:

```
    0 1 2
R
\|/ \|/
0 1 2 3 4 5 6 7 8
P
```

Every bijection $f$ from $X^s$ to $Y^t$ corresponds to a Brauerian split equivalence $R: X \vdash Y$ such that the members of $\text{part}(R)$ are of the form $\{u, f(u)\}$. The composition of such Brauerian split equivalences, which correspond to bijections, is then a simple matter: it amounts to composition of these bijections. If in $Br$ we keep as arrows only such Brauerian split equivalences, then we obtain a subcategory of $Br$ isomorphic to the category $Bij$ whose objects are again the finite ordinals and whose arrows are the bijections between these objects. The category $Bij$ is a subcategory of the category $Rel$ (which played an important role in [22]), whose objects are the finite ordinals and whose arrows are all the relations between these objects. Composition in $Bij$ and $Rel$ is the ordinary composition of relations. The
category $\text{Rel}$ is isomorphic to a subcategory of the category whose arrows are split relations of finite ordinals, of whom $\text{Br}$ is also a subcategory.

We define a functor $G$ from $\text{PN}^-$ to $\text{Br}$ in the following way. On objects, we stipulate that $GA$ is the number of occurrences of letters in $A$. (If $A$ has $n = \{0, 1, \ldots, n-1\}$ occurrences of letters, then the first occurrence corresponds to 0, the second to 1, etc.) On arrows, we have first that $G\alpha$ is an identity arrow of $\text{Br}$ for $\alpha$ being $1_A, b^e_{A,B,C}, b^e_{A,B,C}$ and $d_{A,B,C}$, where $\xi \in \{\land, \lor\}$.

Next, for $i, j \in \{s, t\}$, we have that $\{m_i, n_j\}$ belongs to part($G\hat{\land}_B,A$) iff $\{n_i, m_j\}$ belongs to part($G\hat{\lor}_A,B$), iff $i$ is $s$ and $j$ is $t$, while $m, n < GA + GB$ and

$$(m - n - GA)(m - n + GB) = 0.$$ 

In the following example, we have $G(p \lor q) = 2 = \{0, 1\}$ and $G((q \lor \neg r) \lor q) = 3 = \{0, 1, 2\}$, and we have the diagrams

We have that $\{m_i, n_j\}$ belongs to part($G\hat{\land}_B,A$) iff either

- $i$ is $s$ and $j$ is $t$, while $m, n < GA$ and $m = n$, or
- $i$ and $j$ are both $t$, while $m, n \in \{GA, \ldots, GA + 2GB - 1\}$ and $|m - n| = GB$.

In the following example, for $A$ being $(q \lor \neg r) \lor q$ and $B$ being $p \lor q$, we have
§2.3. The category $\mathcal{B}r$

We have that \{m_i, n_j\} belongs to $\text{part}(G\hat{\Sigma}_{B,A})$ iff either

- $i$ is $s$ and $j$ is $t$, while $m \in \{2GB, \ldots, 2GB + GA - 1\}$, $n < GA$ and $m - 2GB = n$, or
- $i$ and $j$ are both $s$, while $m, n < 2GB$ and $|m - n| = GB$.

For $A$ and $B$ being as in the previous example, we have

Let $G(f \circ g) = Gf \circ Gg$. To define $G(f \xi g)$, for $\xi \in \{\wedge, \vee\}$, we need an auxiliary notion.

Suppose $b_X$ is a bijection from $X$ to $X_1$ and $b_Y$ a bijection from $Y$ to $Y_1$. Then for $R \subseteq (X^s \cup Y^t)^2$ we define $R^{b_X}_{b_Y} \subseteq (X_1^s \cup Y_1^t)^2$ by

\[(u_i, v_j) \in R^{b_X}_{b_Y} \text{ iff } (i(b^{-1}_U(u)), j(b^{-1}_V(v))) \in R,\]

where $(i, U), (j, V) \in \{(s, X), (t, Y)\}$.

If $f : A \vdash D$ and $g : B \vdash E$, then for $\xi \in \{\wedge, \vee\}$ the set of ordered pairs $G(f \xi g)$ is

\[Gf \cup Gg^{+GA}_{+GD}\]
where $+GA$ is the bijection from $GB$ to \{n+GA | n \in GB\} that assigns $n+GA$ to $n$, and $+GD$ is the bijection from $GE$ to \{n+GD | n \in GE\} that assigns $n+GD$ to $n$.

It is not difficult to check that $G$ so defined is indeed a functor from $\mathbf{PN}^\neg$ to $\mathbf{Br}$. For that, we determine by induction on the length of derivation that for every equation $f = g$ of $\mathbf{PN}^\neg$ we have $Gf = Gg$ in $\mathbf{Br}$.

Consider, for example, the following diagram, which illustrates an instance of $(\check{\Sigma}\check{\Delta})$:

This diagram shows that the equation $(\check{\Sigma}\check{\Delta})$, as well as the equation $(\check{\Sigma}'\check{\Delta}')$, which is illustrated by analogous diagrams, is related to triangular equations of adjunctions (cf. [14], Section 4.10, and [16], Section 7). The triangular equations of adjunctions are essentially about “straightening a sinosity”, and this straightening is based on planar ambient isotopies of knot theory (cf. [8], Section 1.A).

We have shown by this induction that $\mathbf{Br}$ is a proof-net category, and the existence of a structure-preserving functor $G$ from $\mathbf{PN}^\neg$ to $\mathbf{Br}$ follows from the freedom of $\mathbf{PN}^\neg$.

We can define analogously to $G$ a functor, which we also call $G$, from the category $\mathbf{DS}$ to $\mathbf{Br}$. We just omit from the definition of $G$ above the clauses involving $\check{\Delta}B,A$ and $\check{\Sigma}B,A$. The image of $\mathbf{DS}$ by $G$ in $\mathbf{Br}$ is the subcategory of $\mathbf{Br}$ isomorphic to $\mathbf{Bij}$, which we mentioned above. The following is proved in [22] (Section 7.6).
§2.4. Some properties of DS

DS Coherence. The functor $G$ from DS to Br is faithful.

It follows immediately from this coherence result that DS is isomorphic to a subcategory of $\text{PN}^-$ (cf. [22], Section 14.4).

Up to the end of §2.7 we will be occupied with proving the following.

PN$^-$ Coherence. The functor $G$ from $\text{PN}^-$ to Br is faithful.

For this proof, we must deal first with some preliminary matters.

§2.4. Some properties of DS

In this section we will prove some results about the category DS, which we will use to ascertain that particular equations hold in $\text{PN}^-$. We need these results also for the proof of $\text{PN}^-$ Coherence.

First we introduce a definition. Suppose $x$ is the $n$-th occurrence of a letter (counting from the left) in a formula $A$ of $\mathcal{L}_{\land,\lor,\neg}$, and $y$ is the $m$-th occurrence of the same letter in a formula $B$ of $\mathcal{L}_{\land,\lor,\neg}$. Then we say that $x$ and $y$ are tied in an arrow $f : A \vdash B$ of $\text{PN}^-$ when in the partition $\text{part}(Gf)$ we have $\{(n-1)s, (m-1)t\}$ as a member. (Note that to find the $n$-th occurrence we count starting from 1, but the ordinal $n > 0$ is \{0, ..., $n-1$\}.) We have an analogous definition of tied occurrences of the same letter for DS: we just replace $\mathcal{L}_{\land,\lor,\neg}$ by $\mathcal{L}_{\land,\lor}$ and $\text{PN}^-$ by DS.

It is easy to establish by induction on the complexity of $f$ that for every arrow term $f : A \vdash B$ of DS we have $GA = GB$. Moreover, every occurrence of letter in $A$ is tied to exactly one occurrence of the same letter in $B$, and vice versa. This is related to the fact that every arrow term $f : A \vdash B$ of DS may be obtained by substituting letters for letters out of an arrow term $f' : A' \vdash B'$ of DS such that every letter occurs in $A'$ at most once, and the same for $B'$ (see [22], Sections 3.3 and 7.6).

Suppose for Lemmata 1D and 2D below that $f : A \vdash B$ is an arrow term of DS such that $A$ has a subformula $D$ in which $\land$ does not occur and $B$ has a subformula $D'$ in which $\land$ does not occur, and suppose that every occurrence of a letter in $D$ is tied to an occurrence of a letter in $D'$ and vice versa. Then we can prove the following.

Lemma 1D. The source $A$ of $f$ is $D$ iff the target $B$ of $f$ is $D'$. 
This follows from the fact, noted above, that $GA = GB$. The arrow term $f$ in this case can have as subterms that are primitive arrow terms only arrow terms of the forms $1_E, \hat{b}_{E,F,G}, \hat{b}_{E,F,G}$ or $\check{c}_{E,F}$. We also have the following.

**Lemma 2D.** If $D \land A'$ or $A' \land D$ is a subformula of $A$, then $D' \land B'$ or $B' \land D'$ is a subformula of $B$ for some $B'$.

We will not go into the inductive proof of this lemma, in which we use Lemma 1D, because we need just a corollary of this lemma (Lemma 2 below), which is more easily proved directly.

Suppose for Lemmata 1C and 2C below that $f : A \vdash B$ is an arrow term of $\mathbf{DS}$ such that $B$ has a subformula $C$ in which $\lor$ does not occur and $A$ has a subformula $C'$ in which $\lor$ does not occur, and suppose that every occurrence of a letter in $C$ is tied to an occurrence of a letter in $C'$ and vice versa. Then we have the following duals of Lemmata 1D and 2D, proved in an analogous manner.

**Lemma 1C.** The target $B$ of $f$ is $C$ iff the source $A$ of $f$ is $C'$.

**Lemma 2C.** If $C \lor B'$ or $B' \lor C$ is a subformula of $B$, then $C' \lor A'$ or $A' \lor C'$ is a subformula of $A$ for some $A'$.

Suppose for the following lemma, which is a corollary of either Lemma 2D or Lemma 2C, that $f : A \vdash B$ is an arrow term of $\mathbf{DS}$ such that an occurrence $x$ of a letter $p$ in $A$ is tied to an occurrence $y$ of $p$ in $B$. This lemma is easily proved by induction on the complexity of $f$.

**Lemma 2.** It is impossible that $A$ has a subformula $x \land A'$ or $A' \land x$ and $B$ has a subformula $y \lor B'$ or $B' \lor y$.

Suppose for Lemmata 3D, 3C, 3 and 4 below that $f : A \vdash B$ is an arrow term of $\mathbf{DS}$, and for $i \in \{1, 2\}$ let $x_i$ in $A$ and $y_i$ in $B$ be occurrences of the letter $p_i$ tied in $f$ (here $p_1$ and $p_2$ may also be the same letter).

**Lemma 3D.** If in $A$ we have a subformula $A_1 \lor A_2$ such that $x_i$ occurs in $A_i$, then in $B$ we have a subformula $B_1 \lor B_2$ or $B_2 \lor B_1$ such that $y_i$ occurs in $B_i$. 

This is easily proved by induction on the complexity of the arrow term $f$. We prove analogously the following.

**Lemma 3C.** If in $B$ we have a subformula $B_1 \land B_2$ such that $y_i$ occurs in $B_1$, then in $A$ we have a subformula $A_1 \land A_2$ or $A_2 \land A_1$ such that $x_i$ occurs in $A_i$.

As a corollary of either Lemma 3D or Lemma 3C we have the following.

**Lemma 3.** It is impossible that $A$ has a subformula $x_1 \lor x_2$ or $x_2 \lor x_1$ and $B$ has a subformula $y_1 \land y_2$ or $y_2 \land y_1$.

The following lemma, dual to Lemma 3, is a corollary of Lemma 2.

**Lemma 4.** It is impossible that $A$ has a subformula $x_1 \land x_2$ or $x_2 \land x_1$ and $B$ has a subformula $y_1 \lor y_2$ or $y_2 \lor y_1$.

Lemma 3 is related to the acyclicity condition of proof nets, while Lemma 4 is related to the connectedness condition (see §7.1).

Next we can prove the following lemma.

**$p$-$q$-$r$ Lemma.** Let $f : A \vdash B$ be an arrow of $\text{DS}$, let $x_i$ for $i \in \{1, 2, 3\}$ be occurrences of the letters $p$, $q$ and $r$, respectively, in $A$, and let $y_i$ be occurrences of the letters $p$, $q$ and $r$, respectively, in $B$, such that $x_i$ and $y_i$ are tied in $f$. Let, moreover, $x_2 \lor x_3$ be a subformula of $A$ and $y_1 \land y_2$ a subformula of $B$. Then there is a $d_{p,q,r}$-term $h : A' \vdash B'$ such that $x'_i$ are occurrences of the letters $p$, $q$ and $r$, respectively, in the source $p \land (q \lor r)$ of the head of $h$ and $y'_i$ are occurrences of the letters $p$, $q$ and $r$, respectively, in the target $(p \land q) \lor r$ of the head of $h$, such that for some arrows $f_x : A \vdash A'$ and $f_y : B' \vdash B$ of $\text{DS}$ we have $f = f_y \circ h \circ f_x$ in $\text{DS}$, and $x_i$ is tied to $x'_i$ in $f_x$, while $y'_i$ is tied to $y_i$ in $f_y$.

**Proof.** The proof of this lemma, of which we give just a sketch, relies on a cut-elimination and related results of [22] (Sections 7.7-8). We first find in the category $\text{GDS}$ introduced in [22] (Section 7.7) a cut-free Gentzen term $f' : X \vdash Y$, which corresponds to $f$, by the relationship that exists between $\text{DS}$ and $\text{GDS}$. According to the equations at the beginning of Section 7.8 of [22], which are used for the proof of the Invertibility Lemmata in the same
section, in GDS we have the equation \( f' = f'' \) for a Gentzen term \( f'' \) that has as a subterm either \( \wedge_{p,q}(1_p, \lor_{q,r}(1_q, 1_r)) \) or \( \lor_{q,r}(\wedge_{p,q}(1_p, 1_q), 1_r) \) both of type \( p \land (q \lor r) \vdash (p \land q) \lor r \). By the relationship that exists between DS and GDS, we can find starting from \( f'' \) an arrow term \( f_y \circ h \circ f_x \) equal to \( f \) in DS, which satisfies the conditions of the lemma.

The full force of the Cut-Elimination Theorem of Section 7.7 of [22] is not essential for this proof, but applying this theorem simplifies the proof.

§2.5. The category PN

We now introduce a category called \( \text{PN} \), which is equivalent to \( \text{PN}^{-} \). In the objects of \( \text{PN} \), the negation connective \( \neg \) will be prefixed only to letters, and hence \( \land_{p,A} \) and \( \lor_{p,A} \) will be primitive only for the crown index \( B \) being a letter. Here is the formal definition of \( \text{PN} \).

For \( \mathcal{P} \) being the set of letters that we used to generate \( L_{\land,\lor} \) and \( L_{\neg,\land,\lor} \) in §§2.1-2, let \( \mathcal{P}^{-} \) be the set \( \{ \neg p \mid p \in \mathcal{P} \} \). The objects of \( \text{PN} \) are the formulae of the propositional language \( L_{\land,\lor}^{\mathcal{P}^{-}} \) generated from \( \mathcal{P} \cup \mathcal{P}^{-} \) with the binary connectives \( \land \) and \( \lor \). To define the arrow terms of \( \text{PN} \), in the inductive definition we had for the arrow terms of DS we assume in addition that for every formula \( A \) of \( L_{\land,\lor}^{\mathcal{P}^{-}} \) and every letter \( p \)

\[
\delta_{p,A}: A \vdash A \land (\neg p \lor p),
\]

\[
\tilde{\delta}_{p,A}: (p \land \neg p) \lor A \vdash A
\]

are primitive arrow terms of \( \text{PN} \).

To define the arrows of \( \text{PN} \), we assume as additional equations in the inductive definition we had for the equations of DS the \( \text{PN} \) equations of §2.2 restricted to the arrow terms \( \delta_{p,A} \) and \( \tilde{\delta}_{p,A} \). This means that in \( (\delta \text{ nat}) \) and \( (\tilde{\delta} \text{ nat}) \) the crown index \( B \) will be \( p \), in \( (\hat{\delta} \Delta) \) and \( (\hat{\tilde{\delta}} \Delta) \) the crown index \( C \) will be \( p \), and in \( (d \tilde{\delta}) \), \( (d \Delta) \), \( (\hat{\tilde{\delta}} \dot{\Delta}) \) and \( (\hat{\delta} \dot{\Delta}) \) the crown index \( A \) will be \( p \). We define \( \tilde{\tilde{\delta}}_{p,A}, \tilde{\delta}_{p,A}, \hat{\delta}_{p,A}, \hat{\tilde{\delta}}_{p,A}, \hat{\delta}_{p,A} \) and \( \hat{\tilde{\delta}}_{p,A} \) for \( \text{PN} \) as they were defined in \( \text{PN}^{-} \) in terms of \( \delta_{p,A} \) and \( \tilde{\delta}_{p,A} \).

The following equations of \( \text{PN} \), and hence also of \( \text{PN}^{-} \), which we call stem-increasing equations, enable us to have in developed arrow terms only \( \hat{\delta}_{A,B} \)-terms and \( \tilde{\delta}_{A,B} \)-terms that coincide with their heads:
\[\Sigma \lor 1 \] 
\[1 \lor \Sigma \] 
\[1 \land \hat{\Delta} \] 
\[\hat{\Delta} \land 1 \] 
\[1 \lor \hat{\Delta} \] 
\[\hat{\Delta} \lor 1 \] 
\[\Sigma \lor \Sigma \] 
\[1 \lor \Sigma \] 
\[\Sigma \land 1 \] 
\[1 \land \Sigma \] 

Note that in the stem-increasing equations the stem index \( B \) of \( \hat{\Delta} \) and \( \hat{\Sigma} \) becomes more complex on the right-hand sides, whereas the crown index \( p \) does not change. We have analogous stem-increasing equations for \( \hat{\Sigma} \), \( \hat{\Delta}' \), \( \hat{\Sigma}' \), \( \hat{\Delta} \), \( \hat{\Sigma} \), and \( \hat{\Delta}' \).

We will next prove several lemmata concerning \( \text{PN} \), which we will find useful for calculations later on. For these lemmata we need the following.

Let \( \text{DS}^-p \) be the category defined as \( \text{DS} \) save that it is generated not by \( \mathcal{P} \), but by \( \mathcal{P} \cup \mathcal{P}^- \). So the objects of \( \text{DS}^-p \) are formulae of \( \mathcal{L}_p^- \), i.e. the objects of \( \text{PN} \). For \( A \) and \( B \) formulae of \( \mathcal{L}_p^- \), we define when an occurrence of \( p \) in \( A \) is tied to an occurrence of \( p \) in \( B \) in an arrow \( f: A \rightarrow B \) of \( \text{DS}^-p \) analogously to what we had at the beginning of the preceding section.

Let \( \hat{\xi} \) for \( \xi \in \{\land, \lor\} \) stand for either \( \hat{\Delta} \), or \( \hat{\Delta}' \), or \( \hat{\Sigma} \), or \( \hat{\Sigma}' \), and let a \( \hat{\xi}_{B,A} \)-term be defined as a \( \beta \)-term in \( \S 2.2 \), save that \( \beta \) is replaced by \( \hat{\xi}_{B,A} \). We use also \( \Theta \) as a variable alternative to \( \Xi \). Then we have the following.
**Chapter 2. Coherence of Proof-Net Categories**

\[ \hat{\Xi} \text{-Permutation Lemma.} \] Let \( g: C \vdash D \) be a \( \hat{\Xi}_{p,B} \)-term of \( \mathcal{PN} \) such that \( x_1 \) and \( \neg x_2 \) are respectively the occurrences within \( D \) of \( p \) and \( \neg p \) in the crown of the head \( \hat{\Xi}_{p,B} \) of \( g \), and let \( f: D \vdash E \) be an arrow term of \( D_{\neg p} \) such that we have an occurrence \( y_1 \) of \( p \) and an occurrence \( \neg y_2 \) of \( \neg p \) within a subformula of \( E \) of the form \( y_1 \lor \neg y_2 \) or \( \neg y_2 \land y_1 \), and \( x_i \) is tied to \( y_i \) for \( i \in \{1, 2\} \) in \( f \). Then there is a \( \hat{\Theta}_{p,B} \)-term \( g': D' \vdash E \) of \( \mathcal{PN} \) the crown of whose head is \( y_1 \lor \neg y_2 \) or \( \neg y_2 \land y_1 \), and there is an arrow term \( f': C \vdash D' \) of \( D_{\neg p} \) such that in \( \mathcal{PN} \) we have \( f \circ g = g' \circ f' \).

**Proof.** By the Development Lemma we can assume that \( f \) is a developed arrow term, and then it is enough to consider the case when \( f \) is either a \( \beta \)-term for \( \beta \) a primitive arrow term of \( D_{\neg p} \) or \( f \) is \( 1_E \). Note that in the developed arrow term \( f_n \circ \ldots \circ f_1 \), which is equal to \( f \), we have that \( f_1 \) is \( 1_D \), and that \( f_2 \), if it exists, cannot be a \( d_{B,p,-p} \)-term or a \( d_{B,-p,p} \)-term such that \( x_1 \) and \( \neg x_2 \) are the occurrences of \( p \) and \( \neg p \) in the right conjunct of the source \( B \land (\neg p \lor p) \) or \( B \lor (p \lor \neg p) \) of the head of \( f_2 \). Otherwise, in the target of the head of \( f_2 \) we would obtain as the left disjunct \( B \land \neg p \) or \( B \land p \), which together with Lemma 2 would contradict the conditions put on \( f \), and hence also on \( f_n \circ \ldots \circ f_1 \), in the formulation of the \( \hat{\Xi} \)-Permutation Lemma.

The case when \( f \) is \( 1_E \) is trivial, and there are also many easy cases settled by bifunctorial and naturality equations. The remaining, more interesting, cases are settled by the following equations of \( \mathcal{PN} \):

\[
\begin{align*}
\hat{b}_{A,B,-p,vp}^{-1} \circ (1_A \land \hat{\Delta}_{p,B}) & = \hat{\Delta}_{p,A \land B}, & \text{by } (\hat{b} \hat{\Delta}), \\
\hat{b}_{B_1,B_2,-p,vp}^{-1} \circ \hat{\Delta}_{p,B_1 \land B_2} & = 1_{B_1} \land \hat{\Delta}_{p,B_2}, & \text{by } (\hat{b} \hat{\Delta}), \\
\hat{b}_{A,-p,vp,B}^{-1} \circ (1_A \land \hat{\Sigma}_{p,B}) & = \hat{\Delta}_{p,A} \land 1_B, & \text{by } (\hat{b} \hat{\Delta} \hat{\Sigma}), \\
\hat{b}_{B,-p,vp,A}^{-1} \circ (\hat{\Delta}_{p,B} \land 1_A) & = 1_B \land \hat{\Sigma}_{p,A}, & \text{by } (\hat{b} \hat{\Delta} \hat{\Sigma}), \\
\hat{b}_{p,vp,B_1,B_2}^{-1} \circ \hat{\Sigma}_{p,B_1 \land B_2} & = \hat{\Sigma}_{p,B_1} \land 1_{B_2}, & \text{by } (\hat{b} \hat{\Sigma}), \\
\hat{b}_{p,vp,B,A}^{-1} \circ (\hat{\Sigma}_{p,B} \land 1_A) & = \hat{\Sigma}_{p,B \land A}, & \text{by } (\hat{b} \hat{\Sigma}), \\
\hat{c}_{B,-p,vp}^{-1} \circ \hat{\Delta}_{p,B} & = \hat{\Sigma}_{p,B}, & \text{by definition}, \\
\hat{c}_{p,vp,B}^{-1} \circ \hat{\Sigma}_{p,B} & = \hat{\Delta}_{p,B}, & \text{by definition and } (\hat{c} \hat{e}).
\end{align*}
\]
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\[(1_B \land \check{\cdot}_{p,-p}) \circ \hat{\Delta}_{p,B} = \hat{\Delta}'_{p,B},\] by definition,

\[(\check{\cdot}_{p,-p} \land 1_B) \circ \hat{\Sigma}_{p,B} = \hat{\Sigma}'_{p,B},\] by definition and (\check{\cdot} nat),

\[d_{p \lor p, B_1, B_2} \circ \hat{\Sigma}_{p,B_1 \lor B_2} = \hat{\Sigma}'_{p,B_1} \lor 1_{B_2},\] by (d $\hat{\Sigma}$).

Besides these equations, we have analogous equations where $\neg p \lor p$ is replaced by $p \lor \neg p$, while $\hat{\Delta}$ and $\hat{\Sigma}$ are replaced by $\hat{\Delta}'$ and $\hat{\Sigma}'$ respectively, and vice versa.

We prove analogously the following dual of the preceding lemma.

\[\hat{\Xi}\text{-Permutation Lemma.} \text{ Let } g: D \vdash C \text{ be a } \hat{\Xi}_{p,B}\text{-term of } \mathbf{PN} \text{ such that } x_1 \text{ and } \neg x_2 \text{ are respectively the occurrences within } D \text{ of } p \text{ and } \neg p \text{ in the crown of the head } \hat{\Xi}_{p,B} \text{ of } g, \text{ and let } f: E \vdash D \text{ be an arrow term of } \mathbf{DS}^{-p} \text{ such that we have an occurrence } y_1 \text{ of } p \text{ and an occurrence } \neg y_2 \text{ of } \neg p \text{ within a subformula of } E \text{ of the form } y_1 \land \neg y_2 \text{ or } \neg y_2 \land y_1, \text{ and } y_i \text{ is tied to } x_i \text{ for } i \in \{1, 2\} \text{ in } f. \text{ Then there is a } \hat{\Theta}_{p,B}\text{-term } g': E \vdash D' \text{ of } \mathbf{PN} \text{ the crown of whose head is } y_1 \land \neg y_2 \text{ or } \neg y_2 \land y_1, \text{ and there is an arrow term } f': D' \vdash C \text{ of } \mathbf{DS}^{-p} \text{ such that in } \mathbf{PN} \text{ we have } g \circ f = f' \circ g'.\]

Next we prove the following lemma, which involves the $p$-$q$-$r$ Lemma of the preceding section.

\[\check{\cdot}p\text{-}\neg p\text{-}p \text{ Lemma.} \text{ Let } x_1, \neg x_2 \text{ and } x_3 \text{ be occurrences of } p, \neg p \text{ and } p, \text{ respectively, in a formula } A \text{ of } \mathcal{L}_{\land \lor}^{p}, \text{ and let } y_1, \neg y_2 \text{ and } y_3 \text{ be occurrences of } p, \neg p \text{ and } p, \text{ respectively, in a formula } B \text{ of } \mathcal{L}_{\land \lor}^{-p}. \text{ Let } \neg x_2 \lor x_3 \text{ or } x_3 \lor \neg x_2 \text{ be a subformula of } A \text{ and } y_1 \land \neg y_2 \text{ or } \neg y_2 \land y_1 \text{ a subformula of } B. \text{ Let } g_1: A' \vdash A \text{ be a } \hat{\Xi}_{p,C}\text{-term of } \mathbf{PN} \text{ such that } \neg x_2 \lor x_3 \text{ or } x_3 \lor \neg x_2 \text{ is the crown of the head of } g_1, \text{ let } g_2: B \vdash B' \text{ be a } \hat{\Theta}_{p,D}\text{-term of } \mathbf{PN} \text{ such that } y_1 \land \neg y_2 \text{ or } \neg y_2 \land y_1 \text{ is the crown of the head of } g_2, \text{ and let } f: A \vdash B \text{ be an arrow term of } \mathbf{DS}^{-p} \text{ such that } x_i \text{ and } y_i \text{ are tied in } f \text{ for } i \in \{1, 2, 3\}. \text{ Then } g_2 \circ f = g_1 \text{ is equal in } \mathbf{PN} \text{ to an arrow term of } \mathbf{DS}^{-p}.\]

\text{Proof.} \text{ By the } p$-$q$-$r \text{ Lemma, } f: A \vdash B \text{ is equal in } \mathbf{DS}^{-p}, \text{ and hence also in } \mathbf{PN}, \text{ to an arrow term of the form } f_y \circ h = f_x, \text{ where } h \text{ is a } d_{p,-p,p}\text{-term, and the other conditions of the } p$-$q$-$r \text{ Lemma are satisfied. So in } \mathbf{PN} \text{ we}
have

\[ g_2 \circ f \circ g_1 = g_2 \circ f_y \circ h \circ f_x \circ g_1 = f'_y \circ g'_2 \circ h \circ g'_1 \circ f'_x, \]

by the Ξ-Permutation Lemmata above. Here the head of \(g'_1\) must be \(\tilde{\Delta}_{p,p} : p \vdash p \land (\neg p \lor p)\), the head of \(h\) is \(d_{p,\neg p,p} : p \land (\neg p \lor p) \lor p\), and the head of \(g'_2\) must be \(\tilde{\Sigma}_{p,p} : (p \land \neg p) \lor p \vdash p\). By applying (\(\tilde{\Sigma}\tilde{\Delta}\)), and perhaps bifunctorial equations, we obtain that \(g'_2 \circ h \circ g'_1\) is equal in \(\text{PN}\) to an arrow term of the form \(1_A\), and hence we have \(g_2 \circ f \circ g_1 = f'_y \circ f'_x\) in \(\text{PN}\), which proves the lemma.

To give an example of the application of the \(p\neg p\neg p\) Lemma, consider the diagram in Figure 1. This diagram corresponds to \(G(\tilde{\Sigma}_{q,p\land q} \circ h \circ \tilde{\Delta}_{q,p\land q})\) for an arrow term \(h\) of \(\text{PN}\), which is of the form \(g_2 \circ f \circ g_1\) for \(g_1\) being \(1_{p\land q} \land (1_{q} \lor \tilde{\Sigma}_{p,q})\), \(g_2\) being \(1_{q} \lor \tilde{\Sigma}_{p,q} \land 1_{p\land q}\) and \(f\) an arrow term of \(\text{DS}^{-}p\). Then by applying the \(p\neg p\neg p\) Lemma we obtain an arrow term \(f'\) of \(\text{DS}^{-}p\) equal to \(g_2 \circ f \circ g_1\) in \(\text{PN}\), and next by applying the \(p\neg p\neg p\) Lemma (as a matter of fact, the \(q\neg q\neg q\) Lemma), we obtain an arrow term \(h'\) of \(\text{DS}^{-}p\) equal to \(\tilde{\Sigma}_{q,p\land q} \circ f' \circ \tilde{\Delta}_{q,p\land q}\) in \(\text{PN}\). By \(\text{DS}\) Coherence of §2.3, we may conclude that \(h'\), and hence also \(\tilde{\Sigma}_{q,p\land q} \circ h \circ \tilde{\Delta}_{q,p\land q}\), is equal to \(1_{p\land q}\) in \(\text{PN}\).

Here is a lemma analogous to the \(p\neg p\neg p\) Lemma.

\(\neg p\neg p\neg p\) LEMMA. Let \(\neg x_1, x_2\) and \(\neg x_3\) be occurrences of \(\neg p\), \(p\) and \(\neg p\), respectively, in a formula \(A\) of \(\mathcal{L}_{\land \lor}^p\), and let \(\neg y_1, y_2\) and \(\neg y_3\) be occurrences of \(\neg p\), \(p\) and \(\neg p\), respectively, in a formula \(B\) of \(\mathcal{L}_{\land \lor}^p\). Let \(g_1 : A' \vdash A\) be a \(\tilde{\Sigma}_{p,c}\)-term of \(\text{PN}\) such that \(x_2 \lor \neg x_3\) or \(\neg x_3 \lor x_2\) is the crown of the head of \(g_1\), let \(g_2 : B \vdash B'\) be a \(\tilde{\Theta}_{p,B}\)-term of \(\text{PN}\) such that \(\neg y_1 \land y_2\) or \(y_2 \land \neg y_1\) is the crown of the head of \(g_2\), and let \(f : A' \vdash B\) be an arrow term of \(\text{DS}^{-}p\) such that \(x_i\) and \(y_i\) are tied in \(f\) for \(i \in \{1,2,3\}\). Then \(g_2 \circ f \circ g_1\) is equal in \(\text{PN}\) to an arrow term of \(\text{DS}^{-}p\).

To prove this lemma we proceed as for the \(p\neg p\neg p\) Lemma, relying on the equation (\(\tilde{\Sigma}\tilde{\Delta}'\)) of \(\text{PN}\).
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\[
p \land q
\]

\[
(p \land q) \land (\neg q \lor q)
\]

\[
(p \land q) \land (\neg q \lor ((\neg p \lor p) \land q))
\]

\[
(p \land q) \land (\neg q \lor (\neg p \lor (p \land q)))
\]

\[
(p \land q) \land (\neg q \lor (\neg p \lor (p \land q)))
\]

\[
(p \land q) \land (\neg q \lor (\neg p \lor (p \land q)))
\]

\[
((p \land q) \land (\neg q \lor \neg p)) \lor (p \land q)
\]

\[
((q \land p) \land (\neg p \lor \neg q)) \lor (p \land q)
\]

\[
(q \land (p \land (\neg p \lor \neg q))) \lor (p \land q)
\]

\[
(q \land (p \land (\neg p \lor \neg q))) \lor (p \land q)
\]

\[
(q \land (p \land (\neg p \lor \neg q))) \lor (p \land q)
\]

\[
(q \land \neg q) \lor (p \land q)
\]

\[
p \land q
\]

\[
\hat{\Delta}_{q,p \land q}
\]

\[
1_{p \land q} \land (1_{\neg q} \lor \hat{\Sigma}_{p,q})
\]

\[
1_{p \land q} \land (1_{\neg q} \lor d^R_{p,p,q})
\]

\[
1_{p \land q} \land b^2_{q,p,-p \land q}
\]

\[
1_{p \land q} \land (\hat{\iota}_{p,-q} \lor 1_{p \land q})
\]

\[
d_{p,q,-p \lor q,p \land q}
\]

\[
(\hat{\iota}_{p,q} \land 1_{q \lor p}) \lor 1_{p \land q}
\]

\[
\hat{b}^2_{q,p,-p \lor q} \lor 1_{p \land q}
\]

\[
(1_{q} \land d_{p,-p,-q}) \lor 1_{p \land q}
\]

\[
(1_{q} \land \hat{\Sigma}_{p,-q}) \lor 1_{p \land q}
\]

\[
(1_{q} \land \hat{\Sigma}_{p,-q}) \lor 1_{p \land q}
\]

\[
\hat{\Sigma}_{q,p \land q}
\]

Figure 1
§2.6. The equivalence of $\text{PN}^-$ and $\text{PN}$

In this section we show that the categories $\text{PN}^-$ and $\text{PN}$ are equivalent categories. We define inductively a functor $F$ from the category $\text{PN}^-$ to $\text{PN}$ in the following manner. On objects we have

$$Fp = p,$$ for $p$ a letter,

$$F(A \xi B) = FA \xi FB,$$ for $\xi \in \{\land, \lor\},$

$$F\neg p = \neg p,$$ for $p$ a letter,

$$F\neg\neg A = FA,$$

$$F\neg(A \land B) = F\neg A \lor F\neg B,$$

$$F\neg(A \lor B) = F\neg A \land F\neg B.$$  

On arrows we have

$$F\alpha_{A_1, \ldots, A_n} = \alpha_{FA_1, \ldots, FA_n},$$ for $\alpha_{A_1, \ldots, A_n}$ being $1_{A_1}, b_{A,B,C}^-, b_{A,B,C}^-, \hat{\epsilon}_{A,B}$ or $d_{A,B,C}$ where $\xi \in \{\land, \lor\},$

$$F\hat{\Delta}_{p,A} = \hat{\Delta}_{p,FA} : FA \vdash FA \land (\neg p \lor p),$$

$$F\hat{\Sigma}_{p,A} = \hat{\Sigma}_{p,FA} : (p \land \neg p) \lor FA \vdash FA,$$

$$F\hat{\Delta}_{\neg B,A} = (1_{FA} \land \hat{\epsilon}_{FB,F\neg B}) \circ F\hat{\Delta}_{B,A} : FA \vdash FA \land (FB \lor F\neg B),$$

$$F\hat{\Sigma}_{\neg B,A} = F\hat{\Sigma}_{B,A} \circ (\hat{\epsilon}_{F-B,FB} \lor 1_{FA}) : (F\neg B \land FB) \lor FA \vdash FA,$$

$$F\hat{\Delta}_{B\land C,A} = (1_{FA} \land (\hat{\epsilon}_{F-B,F-C} \lor 1_{FB\lor FC}) \circ \hat{\Delta}_{B,C,FA} : FA \vdash FA \land ((F\neg B \lor F\neg C) \lor (FB \land FC)),$$

$$F\hat{\Sigma}_{B\land C,A} = F\hat{\Sigma}_{C,A} \circ ((1_{FC} \land (F\hat{\Sigma}_{B,-C} \circ d_{FB,F\neg B,F-C})) \circ \hat{\Delta}_{\neg C,FA} : \hat{\Delta}_{B,C,FA} \circ (FB \land FC) \land (F\neg B \lor F\neg C) \lor FA \vdash FA,$$

$$F\hat{\Delta}_{B\lor C,A} = (1_{FA} \land (\hat{\epsilon}_{F-B,F-C} \lor 1_{FB\lor FC}) \circ \hat{\Delta}_{B,C,FA} : FA \vdash FA \land ((F\neg B \lor F\neg C) \lor (FB \land FC)),$$

$$F\hat{\Sigma}_{B\lor C,A} = F\hat{\Sigma}_{C,A} \circ ((1_{FC} \land (F\hat{\Sigma}_{B,-C} \circ d_{FB,F\neg B,F-C})) \circ \hat{\Delta}_{\neg C,FA} : \hat{\Delta}_{B,C,FA} \circ (FB \land FC) \land (F\neg B \lor F\neg C) \lor FA \vdash FA,$$
§2.6. The equivalence of \( \mathbf{PN}^- \) and \( \mathbf{PN} \)

\[
F\hat{\Delta}_{B\lor C,A} = (1_{FA} \land ((c_{F-C,F-B} \lor 1_{FB\lor FC}) \ast \hat{b}_{F-C,F-B,F,B,FC} \ast
\ast ((d_{F-C,F-B,F} \ast F\hat{\Delta}_{B,-C} \lor 1_{FC}))) \ast F\hat{\Delta}_{C,A}:
FA \vdash FA \land ((F\neg B \land F\neg C) \lor (FB \lor FC)),
F\hat{\Sigma}_{B\lor C,A} = F\hat{\Sigma}_{C,A} \ast (((F\hat{\Sigma}_{B,C} \ast \hat{c}_{FB,F-B,FC} \ast d^R_{FC,F,B,F-B}) \land 1_{F-C}) \ast
\ast \hat{b}_{FC\lor FB,F-B,F-C} \ast (\hat{c}_{FC,FB} \land 1_{F-B,F-C}) \lor 1_{FA}):
((FB \lor FC) \land (F\neg B \land F\neg C)) \lor FA \vdash FA,
F(f \circ g) = Ff \circ Fg,
F(f \xi g) = Ff \xi Fg, \quad \text{for } \xi \in \{\land, \lor\}.
\]

It is easy to infer

\[
F\hat{\Delta}_{-,B,A} = F\hat{\Delta}'_{B,A}, \quad F\hat{\Sigma}_{-,B,A} = F\hat{\Sigma}'_{B,A},
F\hat{\Delta}'_{-,B,A} = F\hat{\Delta}_{B,A}, \quad F\hat{\Sigma}'_{-,B,A} = F\hat{\Sigma}_{B,A},
F\hat{\Delta}_{B,A} = F\hat{\Delta}_{B,FA}, \quad F\hat{\Sigma}_{B,A} = F\hat{\Sigma}_{B,FA}.
\]

To ascertain that \( F \) so defined is indeed a functor, we have to verify that if \( f = g \) is an instance of one of the \( \mathbf{PN} \) equations, then \( Ff = Fg \) holds in \( \mathbf{PN} \). This is done by induction on the number of occurrences of connectives in the crown indices occurring in these equations.

For \( (\land \Delta \text{ nat}) \) and \( (\lor \Sigma \text{ nat}) \) this is a very easy matter. For \( (\land \Delta), (\land \Sigma), (d\hat{\Sigma}) \) and \( (d\Delta) \) we use essentially naturality equations. (In that context, it might be easier to rely on the equations \( (d\hat{\Sigma}) \) and \( (d\Delta) \), which are alternative to \( (d\hat{\Sigma}) \) and \( (d\Delta) \).)

To verify \( (\hat{\Sigma}\Delta) \) in cases where \( A \) is of the form \( B \land C \) or \( B \lor C \), we rely on the induction hypothesis that if \( f = g \) is an instance of a \( \mathbf{PN} \) equation such that the crown indices are \( B \) and \( C \), then we have \( Ff = Fg \) in \( \mathbf{PN} \). This induction hypothesis entails that we can proceed as in the proof of the \( p{-}\neg p{-}p \) Lemma in the preceding section, first for \( p \) replaced by \( B \), and then for \( p \) replaced by \( C \). Finally, we apply \textbf{DS Coherence} (see the example at the end of the preceding section). To verify \( (\hat{\Sigma}\Delta) \) in case \( A \) is of the form \( \neg B \), we rely on the induction hypothesis for the equation \( (\hat{\Sigma}'\Delta') \).

To verify \( (\hat{\Sigma}'\Delta') \) we proceed analogously. In case \( A \) is \( B \land C \) or \( B \lor C \), we rely on the proof of the \( \neg p{-}p{-}\neg p \) Lemma in the preceding section, and
in case $A$ is $\neg B$ we rely on the induction hypothesis for the equation $(\check{\Sigma}\check{\Delta})$.
This concludes the verification that $F$ is a functor from $\text{PN}^\neg$ to $\text{PN}$.

(To verify that the functor $F$ from $\text{PN}^\neg$ to $\text{PN}$ is a functor we could have proceeded by establishing $\text{PN}$ Coherence first, before introducing the functor $F$. We do not need the functor $F$ to prove $\text{PN}$ Coherence in the next section. From $f = g$ in $\text{PN}^\neg$ we pass to $Gf = Gg$, from which by relying on the first paragraph of §2.7 we pass to $GFf = GFg$, which by $\text{PN}$ Coherence implies $Ff = Fg$.)

In the definition of $F$, there is some freedom in choosing the clauses for $F\check{\Xi}_{B\in C,A}$, where $\Xi \in \{\Delta, \Sigma\}$ and $\xi, \psi \in \{\land, \lor\}$. We chose ours to be able to apply easily the $p-\neg p-p$ and $\neg p-p-\neg p$ Lemmata in verifying that $F$ is a functor.

We define a functor $F^\neg$ from $\text{PN}$ to $\text{PN}^\neg$ by stipulating that $F^\neg A = A$ and $F^\neg f = f$. It is clear that if $f = g$ in $\text{PN}$, then $F^\neg f = F^\neg g$ in $\text{PN}^\neg$; so $F^\neg$ is indeed a functor.

Our purpose is to show that $\text{PN}^\neg$ and $\text{PN}$ are equivalent categories via the functors $F$ and $F^\neg$. It is clear that $F F^\neg A = A$ and $F F^\neg f = f$. Since $F^\neg FA = FA$, we have to define in $\text{PN}^\neg$ an isomorphism $i_A : A \vdash FA$. For that we need the following auxiliary definitions in $\text{PN}^\neg$:

\[
\begin{align*}
n^\neg_A &= d_{\neg\neg A} \circ d_{\neg A, \neg A} \circ d_{A, \neg A} \circ d_{\neg A, \neg A} : \neg\neg A \vdash A, \\
n^\neg_A &= d_{\neg\neg A} \circ d_{\neg A, \neg A} \circ d_{A, \neg A} \circ d_{\neg A, \neg A} : A \vdash \neg\neg A, \\
n_1^\neg_{A,B} &= d_{A, \neg A} \circ d_{\neg\neg A, \neg A} \circ d_{A, \neg A} \circ d_{\neg A, \neg A} \circ d_{A, \neg A} \circ d_{A, \neg A} \circ d_{\neg A, \neg A} : (\neg\neg A \land \neg A) \vdash \neg\neg A \lor \neg A, \\
\check{\phi}^\neg_{A,B} &= d_{A, \neg A} \circ d_{\neg\neg A, \neg A} \circ d_{A, \neg A} \circ d_{\neg A, \neg A} \circ d_{A, \neg A} \circ d_{A, \neg A} \circ d_{\neg A, \neg A} : ((\neg\neg A \land \neg A) \lor 1_{\neg A}) \vdash \neg\neg A \lor \neg A, \\
\check{\psi}^\neg_{A,B} &= d_{A, \neg A} \circ d_{\neg\neg A, \neg A} \circ d_{A, \neg A} \circ d_{\neg A, \neg A} \circ d_{A, \neg A} \circ d_{A, \neg A} \circ d_{\neg A, \neg A} : ((\neg\neg A \land \neg A) \lor 1_{\neg A}) \vdash \neg\neg A \lor \neg A.
\end{align*}
\]
§2.6. The equivalence of $\mathbf{PN}^\neg$ and $\mathbf{PN}$

$\mathcal{L}_{\neg,\wedge,\vee}$ holds in $\mathbf{PN}^\neg$; this enables us to apply the $p\neg\neg p$ and $\neg p p \neg p$ Lemmata. (If an equation holds in $\mathbf{PN}$, then every substitution instance of it obtained by replacing letters uniformly by formulae of $\mathcal{L}_{\neg,\wedge,\vee}$ holds in $\mathbf{PN}^\neg$.) The definitions of $n\neg$, $n\neg$, $\hat{\varphi}$ and $\hat{\varphi}$, for $\varpi \in \{\wedge,\vee\}$, are such that they enable an easy application of the $p\neg\neg p$ and $\neg p p \neg p$ Lemmata.

Then we define $i_A : A \vdash FA$ and its inverse $i_A^{-1} : FA \vdash A$ by induction on the complexity of the formula $A$ of $\mathcal{L}_{\neg,\wedge,\vee}$ (cf. [22], Section 14.1):

$$i_A = i_A^{-1} = 1_A, \text{ if } A = p \text{ or } \neg p, \text{ for } p \text{ a letter},$$
$$i_{A_1 \wedge A_2} = i_{A_1} \wedge i_{A_2}, \quad i_{A_1 \vee A_2} = i_{A_1} \vee i_{A_2}, \quad i_{\neg A_1} = i_{A_1} = i_{\neg A_2} = i_{A_2}, \text{ for } \varpi \in \{\wedge,\vee\},$$
$$i_{\neg \neg B} = i_B \circ n_B^{-1}, \quad i_{\neg A_1 \wedge A_2} = (i_{A_1} \wedge i_{A_2}) \circ i_{A_1}, \quad i_{\neg A_1 \vee A_2} = (i_{A_1} \vee i_{A_2}) \circ i_{A_1},$$
$$i_{\neg (A_1 \wedge A_2)} = (i_{A_1} \wedge i_{A_2}) \circ i_{A_1}, \quad i_{\neg (A_1 \vee A_2)} = (i_{A_1} \vee i_{A_2}) \circ i_{A_1}, \quad i_{\neg A_1 \wedge A_2} = i_{A_1 \wedge A_2} \circ i_{A_1}, \quad i_{\neg A_1 \vee A_2} = i_{A_1 \vee A_2} \circ i_{A_1}.$$

We can then prove the following (cf. [22], Section 14.1).

**Auxiliary Lemma.** For every arrow term $f : A \vdash B$ of $\mathbf{PN}^\neg$ we have $f = i_B^{-1} \circ F f \circ i_A$ in $\mathbf{PN}^\neg$. 
PROOF. We proceed by induction on the complexity of the arrow term \( f \). If \( f \) is a primitive arrow term \( 1_A, \tilde{b}_{A,B,C}, \tilde{b}'_{A,B,C}, \tilde{c}_{A,B,C} \) or \( d_{A,B,C} \), for \( \xi \in \{\land, \lor\} \), then we use naturality equations, and the fact that \( i_D \) is an isomorphism.

If \( f \) is \( \Delta_{D,A} \), then we proceed by induction on the complexity of \( D \). (This is an auxiliary induction in the basis of the main induction.) If \( D \) is \( p \), then we use \( (\tilde{\Delta} \text{ nat}) \) and the fact that \( i_A \) is an isomorphism.

If \( D \) is \( \neg B \), then we rely on the following equation of \( \text{PN}^- \), analogous to the clause defining \( F(\tilde{\Delta}_{B,A}) \) above:

\[
(\tilde{\Delta} n) \quad \Delta_{-B,A} = (1_A \wedge (n_B \lor 1_B)) \circ \Delta_{B,A},
\]

together with the induction hypothesis. To derive \( (\tilde{\Delta} n) \) we have

\[
(1_A \wedge (n_B \lor 1_B)) \circ \Delta_{B,A}'
\]

\[
= (1_A \wedge (\tilde{\Delta}_{B,-B} \lor 1_B)) \circ (1_A \wedge (d_{B,-B,-B} \lor 1_B)) \circ \Delta_{B,A}, \quad \text{by bifunctorial equations,}
\]

\[
= (1_A \wedge (\tilde{\Delta}_{B,-B} \lor 1_B)) \circ (1_A \wedge (d_{B,-B,-B} \lor 1_B)) \circ \Delta_{B,A}, \quad \text{by stem-increasing equations involving \( \Delta' \) analogous to \( 1 \lor \tilde{\Delta} \) and \( 1 \land \tilde{\Delta} \) of the preceding section, and also \( \tilde{\Delta}' \text{ nat} \).}
\]

The equation \( (\tilde{\Delta} n) \) follows by applying the \( \neg p - p - p \) Lemma (with \( p \) replaced by \( B \)), and \( \text{DS Coherence} \).

If \( D \) is \( B \land C \), then we rely on the following equation of \( \text{PN}^- \), analogous to the clause defining \( F(\tilde{\Delta}_{B\land C,A}) \) above:

\[
(\tilde{\Delta} r) \quad \Delta_{B\land C,A} = (1_A \wedge ((d_{B,\neg B,-B} \lor 1_{B\land C}) \circ \tilde{\Delta}_{C,B\land C}) \circ (1_C \lor (d_{B,B,C} \circ \tilde{\Sigma}_{B,C})) \circ \Delta_{C,A}
\]

To show that \( (\tilde{\Delta} r) \) holds in \( \text{PN}^- \) we proceed as above, by applying essentially stem-increasing equations together with the \( p - p - p \) Lemma. We proceed analogously when \( D \) is \( B \lor C \).

The cases we have if \( f \) is \( \tilde{\Sigma}_{D,A} \) are dual to those we had above for \( f \) being \( \Delta_{D,A} \). In all these cases we proceed in an analogous manner. This concludes the basis of the induction.
§2.7. PN Coherence

If \( f \) is \( f_2 \circ f_1 \), then by the induction hypothesis we have

\[ f_2 \circ f_1 = i_B^{-1} \circ F f_2 \circ i_C^{-1} \circ F f_1 \circ i_A \]

which yields \( f = i_B^{-1} \circ F f \circ i_A \), by the fact that \( i_C \) is an isomorphism and by the functoriality of \( F \).

If \( f \) is \( f_1 \xi f_2 \), for \( \xi \in \{\land, \lor\} \), then \( i_A^{-1} \xi i_A \) and \( i_B^{-1} \xi i_B \) yield \( f = i_B^{-1} \circ F f \circ i_A \) by using bifunctorial equations.

The Auxiliary Lemma shows that \( i_A \) is an isomorphism natural in \( A \), and so we may conclude that \( \text{PN}^- \) and \( \text{PN} \) are equivalent categories.

§2.7. PN Coherence

We define a functor \( G \) from \( \text{PN} \) to \( \text{Br} \) as we defined it from \( \text{PN}^- \) to \( \text{Br} \). In the clauses for \( \land \Delta \) and \( \lor \Sigma \) we just restrict \( B \) to a letter \( p \). For \( f \) an arrow term of \( \text{PN}^- \) we have that \( GF f \) coincides with \( Gf \) where \( F \) is the functor from \( \text{PN}^- \) to \( \text{PN} \) of the preceding section, \( G \) in \( GF f \) is the functor \( G \) from \( \text{PN} \) to \( \text{Br} \) and \( G \) in \( Gf \) is the functor \( G \) from \( \text{PN}^- \) to \( \text{Br} \).

To show that, it is essential to check that \( GF \land \Delta \) and \( GF \lor \Sigma \) coincide with \( G \land \Delta \) and \( G \lor \Sigma \) respectively.

In this section we will prove that \( G \) from \( \text{PN} \) to \( \text{Br} \) is faithful. This will imply that \( G \) from \( \text{PN}^- \) to \( \text{Br} \) is faithful too.

Analogously to what we had at the beginning of §2.4, we define when an occurrence \( x \) of a letter \( p \) in \( A \) is tied to an occurrence \( y \) of the same letter \( p \) in \( B \) in an arrow \( f : A \rightarrow B \) of \( \text{PN} \). We say that \( x \) and \( y \) are directly tied in a headed factorized arrow term \( f_n \circ \ldots \circ f_1 \) of \( \text{PN} \) when \( x \) and \( y \) are tied in the arrow \( f_n \circ \ldots \circ f_1 \), and for every \( i \in \{2, \ldots, n\} \) if \( f_i \) is a \( \Sigma_{p,C} \)-term and \( z \) is one of the two occurrences of \( p \) in the crown \( p \land \neg p \) of the head of \( f_i \), then \( x \) and \( z \) are not tied in the arrow \( f_i^{-1} \circ \ldots \circ f_1 \) (see the end of §2.2 for the definition of headed factorized arrow term).

An alternative definition of directly tied \( x \) and \( y \) in a headed factorized arrow term \( f_1 \circ \ldots \circ f_n \) of \( \text{PN} \) is obtained by stipulating that \( x \) and \( y \) are tied in the arrow \( f_1 \circ \ldots \circ f_n \), and for every \( i \in \{2, \ldots, n\} \) if \( f_i \) is a \( \land \Delta_{p,D} \)-term and \( z \) is one of the two occurrences of \( p \) in the crown \( \neg p \lor p \) of the head of \( f_i \), then \( z \) and \( y \) are not tied in the arrow \( f_1 \circ \ldots \circ f_{i-1} \).
For example, the occurrence of \(q\) in the source \(p \land q\) and the occurrence of \(q\) in the target \(q \land p\) of

\[
\hat{c}_{p,q} \cdot (\hat{\Sigma}_{p,p} \land 1_q) \cdot (d_{p\land \neg p,p} \land 1_q) \cdot (\hat{\Delta}_{p,p} \land 1_q)
\]

are directly tied in this headed factorized arrow term of \(\mathbf{PN}\), while the two occurrences of \(p\) in its source and target are not directly tied.

Take a headed factorized arrow term of \(\mathbf{PN}\) of the form \(g_2 \cdot f \cdot g_1\) where \(g_1\) is a \(\land \Delta\)-term and \(g_2\) is a \(\lor \Sigma\)-term. Let \(\neg x_1 \lor x_2\) be the crown of the head of \(g_1\) (so \(x_1\) and \(x_2\) are both occurrences of \(p\)) and let \(y_2 \land \neg y_1\) be the crown of the head of \(g_2\) (so \(y_1\) and \(y_2\) are also occurrences of the same letter \(p\)). We say that \(g_1\) and \(g_2\) are confronted through \(f\) when \(x_i\) and \(y_i\) are directly tied for some \(i \in \{1, 2\}\) in the arrow term \(f\).

Let a \(\land \Delta\)-factor that is a factor of a factorized arrow term \(f\) be called a \(\land \Delta\)-factor. We have an analogous definition of \(\lor \Sigma\)-factor obtained by replacing \(\land \Delta\) by \(\lor \Sigma\). We can then prove the following lemma.

**Confrontation Lemma.** For every headed factorized arrow term \(g_2 \cdot f \cdot g_1\) of \(\mathbf{PN}\) such that \(g_1\) and \(g_2\) are confronted through \(f\) there is a headed factorized arrow term \(h\) of \(\mathbf{PN}\) with a subterm of the form \(g_2' \cdot f' \cdot g_1'\) such that \(g_1'\) is a \(\Delta\)-factor, \(g_2'\) is a \(\Sigma\)-factor, \(g_1'\) and \(g_2'\) are confronted through \(f'\), and, moreover,

1. \(f'\) is an arrow term of \(\mathbf{DS}^{-p}\),
2. \(g_2 \cdot f \cdot g_1 = h\) in \(\mathbf{PN}\),
3. the number of \(\Delta\)-factors is equal in \(g_2 \cdot f \cdot g_1\) and \(h\), and the same for the number of \(\Sigma\)-factors.

**Proof.** We proceed by induction on the number \(n\) of factors of \(f\) that are \(\Delta\)-factors or \(\Sigma\)-factors. If \(n = 0\), then the arrow term \(f'\) coincides with the arrow term \(f\).

If \(n > 0\), then let \(g_2 \cdot f \cdot g_1\) be of the form \(f_2 \cdot g \cdot f_1\) for \(g\) a \(\Delta_{q,E}\)-term (we proceed analogously when \(g\) is a \(\Sigma_{q,E}\)-term). According to the stem-increasing equations of §2.5, we may assume that \(g\) coincides with its head \(\Delta_{q,E}\). Then by \((\Delta \text{ nat})\) we obtain in \(\mathbf{PN}\)

\[
g_2 \cdot f \cdot g_1 = f_2 \cdot (f_1 \land 1_{\neg q \lor q}) \cdot \Delta_{q,E}.
\]
After $f_1 \land 1_{\neg q \lor q}$ in $f_2 \ast (f_1 \land 1_{\neg q \lor q})$ is replaced by a headed factorized arrow term $g_2 \ast f'' \ast (g_1 \land 1_{\neg q \lor q})$, we may apply the induction hypothesis to this arrow term, because it can easily be seen that $g_1 \land 1_{\neg q \lor q}$ and $g_2$ are confronted through $f''$, and $f''$ has one $\Delta$-factor less than $f$.

A headed factorized arrow term of $\text{PN}$ that has no subterm of the form $g_2 \ast f \ast g_1$ with $g_1$ and $g_2$ confronted through $f$ is called pure. For a pure arrow term $f$ there is a one-to-one correspondence, which we call the $\Delta$-cap bijection, between the $\Delta$-factors of $f$ and the caps of the partition $\text{part}(Gf)$. In this bijection, a cap ties, in an obvious sense, the occurrences of $p$ in the crown $\neg p \lor p$ of the head of the corresponding $\Delta$-factor. There is an analogous one-to-one correspondence, which we call the $\Sigma$-cup bijection, between the $\Sigma$-factors of $f$ and the cups of $\text{part}(Gf)$ (see §2.3 for the notions of cup and cap). Intuitively speaking, this follows from the fact that in a sequence of cups and caps tied to each other as in the following example:

```
  *
  \
  *
```

cups and caps must alternate. For a pair made of a cap and a cup that is its immediate neighbour, like those marked with * in the picture, we can find a subterm $g_2 \ast f \ast g_1$ such that $g_1$ and $g_2$ are confronted through $f$.

We can then prove the following.

**Purification Lemma.** Every arrow term of $\text{PN}$ is equal in $\text{PN}$ to a pure arrow term of $\text{PN}$.

**Proof.** We apply first the Development Lemma of §2.2. If in the resulting developed arrow term $h$ we have a subterm $g_2 \ast f \ast g_1$ with $g_1$ and $g_2$ confronted through $f$, then we apply first the Confrontation Lemma to obtain
a developed arrow term \( h' \) with a subterm of the form \( g'_2 \circ f' \circ g'_1 \) where \( g'_1 \) and \( g'_2 \) are confronted through \( f' \), and \( f' \) is an arrow term of DS\(^{\neg p}\).

Suppose that \( \neg x_2 \lor x_3 \) is the crown of the head of \( g'_1 \), and \( y_1 \land \neg y_2 \) is the crown of the head of \( g'_2 \). Suppose \( x_2 \) is tied to \( y_2 \) in \( f' \). Then, by Lemma 3 of §2.4, it is impossible that \( x_3 \) is tied to \( y_1 \), and so there must be an occurrence \( x_1 \) of \( p \) different from \( x_3 \) in the source of \( f' \) such that \( x_1 \) is tied to \( y_1 \) in \( f' \). Next we apply the \( p \land \neg p \land p \) Lemma of §2.5 to conclude that \( g'_2 \circ f' \circ g'_1 \) is equal to an arrow term \( h'' \) of DS\(^{\neg p}\).

After replacing \( g'_2 \circ f' \circ g'_1 \) in \( h' \) by \( h'' \), we obtain a headed factorized arrow term in which there is one \( \Delta \)-factor and one \( \Sigma \)-factor less than in \( h' \), and hence also than in \( h \), by clause (3) of the Confrontation Lemma.

If \( x_3 \) is tied to \( y_1 \), then we reason analogously by applying Lemma 3 of §2.4 and the \( \neg p \land p \land \neg p \) Lemma of §2.5.

We can iterate this procedure, which must terminate, because the number of \( \Delta \)-factors and \( \Sigma \)-factors in \( h \) is finite.

We can then prove the following.

**PN Coherence.** The functor \( G \) from PN to Br is faithful.

**Proof.** Suppose for \( f \) and \( g \) arrow terms of PN of the same type \( A \vdash B \) we have \( Gf = Gg \). By the Purification Lemma, we can assume that \( f \) and \( g \) are pure arrow terms. Since \( Gf = Gg \), by the \( \Delta \)-cap and \( \Sigma \)-cup bijections we must have the same number \( n \geq 0 \) of \( \Delta \)-factors in \( f \) and \( g \) and the same number \( m \geq 0 \) of \( \Sigma \)-factors in \( f \) and \( g \). We proceed by induction on \( n+m \).

If \( n+m = 0 \), then we just apply DS Coherence. Suppose now \( n > 0 \). So there is a \( \Delta \)-factor in \( f \) and a \( \Delta \)-factor in \( g \) that correspond by the \( \Delta \)-cap bijections to the same cap of part(\( Gf \)), which is equal to part(\( Gg \)). By using the stem-increasing equations of §2.5, together with (\( \Delta \) nat), we obtain in PN

\[
f = f' \circ \Delta_{p,A}, \quad g = g' \circ \Delta_{p,A}
\]

for \( f' \) and \( g' \) pure arrow terms of the same type \( A \land (\neg p \lor p) \vdash B \), and such that the number of \( \Delta \)-factors in \( f' \) and \( g' \) is \( n-1 \) in each, and the number
2.8. The contravariant functor $\neg$

of $\Sigma$-factors in $f'$ and $g'$ is $m$ in each, the same number we had for the $\Sigma$-factors in $f$ and $g$. So we have

$$G(f' \circ \hat{\Delta}_{p,A}) = Gf = Gg = G(g' \circ \hat{\Delta}_{p,A}).$$

We can show that $Gf' = Gg'$. This is because we obtain $Gf'$ from $G(f' \circ \hat{\Delta}_{p,A})$ in the following manner. We first remove from the partition $\text{part}(G(f' \circ \hat{\Delta}_{p,A}))$ a cap $\{k_t, l_t\}$, where the $k+1$-th occurrence of letter in $B$ is an occurrence of $p$ in a subformula $\neg p$ of $B$, and the $l+1$-th occurrence of letter in $B$ is an occurrence of $p$ that is not in a subformula $\neg p$ of $B$ (here we have either $k < l$ or $l < k$). After this removal, we add two new transversals:

$$\{GA_s, k_t\}, \quad \{(GA+1)_s, l_t\},$$

and this yields $\text{part}(Gf')$. Since $Gg'$ is obtained from $G(g' \circ \hat{\Delta}_{p,A})$, which is equal to $G(f' \circ \hat{\Delta}_{p,A})$ in exactly the same manner, we obtain that $Gf' = Gg'$.

Then, by the induction hypothesis, we have that $f' = g'$ in $\mathbf{PN}$, which implies that $f = g$ in $\mathbf{PN}$. We proceed analogously in the induction step when $m > 0$, via $\Sigma$-factors:

From $\mathbf{PN}$ Coherence and the equivalence between the categories $\mathbf{PN}^\neg$ and $\mathbf{PN}$, proved in the preceding section, we may conclude in the following manner that the functor $G$ from $\mathbf{PN}^\neg$ to $B_{\neg}$ is faithful.

**Proof of $\mathbf{PN}^\neg$ Coherence.** Suppose that for $f$ and $g$ arrows of $\mathbf{PN}^\neg$ of the same type we have $Gf = Gg$. Then, as we noted at the beginning of this section, we have $GFf = GFg$, and hence $Ff = Fg$ in $\mathbf{PN}$ by $\mathbf{PN}$ Coherence. It follows that $f = g$ in $\mathbf{PN}^\neg$ by the equivalence of the categories $\mathbf{PN}^\neg$ and $\mathbf{PN}$. 

So we have proved $\mathbf{PN}^\neg$ Coherence, announced at the end of §2.3.

2.8. The contravariant functor $\neg$

In every proof-net category $\mathcal{A}$ we can define a contravariant $\neg$ functor from $\mathcal{A}$ to $\mathcal{A}$ by relying on the following definition. For $f: A \vdash B$, we have
\[ -f = df \hat{\Sigma}'_{B,-A} \circ d_{-B,B,A} \circ (1_{-B} \land (f \lor 1_{-A})) \circ \hat{\Delta}'_{A,-B} : \neg B \vdash \neg A. \]

It is easy to check that we have
\[ \neg f = \hat{\Delta}_{B,-A} \circ d_{-B,-B,-B} \circ ((1_{-A} \lor f) \land 1_{-B}) \circ \hat{\Sigma}_{A,-B}, \]
which gives an alternative definition of \(-f\).

To verify that \(\neg\) is a contravariant functor we have first
\[ (\neg 1) \neg 1 = 1 \neg A \]
by \(\text{PN}^-\) Coherence (namely, we use here the equation \((\hat{\Sigma}' \hat{\Delta}')\), or alternatively \((\hat{\Delta} \hat{\Sigma})\), of \(\S 2.2\)).

Next, for \(f : A \vdash B\) we have the following equation:
\[ (\hat{\Sigma} \text{ dinat}) \quad \hat{\Sigma}_{A,C} \circ ((1_A \land \neg f) \lor 1_C) = \hat{\Sigma}_{B,C} \circ ((1_A \land (1_{-A} \lor f) \land 1_{-B}) \circ \hat{\Sigma}_{A,-B}) \lor 1_C, \]
which together with \((\hat{\Sigma} \text{ nat})\) says that \(\hat{\Sigma}\) is a dinatural transformation in the sense of \([38]\) (Section IX.4). To verify \((\hat{\Sigma} \text{ dinat})\) we have
\[ \hat{\Sigma}_{A,C} \circ ((1_A \land \neg f) \lor 1_C) = \hat{\Sigma}_{B,C} \circ ((1_A \land (1_{-A} \lor f) \land 1_{-B}) \circ \hat{\Sigma}_{A,-B}) \lor 1_C, \]
by \(\text{PN}^-\) Coherence,
\[ = \hat{\Sigma}_{B,C} \circ ((1_A \land (1_{-A} \lor f) \land 1_{-B}) \circ \hat{\Sigma}_{A,-B}) \lor 1_C, \]
by bifunctorial and naturality equations, and \(\text{PN}^-\) Coherence. We verify analogously the equation
\[ (\hat{\Delta} \text{ dinat}) \quad (1_C \land (\neg f \lor 1_B)) \circ \hat{\Delta}_{B,C} = (1_C \land (1_{-A} \lor f)) \circ \hat{\Delta}_{A,C}. \]

From these equations we derive easily other analogous equations, which we call dinaturality equations, for \(\hat{\Sigma}, \hat{\Delta}, \hat{\Delta}', \hat{\Sigma}', \hat{\Sigma}'\) and \(\hat{\Delta}'\). For example, we have
\[ (\hat{\Sigma} \text{ dinat}) \quad (\neg f \lor 1_B) \circ \hat{\Sigma}_{B,C} = (1_{-A} \lor f) \circ \hat{\Sigma}_{A,C}, \]
\[ (\hat{\Sigma}' \text{ dinat}) \quad \hat{\Sigma}'_{A,C} \circ ((\neg f \land 1_A) \lor 1_C) = \hat{\Sigma}'_{B,C} \circ ((1_{-B} \land f) \lor 1_C), \]
§2.8. The contravariant functor \( \neg \)

We can derive now the following equation:

\[
(\neg 2) \quad \neg(f_1 \circ f_2) = \neg f_2 \circ \neg f_1
\]

for \( f_1 : A \vdash B \) and \( f_2 : C \vdash A \). We have

\[
\neg f_2 \circ \neg f_1 = \sum_{A,-C}((\neg f_1 \land 1_A) \lor 1_{-C}) \circ \Delta_{B,-A,-C} \circ (1_{-B} \land (\neg f_2 \lor 1_{-C}) \circ \Delta_{C,-B}^t,
\]

by \( \Delta^t \) \( \text{nat} \), \( \land (2) \) and \( (d \ \text{nat}) \),

\[
= \sum_{B,-C}((1_{-B} \land f_1) \lor 1_{-C}) \circ \Delta_{B,-A,-C} \circ (1_{-B} \land (\neg f_2 \lor 1_{-C}) \circ \Delta_{C,-B}^t,
\]

by \( \Delta^t \) \( \text{dinat} \),

\[
= \neg(f_1 \circ f_2), \quad \text{by} \ (d \ \text{nat}) \text{ and } (\land 2).
\]

This proves that for every proof-net category \( \mathcal{A} \) we have that \( \neg \) is a contravariant functor from \( \mathcal{A} \) to \( \mathcal{A} \).

In every proof-net category \( \mathcal{A} \), for every object \( A \) and for \( \xi \in \{\land, \lor\} \), we have a functor \( A \xi \) from \( \mathcal{A} \) to \( \mathcal{A} \), i.e. an endofunctor (1-endofunctor in the terminology of [22], Section 2.4), which on arrows is the operation \( 1_A \xi \). It can easily be shown with the help of \( \text{PN}^- \) Coherence that in every proof-net category \( A \land \) is left adjoint to \( \neg A \lor \), and \( \neg A \land \) is left adjoint to \( A \lor \) (cf. §3.6 below; see [38], Section IV.1, or §5.1 below, for the notion of adjunction).
Chapter 3

Star-Autonomous Categories

The goal of this chapter is to prove the categorial equivalence between the star-autonomous category $\mathbf{SA}_s$ freely generated by a set of objects and the proof-net category with units $\mathbf{SA}'_s$ freely generated by the same set of objects. Our notion of proof-net category with units, which is obtained by extending the notion of proof-net category with unit objects and appropriate arrows and equations between these arrows, amounts to the notion of linearly distributive category with negation introduced in [11].

This chapter is rather technical. It demonstrates the equivalence of two notions of category formulated in two different languages, which happens to involve some pretty complex syntactical matters.

Since the language for star-autonomous categories we rely on only overlaps with the language for proof-net categories with units, we introduce two auxiliary freely generated categories called $\mathbf{SA}$ and $\mathbf{SA}'$, for which we have the union of these two different languages. The proof of equivalence of $\mathbf{SA}_s$ with $\mathbf{SA}'_s$ is broken into proofs of equivalence of $\mathbf{SA}_s$ with $\mathbf{SA}$ and $\mathbf{SA}'_s$ with $\mathbf{SA}'$, and a proof of isomorphism of $\mathbf{SA}$ with $\mathbf{SA}'$.

In the proofs of these equivalences and of this isomorphism, we rely heavily on Kelly’s and Mac Lane’s coherence theorem for symmetric monoidal closed categories, on our coherence theorem for $\mathbf{PN}^-$ from the preceding chapter, and on a coherence theorem for a freely generated category $\mathbf{PN}^-_{-1}$, intermediary between $\mathbf{PN}^-$ and $\mathbf{SA}'$, which is equivalent with $\mathbf{PN}^-$. Without these tools, the computations needed would be excessively difficult, as it was prefigured in the literature (see [11], Section 4, Theorem 4.5).
In the course of these proofs, we define in §3.4 a nonsymmetric net structure in the sense of [22] (Section 7.2) in the category $\text{Set}$ of sets with functions, and a nonassociative semiassociative structure in the sense of [22] (Section 4.2) in the same category.

§3.1. The category $\text{SMC}$

First we define the category $\text{SMC}$, which is the symmetric monoidal closed category (see [38], Section VII.7) freely generated by the set of letters $\mathcal{P}$. The objects of the category $\text{SMC}$ are the formulae of the propositional language $\mathcal{L}_\top, \land, \rightarrow$ generated by $\mathcal{P}$ with the nullary connective (i.e. propositional constant) $\top$ and the binary connectives $\land$ and $\rightarrow$.

To define the arrows of $\text{SMC}$, we define inductively the arrow terms of $\text{SMC}$ by assuming as primitive arrow terms $1_A: A \rightarrow 1_B, \hat{\delta}^-_{A}: A \rightarrow A \land \top, \epsilon_{A,B}: A \land (A \rightarrow B) \rightarrow B, \eta_{A,B}: B \rightarrow A \rightarrow (A \land B)$; as operations on arrow terms we have $\cd$ and $\land$ (which we know from DS; see §2.1) and the unary operations $A \rightarrow$, for every object $A$, such that for $f: B \rightarrow C$ we have the arrow term $A \rightarrow f: A \rightarrow B \rightarrow A \rightarrow C$. This concludes the definition of the arrow terms of $\text{SMC}$.

The equations of $\text{SMC}$ are obtained by assuming besides $f = f$ the following equations: (cat 1), (cat 2), (\land 1), (\land 2) (see §2.1), plus

\[
\begin{align*}
(A \rightarrow 1) & \quad A \rightarrow 1_B = 1_A \rightarrow B, \\
(A \rightarrow 2) & \quad A \rightarrow (f \cdot g) = (A \rightarrow f) \cdot (A \rightarrow g), \\
(\hat{\delta}^- \text{ nat}) & \quad (\hat{\delta}^-_A) f \cdot \hat{\delta}^-_B = \hat{\delta}^-_{A \land B}, \\
(\epsilon \text{ nat}) & \quad f \cdot \epsilon_{C,A} = \epsilon_{C,B} \cdot (1_C \land (C \rightarrow f)), \\
(\eta \text{ nat}) & \quad (C \rightarrow (1_C \land f)) \cdot \eta_{C,A} = \eta_{C,B} \cdot f,
\end{align*}
\]

(\hat{\delta} \hat{b}), (\hat{b} 5), (\hat{\delta} \hat{\epsilon}), (\hat{b} \hat{\epsilon}) (see §2.1), plus
§3.1. The category SMC

\[
\begin{align*}
(\delta \delta) & \quad \delta^-_A \circ \delta^-_A = 1_A, \quad \delta^-_A \circ \delta^-_A = 1_{A \land T}, \\
(b\delta) & \quad b^-_{A,B,T} \circ \delta^-_{A \land B} = 1_A \land \delta^-_B, \\
(\varepsilon \eta \land) & \quad \varepsilon_{A,A \land B} \circ (1_A \land \eta_{A,B}) = 1_{A \land B}, \\
(\varepsilon \eta \rightarrow) & \quad (A \rightarrow \varepsilon_{A,B}) \circ \eta_{A,A \rightarrow B} = 1_{A \rightarrow B}.
\end{align*}
\]

The equations \((A \rightarrow 1)\) and \((A \rightarrow 2)\) say that \(A \rightarrow\) is a functor, while the last two equations are the triangular equations of an adjunction (see [38], Section IV.1, or §5.1 below).

The set of equations of SMC is closed under symmetry and transitivity of equality, under the rules \((\text{cong } \xi )\) for \(\xi \in \{\circ, \land\}\) (see §2.1), and also under the rules

\[
\begin{align*}
f = g \\
\frac{A \rightarrow f = A \rightarrow g}
\end{align*}
\]

This defines the equations of SMC.

It is easy to see that for SMC we have the naturality equation

\[
(\delta^- \land \text{ nat}) \quad (f \land 1_T) \circ \delta^-_A = \delta^-_B \circ f.
\]

With the definitions

\[
\delta^-_A = df \circ \delta^-_A \circ \hat{\epsilon}_{T,A}, \quad \delta^-_A = df \circ \hat{\epsilon}_{A,T} \circ \delta^-_A,
\]

we obtain that \(\delta^-_A\) and \(\delta^-_A\) are inverse to each other. Note that there is an analogy between \(\hat{\Delta}_{B,A}: A \vdash A \land (\neg B \lor B)\) and \(\hat{\delta}^-: A \vdash A \land T,\) and between \(\hat{\Sigma}_{B,A}: A \vdash (\neg B \lor B) \land A\) and \(\hat{\sigma}^-: A \vdash T \land A,\) though \(\hat{\Delta}_{B,A}\) and \(\hat{\Sigma}_{B,A}\) are not isomorphisms. This analogy, which is the reason for our notation, is manifested by comparing the equation \((\delta^- \Delta)\) of §2.2 and \((b\delta)\) above. (Note that the equation \((b\delta)\) above is not exactly the equation \((\hat{b}\delta)\) of Section 4.6 of [22], but the two equations follow from each other in the presence of isomorphism equations.)

For \(g: A \vdash D\) and \(f: B \vdash C,\) we introduce the following definition in SMC:

\[
g \rightarrow f = df \circ (\varepsilon_{D,C} \circ (g \land (D \rightarrow f))) \circ \eta_{A,D \rightarrow B} : D \rightarrow B \vdash A \rightarrow C.
\]
CHAPTER 3. STAR-AUTONOMOUS CATEGORIES

With the help of $(\varepsilon \text{ nat})$, $(A \rightarrow 2)$ and $(\varepsilon \eta \rightarrow)$ we obtain that

$$1_A \rightarrow f = A \rightarrow f.$$  

(For the sake of uniformity, we will later prefer to write $1_A \rightarrow f$, rather than $A \rightarrow f$.) We then obtain

$$\begin{align*}
(\rightarrow 1) & \quad 1_A \rightarrow 1_B = 1_{A \rightarrow B}, \\
(\rightarrow 2) & \quad (f_1 \circ g_1) \rightarrow (g_2 \circ f_2) = (g_1 \rightarrow g_2) \circ (f_1 \rightarrow f_2);
\end{align*}$$

for $(\rightarrow 1)$ we use $(A \rightarrow 1)$, while for $(\rightarrow 2)$ we use essentially $(A \rightarrow 2)$, $(\wedge 2)$, $(\varepsilon \eta \wedge)$, $(\varepsilon \text{ nat})$ and $(\eta \text{ nat})$. So $\rightarrow$ is a bifunctor from $\text{SMC}^{op} \times \text{SMC}$ to $\text{SMC}$ (see [38], Section IV.7, Theorem 3).

For $f : A \vdash B$, we derive in $\text{SMC}$

$$\begin{align*}
(\varepsilon \text{ dinat}) & \quad \varepsilon_{A,C} \circ (1_A \wedge (f \rightarrow 1_C)) = \varepsilon_{B,C} \circ (f \wedge 1_{B \rightarrow C}), \\
(\eta \text{ dinat}) & \quad (1_A \rightarrow (f \wedge 1_C)) \circ \eta_{A,C} = (f \rightarrow 1_{B \wedge C}) \circ \eta_{B,C};
\end{align*}$$

for $(\varepsilon \text{ dinat})$ we use $(\varepsilon \text{ nat})$ and $(\varepsilon \eta \wedge)$, while for $(\eta \text{ dinat})$ we use $(\eta \text{ nat})$, $(\varepsilon \eta \wedge)$ and bifunctorial equations. We call these two equations dinaturality equations for $\varepsilon$ and $\eta$. Together with $(\varepsilon \text{ nat})$ and $(\eta \text{ nat})$, these dinaturality equations show that $\varepsilon$ and $\eta$ are dinatural transformations in the sense of [38] (Section IX.4).

We define a functor $G$ from $\text{SMC}$ to $\text{Br}$ by using the appropriate clauses we had for the functor $G$ from $\text{PN}^\neg$ to $\text{Br}$ in §2.3, to which we add that $G\alpha$ is an identity arrow of $\text{Br}$ also when $\alpha$ is $\delta_\wedge^{-}$ and $\delta_\wedge^{-}$, and the following clauses:

$$\begin{align*}
G\varepsilon_{A,B} & = G\Sigma_{A,B}^\vee, \\
G\eta_{A,B} & = G\Sigma_{A,B}^\wedge
\end{align*}$$

(see §2.3). We define $G(f \circ g)$ and $G(f_1 \wedge f_2)$ as we did in §2.3. For $f : B \vdash C$, the set of ordered pairs of $G(A \rightarrow f)$ is

$$G1_A \cup Gf^+_{+G^A}$$

(see §2.3 for $Gf^+_{+G^A}$).

For a Brauerian split equivalence $R \subseteq (X^s \cup Y^t)^2$, we define the Brauerian split equivalence $R^{op} \subseteq (Y^s \cup X^t)^2$ by replacing every ordered pair
§3.2. The category $\text{SA}$

The objects of the category $\text{SA}$ are the formulae of the propositional language $\mathcal{L}_{\top,\bot,\neg,\wedge,\vee,\rightarrow}$ generated by $\mathcal{P}$ with the nullary connectives (i.e. propositional constants) $\top$ and $\bot$, the unary connective $\neg$ and the binary connectives $\wedge$, $\vee$ and $\rightarrow$.

To define the arrows of $\text{SA}$, we define inductively the arrow terms of $\text{SA}$ by assuming as primitive arrow terms all the primitive arrow terms we had for $\text{SMC}$ (with $A$, $B$ and $C$ ranging over the formulae of $\mathcal{L}_{\top,\bot,\neg,\wedge,\vee,\rightarrow}$) plus

- $\nu^\top_A : (A \rightarrow \bot) \rightarrow \bot \vdash A$,
- $\lambda^\top_A : \neg A \vdash A \rightarrow \bot$,
- $\nu^\bot_A,B : A \vee B \vdash (A \rightarrow \bot) \rightarrow B$,
- $\lambda^\bot_A,B : (A \rightarrow \bot) \rightarrow B \vdash A \vee B$;

$$Gg^{op} \cup Gf^{\downarrow GA}.$$

We call a formula $A$ of $\mathcal{L}_{\top,\wedge,\rightarrow}$ consequential when for every subformula $B \rightarrow C$ of $A$ we have that either $B$ is letterless or $C$ has letters occurring in it. An alternative way to characterize consequential formulae is to say that these are formulae $A$ of $\mathcal{L}_{\top,\wedge,\rightarrow}$ for which there is an isomorphism of type $A \vdash A'$ of $\text{SMC}$ such that either $\top$ does not occur in $A'$ or $A'$ is $\top$. (To establish the equivalence of these two characterizations, one may rely on the results of [17].)

Let $\text{SMC}^c$ be the full subcategory of $\text{SMC}$ whose objects are consequential formulae. (The category $\text{SMC}^c$ is a replete subcategory of $\text{SMC}$ in the sense of [29], Section A1.1; namely, every object of $\text{SMC}$ isomorphic to an object of $\text{SMC}^c$ is in $\text{SMC}^c$.) The functor $G$ from $\text{SMC}$ to $\text{Br}$ may be restricted to a functor $G$ from $\text{SMC}^c$ to $\text{Br}$. The following result is proved by Kelly and Mac Lane in [32].

$\text{SMC}^c$ Coherence. The functor $G$ from $\text{SMC}^c$ to $\text{Br}$ is faithful.

§3.2. The category $\text{SA}$

(\(u_i, v_j\) of $R$ by (\(u_{i'}, v_{j'}\)) where $i, j \in \{s, t\}$, while $s' = t$ and $t' = s$. With that, for $g : A \vdash D$ and $f : B \vdash C$ it can be checked that the set of ordered pairs of $G(g \rightarrow f)$ is

$$Gg^{op} \cup Gf^{\downarrow GA}.$$
the operations on these arrow terms are as for SMC.

The equations of SA are obtained by assuming all the equations we have assumed for SMC, plus for

\[ 
\nu_{A,B}^{\rightarrow} = \text{df } (1_{A\rightarrow B} \rightarrow (\varepsilon_{A,B} \circ c_{A\rightarrow B,A})) \circ \eta_{A\rightarrow B,A} : A \vdash (A \rightarrow B) \rightarrow B
\]

the equations

\[ 
(\nu\nu) \quad \nu_{A,\bot}^{\rightarrow} \circ \nu_{A,\bot}^{-\rightarrow} = 1_A, \quad \nu_{A,\bot}^{-\rightarrow} \circ \nu_{A,\bot}^{\rightarrow} = 1_{(A \rightarrow \bot) \rightarrow \bot},
\]

\[ 
(\lambda\lambda) \quad \lambda_A^{-\rightarrow} \circ \lambda_A^{\rightarrow} = 1_{A \rightarrow \bot}, \quad \lambda_A^{\rightarrow} \circ \lambda_A^{-\rightarrow} = 1_{\neg A},
\]

\[ 
(\nu\nu) \quad \nu_{A,B}^{-\rightarrow} \circ \nu_{A,B}^{\rightarrow} = 1_{(A \rightarrow \bot) \rightarrow B}, \quad \nu_{A,B}^{\rightarrow} \circ \nu_{A,B}^{-\rightarrow} = 1_{A \lor B}.
\]

The set of equations of SA is closed under the same rules as the set of equations of SMC.

The equations (\nu\nu) assert that \( \nu_{A,\bot}^{\rightarrow} \) is an isomorphism. This is the assumption used by Barr in [2] (Section 2) to define star-autonomous categories. The isomorphism equations (\lambda\lambda) and (\nu\nu) are auxiliary, and will be discarded in a language where \( \neg \) and \( \lor \) are not primitive (see §3.8).

\section{The category SA'}

The objects of the category SA' are as for SA the formulae of the propositional language \( \mathcal{L}_{\top,\bot,\neg,\land,\lor} \) generated by \( \mathcal{P} \). As primitive arrow terms we have \( 1_A, \hat{b}_{A,B,C}, \hat{b}_{A,B,C}^{-\rightarrow}, \hat{c}_{A,B}, \hat{b}_{A,B,C}^{-\rightarrow}, \hat{c}_{A,B}, d_{A,B,C} \) (see §2.1), \( \hat{\Delta}_{B,A}, \hat{\Sigma}_{B,A} \) (see §2.2), \( \hat{\delta}_A^{-\rightarrow}, \hat{\delta}_A^{\rightarrow} \) (see §3.1), plus

\[ 
\hat{\delta}_A^{-\rightarrow} : A \lor \bot \vdash A, \quad \hat{\delta}_A^{\rightarrow} : A \vdash A \lor \bot,
\]

\[ 
\pi_{A,B}^{\rightarrow} : A \rightarrow B \vdash \neg A \lor B, \quad \pi_{A,B}^{-\rightarrow} : \neg A \lor B \vdash A \rightarrow B.
\]

These primitive arrow terms together with the operations on arrow terms \( \circ, \land \) and \( \lor \) (the same we had for DS and PN in §§2.1-2) define the arrow terms of SA'.

The equations of SA' are obtained by assuming all the equations we have assumed for PN (which are the equations of DS of §2.1 plus the PN equations of §2.2), plus (\hat{\delta} nat), (\hat{\delta} \hat{\delta}), (\hat{b} \hat{\delta}) (see §3.1), with the dual equations
§3.3. The category $\mathbf{SA}'$

\[
(\hat{\delta} \dashv \text{nat}) \quad f \circ \hat{\delta}_{A}^{-} = \hat{\delta}_{B}^{-} \circ (f \lor \mathbf{1}_{\bot}),
\]

\[
(\hat{\delta} \delta) \quad \hat{\delta}_{A}^{-} \circ \hat{\delta}_{A}^{-} = \mathbf{1}_{A}, \quad \hat{\delta}_{A}^{-} \circ \hat{\delta}_{A}^{-} = 1_{A \lor \bot},
\]

\[
(\hat{b} \hat{\delta}) \quad \hat{b}^{-}_{A,B,\bot} \circ \hat{\delta}_{A \lor B} = \mathbf{1}_{A \lor \bot},
\]

and, finally, with $\hat{\sigma}_{A}$ defined as in §3.1, the following equations:

\[
(d \hat{\delta}) \quad d_{T,B,C} \circ \hat{\delta}_{B \lor C} = \hat{\delta}_{B}^{-} \lor \mathbf{1}_{C},
\]

\[
(d \hat{\delta}) \quad \hat{\delta}_{C \land B} \circ d_{C,B,\bot} = \mathbf{1}_{C} \land \hat{\delta}_{B}^{-},
\]

\[
(\pi \pi) \quad \pi_{A,B}^{-} \circ \pi_{A,B}^{-} = \mathbf{1}_{A \land B}, \quad \pi_{A,B}^{-} \circ \pi_{A,B}^{-} = 1_{A \to B}.
\]

The set of equations of $\mathbf{SA}'$ is closed under symmetry and transitivity of equality and under the rules ($\text{cong } \xi$) for $\xi \in \{\circ, \land, \lor\}$ (see §2.1). This defines the equations of $\mathbf{SA}'$.

It is clear that in $\mathbf{SA}'$ we have the naturality equations ($\hat{\delta} \dashv \text{nat}$) (see §3.1) and

\[
(\hat{\delta} \dashv \text{nat}) \quad (f \lor \mathbf{1}_{\bot}) \circ \hat{\delta}_{A}^{-} = \hat{\delta}_{B}^{-} \circ f.
\]

Analogously to what we had in §3.1, we define $\hat{\sigma}_{A}^{-}$ and

\[
\hat{\sigma}_{A}^{-} = df \circ \hat{\delta}_{A}^{-} \circ \hat{\sigma}_{A,\bot}, \quad \hat{\sigma}_{A}^{-} = df \circ \hat{\delta}_{A,\bot} \circ \hat{\sigma}_{A}^{-},
\]

which give isomorphisms in $\mathbf{SA}'$. Note that $\hat{\sigma}_{A}^{-}$: $\bot \lor A \vdash A$ is analogous to $\hat{\Sigma}_{B,A}$: $(B \land \neg B) \lor A \vdash A$, though $\hat{\Sigma}_{B,A}$ is not an isomorphism. The equation ($\hat{b} \hat{\Sigma}$) of §2.2 is analogous to the following equation of $\mathbf{SA}'$ (an equation of monoidal categories):

\[
\hat{\delta}_{B \lor A}^{-} \circ \hat{b}^{-}_{B,A} = \hat{\sigma}_{B}^{-} \lor \mathbf{1}_{A}.
\]

The equations ($d \hat{\sigma}$) and ($d \hat{\delta}$), which amount to the equations ($\hat{\sigma} d^{L}$) and ($\hat{\delta} d^{L}$) of Section 7.9 of [22] (these equations stem from [11], Section 2.1), are analogous to the equations ($d \hat{\Sigma}$) and ($d \hat{\Delta}$) of §2.2. The equations ($\pi \pi$) are auxiliary, and will be discarded in a language where $\to$ is not primitive (see §3.8).
CHAPTER 3. STAR-AUTONOMOUS CATEGORIES

§3.4. SA' in SA

Our purpose now is to define the SA' structure in SA, and then show that the equations of SA' hold in SA for this defined structure.

To define \( \bar{i}_{A,B,C} \) and \( \bar{\nu}_{A,B,C} \) in SA we need some preliminary definitions. We note first that in SMC, and hence also in SA, we can introduce the following definitions:

\[
\begin{align*}
\bar{i}_{A,B,C} &= \text{df} \ (1_A \to ((1_B \to (\varepsilon_{A\land B,C} \cdot (\hat{\varepsilon}_{B,A} \land 1_{(A\land B)\to C}) \cdot \hat{\nu}_{B,A,(A\land B)\to C}) \cdot \eta_{B,A,(A\land B)\to C}) : (A \land B) \to C) \vdash A \to (B \to C), \\
\bar{\nu}_{A,B,C} &= \text{df} \ (1_A \to (1_B \to (\varepsilon_{B,C} \cdot (1_B \land \varepsilon_{A,B,C} \to C) \cdot \hat{\nu}_{B,A,(B\to C)}) \cdot (\hat{\varepsilon}_{A,B} \land 1_{C})) \cdot \eta_{A\land B,A,(B\to C)} : A \to (B \to C) \vdash (A \land B) \to C.
\end{align*}
\]

By SMC Coherence of §3.1, we can immediately conclude that in SMC, and hence also in SA, the arrows \( \bar{i}_{A,B,C} \) and \( \bar{\nu}_{A,B,C} \) are isomorphisms inverse to each other. By applying naturality and dinaturality equations, we can also conclude that \( \bar{i} \) and \( \bar{\nu} \) are natural transformations of SMC and SA in all their three indices \( A, B \) and \( C \).

In SA we have the following definitions:

\[
\begin{align*}
\bar{j}_{A,B} &= \text{df} \ \nu_{A\land B\to \bot} \cdot (\bar{i}_{A,B\to \bot,\bot} \to 1_\bot) \cdot ((1_A \to \nu_{B\to \bot} \to 1_\bot) : (A \to B) \to \bot \vdash A \land (B \to \bot), \\
\bar{j}_{A,B,C} &= \text{df} \ \nu_{A\land (B\to C)} \cdot (\bar{i}_{A,B\to \bot,\bot,\bot} \to 1_\bot) \cdot ((1_A \to \nu_{B\to \bot} \to 1_\bot) \cdot \nu_{A\land (B\to C),C} : A \land (B \to C) \vdash (A \to B) \to C.
\end{align*}
\]

The definition of \( \bar{j}_{A,B,C} \) can be given already in SMC, but not the definition of \( \bar{j}_{A,B} \). It is easy to see that in SA we have that \( \bar{j}_{A,B} \) and \( \bar{j}_{A,B,\bot} \) are isomorphisms inverse to each other.

We also have the following definitions in SA:

\[
\begin{align*}
\bar{b}_{A,B,C} &= \text{df} \ \bar{j}_{A,B} \to 1_C \cdot \bar{i}_{A,B\to \bot,\bot,C} : A \to (B \to \bot) \to C) \vdash (A \to B) \to \bot) \to C, \\
\bar{b}_{A,B,C,D} &= \text{df} \ \bar{i}_{A,B\to D,\bot,C} \cdot (\bar{j}_{A,B,D} \to 1_C) : ((A \to B) \to D) \to C \vdash A \to ((B \to D) \to C).
\end{align*}
\]
The second of these definitions can be given already in SMC. It is easy to see that in SA we have that \( b_{A,B,C}^\perp \) and \( b_{A,B,C,\perp}^\perp \) are isomorphisms inverse to each other.

If \( \perp \) is an arbitrary object, and we define \( A \lor B \) as \( (A \to \perp) \to B \) and \( f \lor g \) as \( (f \to 1_\perp) \to g \), then we can check that in SMC, and hence also in every symmetric monoidal closed category, we have for this defined \( \lor \) the bifunctorial equations (\( \lor 1 \)) and (\( \lor 2 \)) (see §2.1), while for \( b_{A,B,C}^- \) replaced by \( b_{A^{-1},B,C,\perp}^- \) we have \( (b^- \text{ nat}) \) and (\( b^- 5 \)) of §2.1. So every symmetric monoidal closed category is a semiassociative category in the sense of Section 4.2 of [22]. Therefore, every cartesian closed category, and in particular monoidal closed category is a semiassociative category in the sense of Section 4.3 of [22]. We can check that in Set of sets with functions, is a semiassociative category with \( A \lor B \) being \( (A \to \perp) \to B \), commonly written \( B^{(\perp \perp)} \), where \( \perp \) is an arbitrary set, not necessarily the initial object \( \emptyset \) of Set. For \( \perp \) distinct from \( \emptyset \) and from a singleton, we have that \( A \lor (B \lor C) \) need not be isomorphic to \( (A \lor B) \lor C \).

With \( A \lor B \) being \( (A \to \emptyset) \to B \), the category \( \text{Set} \) is an associative category in the sense of Section 4.3 of [22]. We can check that in \( \text{Set} \) the arrow \( b_{A,B,C}^- \), defined as \( \lambda_{A \to \emptyset,B,C,\emptyset} \), is an isomorphism, and hence in \( \text{Set} \) we have a natural transformation whose members are of type

\[
(A \to \emptyset) \to ((B \to \emptyset) \to C) \vdash (((A \to \emptyset) \to B) \to \emptyset) \to C;
\]

this defines the inverse \( \tilde{b}_{A,B,C} \) of \( b_{A,B,C}^- \), i.e. of \( \lambda_{A \to \emptyset,B,C,\emptyset} \).

To define \( \tilde{c}_{A,B} : B \lor A \vdash A \lor B \) in SA, we need some further preliminary definitions. In SMC we have

\[
s_{A,B,C} = df \ (1_{B \to C} \to (1_A \to (\varepsilon_{B,C} \land 1_{B \to C}) \to \tilde{b}_{A,A \to B,B \to C})) \to
\]

\[
\eta_{A,A \to B,B \to C} \to \tilde{b}_{B,C,A,B} \to \tilde{c}_{B,C,A,B} \to \eta_{B \to C,A \to B} : A \to B \vdash (B \to C) \to (A \to C).
\]
By applying naturality and dinaturality equations, we can verify that \( s \) is a natural transformation of \( \text{SA} \) in its first two indices \( A \) and \( B \), and a dinatural transformation in its third index \( C \).

We have in \( \text{SA} \) the following definitions based on \( s \):

\[
\check{c}_{A,B} = \text{df} (1_{A \rightarrow \bot} \rightarrow \nu_{B \rightarrow \bot})^* s_{B \rightarrow \bot,A \rightarrow \bot} : (B \rightarrow \bot) \rightarrow A \vdash (A \rightarrow \bot) \rightarrow B,
\]

\[
\check{c}_{A,B} = \text{df} \nu_{A,B}^{-1} \ast \check{c}_{A,B} \ast \nu_{B,A}^{-1} : B \lor A \vdash A \lor B.
\]

Next we have in \( \text{SMC} \):

\[
d_{A,B,C,D} = \text{df} (i_B \rightarrow A,B,D \rightarrow 1_C) \ast (j_B \leftarrow A,B \rightarrow D,C) : A \land ((B \rightarrow D) \rightarrow C) \vdash ((A \land B) \rightarrow D) \rightarrow C,
\]

\[
d_{C,B,A,D}^R = \text{df} (1_C \rightarrow D) \rightarrow ((\varepsilon_{C \rightarrow D,B} \land 1_A) \ast (\beta_{C \rightarrow D,B,C \rightarrow B,A}^{-1})^*)
\ast (\beta_{C \rightarrow D,B,C \rightarrow B,A}^{-1})^{-1} : ((C \rightarrow D) \rightarrow B) \land A \vdash (C \rightarrow D) \rightarrow (B \land A).
\]

Note that both \( d \) and \( d^R \) are natural transformations in all their four indices.

If, as above, we define \( A \lor B \) in \( \text{Set} \) as \( (A \rightarrow \emptyset) \rightarrow B \), and \( b_{A,B,C}^{-1} \) as \( b_{A,B,C}^{-1,\emptyset} \), while \( b_{A,B,C}^{-1,\emptyset} \) is its inverse, and if, moreover, \( \land \) is cartesian product, while \( d_{A,B,C}^L \) is \( d_{A,B,C,\emptyset} \) and \( d_{C,B,A}^R \) is \( d_{C,B,A,\emptyset}^R \), then we can check that \( \text{Set} \) with this structure is a net category in the sense of Section 7.2 of [22]. To verify the equations of net categories (which stem from [11]), it here helps a lot to apply \( \text{SMC}^c \) Coherence. In this net structure of \( \text{Set} \) all arrows with the same source and target are equal, i.e. all diagrams commute; this follows from the Net Coherence of [22] (Section 7.3). This net structure of \( \text{Set} \) is not symmetric, because \( (A \rightarrow \emptyset) \rightarrow B \) need not be isomorphic to \( (B \rightarrow \emptyset) \rightarrow A \). Note that \( \text{Set} \) with \( \land \) being cartesian product and \( \lor \) being disjoint union cannot be a net category for any definition of \( d^L \) and \( d^R \) (see [11], Section 3, and [22], Section 11.3).

With the help of \( d_{A,B,C,\bot} \), we have the following definition in \( \text{SA} \):

\[
d_{A,B,C} = \text{df} \nu_{A,B,C}^{-1} \ast d_{A,B,C,\bot} \ast (1_A \land \nu_{B,C}^{-1}) : A \land (B \lor C) \vdash (A \land B) \lor C,
\]

and for \( f : A \vdash D \) and \( g : B \vdash E \) also the following:
§3.4. SA' in SA

\[ f \lor g \overset{df}{=} v_{D,E}^{-1}((f \to 1_\bot) \to g) \circ v_{A,B}^{-1} : A \lor B \vdash D \lor E. \]

With that we have finished defining what was missing to obtain the DS structure in SA. (We already have in SA the arrows \(1_A, \hat{b}_{A,B,C}^-, \hat{b}_{A,B,C}^+\) and \(\hat{c}_{A,B}\), and the operations on arrows \(\circ\) and \(\land\).)

To define \(\hat{\Delta}_{B,A} : A \vdash A \land (\neg B \lor B)\) and \(\hat{\Sigma}_{B,A} : (B \land \neg B) \lor A \vdash A\) in SA we introduce first the following preliminary definitions in SMC:

\[ \Delta_{B,A} = df (1_A \land ((1_B \to \hat{\delta}_B^{-1} \circ \eta_{B,\top})) \circ \hat{\delta}_A^{-1} : A \vdash A \land (B \to B), \]

for \(E\) being \(B \land (B \to C)\),

\[ \Sigma_{B,A,C} = df \varepsilon_{E\to E,A} \circ \hat{c}_{E\to E} \circ \Delta_{E\to E\to E} \circ \Delta_{E\to E\to A} \circ ((1_E \to \varepsilon_{B,C}) \to 1_A) : (B \land (B \to C)) \to A \vdash A. \]

These definitions are not the only possible. It is easy to see with the help of SMC coherency that many other definitions would do, and, in particular, shorter definitions of \(\Sigma_{B,A,C}\) are possible. (The present one was chosen to facilitate calculation in §3.7 below.)

Then we have the following definitions in SA:

\[ \hat{\Delta}_{B,A} = df (1_A \land (v_{\neg B,\bot}^{-1} \circ ((v_B^{-1} \circ (\lambda_B^{-1} \to 1_\bot)) \to 1_B))) \circ \hat{\Delta}_{B,A} : A \vdash A \land (\neg B \lor B), \]

\[ \hat{\Sigma}_{B,A} = df \Sigma_{B,A,\bot} \circ ((1_B \land \lambda_B^{-1} \to 1_\bot) \to 1_A) \circ v_{B\land \neg B,\bot}^{-1} : (B \land \neg B) \lor A \vdash A. \]

For the remainder of the SA' structure we have the following definitions in SA:

\[ \hat{\delta}_A^- = df v_A^{-1} \circ v_{A,\bot}^{-1} : A \lor \bot \vdash A, \]

\[ \hat{\delta}_A^+ = df v_{A,\bot}^{-1} \circ v_{A,\bot}^{-1} : A \vdash A \lor \bot, \]

\[ \pi_{A,B}^- = df v_{\neg A,B}^{-1} \circ ((\nu_A^{-1} \circ (\lambda_A \to 1_\bot)) \to 1_B) : A \to B \vdash \neg A \lor B, \]

\[ \pi_{A,B}^+ = df ((\lambda_A \to 1_\bot) \circ v_A^{-1} \circ 1_B) \circ v_{\neg A,B}^{-1} : \neg A \lor B \vdash A \to B. \]
Note that with \( \pi_{A,B} \) defined as above we have

\[
\hat{\Delta}_{B,A} = df \left( 1_{A} \land \pi_{B,B} \right) \ast \hat{\Delta}_{B,A}.
\]

With that we have finished defining what was missing to obtain the \( \mathbf{SA}' \) structure in \( \mathbf{SA} \).

It remains now to verify that the equations of \( \mathbf{SA}' \) hold for this defined structure in \( \mathbf{SA} \). The equations \((\lor 1)\), \((\lor 2)\), \((\check{b} \text{ nat})\), \((\check{c} \text{ nat})\) and \((\check{b} \check{b})\) of §2.1 are trivial to check. (We rely here and later on isomorphism equations without mentioning that explicitly.) For \((\check{b} \check{c})\) we appeal to \( \mathbf{SMC}^c \) Coherence, while for \((\check{c} \check{c})\) we need some preparation.

We have the following equations in \( \mathbf{SA} \):

\[
(\nu_{A\rightarrow \bot,\bot}^A) \quad \nu_{A\rightarrow \bot,\bot}^A = \nu_A^A \rightarrow 1_{\bot,\bot},
\]

\[
(\nu_{A\rightarrow \bot}^A) \quad \nu_{A\rightarrow \bot}^A = \nu_{A,\bot}^A \rightarrow 1_{\bot}.
\]

To prove these equations it is enough to establish

\[
(\nu_{A,B}^A \rightarrow 1_{B}) \ast \nu_{A\rightarrow B,B}^A = 1_{A\rightarrow B},
\]

which follows from \( \mathbf{SMC}^c \) Coherence, and then use isomorphism equations.

To verify \((\check{c} \check{c})\) we use the following:

\[
\check{c}_{A,B} \ast \check{c}_{B,A} = (1_{A\rightarrow \bot} \rightarrow \nu_B^A) \ast s_{B\rightarrow \bot,A,\bot} \ast (1_{B\rightarrow \bot} \rightarrow \nu_A^B) \ast s_{A\rightarrow \bot,B,\bot}
\]

\[
= (1_{A\rightarrow \bot} \rightarrow \nu_B^A) \ast (\nu_A^A \rightarrow 1_{\bot}) \rightarrow 1_{(B\rightarrow \bot)\rightarrow \bot} \ast s_{B\rightarrow \bot,(A\rightarrow \bot)\rightarrow \bot} \ast s_{A\rightarrow \bot,B,\bot}, \text{ by the naturality of } s,
\]

\[
= (1_{A\rightarrow \bot} \rightarrow \nu_B^A) \ast (\nu_{A\rightarrow \bot,\bot}^A \rightarrow 1_{(B\rightarrow \bot)\rightarrow \bot}) \ast s_{B\rightarrow \bot,(A\rightarrow \bot)\rightarrow \bot} \ast s_{A\rightarrow \bot,B,\bot}, \text{ by } \mathbf{SMC}^c \text{ Coherence},
\]

\[
= 1_{1_{(A\rightarrow \bot)\rightarrow \bot}}.
\]

To verify \((\check{b} \check{c})\) we use the following equation of \( \mathbf{SA} \):

\[
(1_{B\rightarrow \bot} \rightarrow \check{c}_{A,C}) \ast \check{b}_{B\rightarrow \bot,c,A,\bot} \ast \check{c}_{(B\rightarrow \bot)\rightarrow C,A} \ast \check{b}_{A\rightarrow \bot,B,C,\bot} \ast ((\check{c}_{A,B} \rightarrow 1_{\bot}) \rightarrow 1_{C}) = \check{b}_{B\rightarrow \bot,A,C,\bot}.
\]
To verify this equation we use the naturality of $\mathcal{b}^\rightarrow$ and $s$, and the equation $(\nu^\leftarrow_{A \rightarrow \bot, \bot})$ where $A$ is instantiated by $A$ and $B$. In this verification, for the arrow $g(p, q, r)$ of $\text{SMC}$ defined as

$$(1_{q \rightarrow \bot} \to ((\nu^\leftarrow_{p \rightarrow \bot, \bot} \to 1_{(r \rightarrow \bot) \to \bot})) \circ 1_{q \rightarrow \bot} \to r),$$

where $\bot$ is an arbitrary letter, we have that $Gg(p, q, r)$ corresponds to the diagram

$$(((q \to \bot) \to p) \to \bot) \to r$$

and so, by $\text{SMC}^c$ Coherence, the following holds:

$$g(A, B, C) = (1_{B \rightarrow \bot} \to (1_{A \rightarrow \bot} \to \nu^\leftarrow_{C, \bot})) \circ 1_{B \rightarrow \bot} \to A, C, \bot.$$ 

The equation $(d \text{ nat})$ is easily inferred from the naturality of $d$ in all its four indices. To verify the equations $(d \wedge)$ and $(d \vee)$ we apply $\text{SMC}^c$ Coherence.

For the equations $(d \hat{b})$ and $(d \hat{b})$ we verify first that for

$$d^R_{C, B, A} = d^R_{C, B, A} \circ (\hat{c}_{A, B} \lor \hat{1}_C) \circ d_{A, B, C} \circ (\hat{1}_A \wedge \hat{c}_{B, C}) \circ \hat{c}_{C \lor B, A}$$

of type $(C \lor B) \land A \vdash C \lor (B \land A)$ (see §2.1) we have in $\text{SA}$

$$d^R_{C, B, A} = \nu^\leftarrow_{C, B, \land A} \circ d^R_{C, B, A, \bot} \circ (\nu^\leftarrow_{C, B, \bot} \land 1_A).$$

For that we use the naturality of $s$ and $d$, the equation $(\nu^\leftarrow_{A \rightarrow \bot, \bot})$, and $\text{SMC}^c$ Coherence. After this verification, we use essentially $\text{SMC}^c$ Coherence to obtain $(d \hat{b})$ and $(d \hat{b})$ in $\text{SA}$. With that we have all the equations of $\text{DS}$ in $\text{SA}$.

We pass now to the $\text{PN}$ equations of §2.2. It is trivial to check in $\text{SA}$ the equations $(\hat{\Delta} \text{ nat})$ and $(\hat{\Sigma} \text{ nat})$. For the equations $(\hat{b} \hat{\Delta})$, $(\hat{b} \hat{\Sigma})$ and $(d \hat{\Sigma})$ we apply various naturality equations and $\text{SMC}^c$ Coherence.
For the equation $(d \Delta)$ we verify first that we have
\[
\nu_{C \land B, \bot} = d_{C, B, \bot, \bot} \circ (1_C \land \nu_{B, \bot}',)
\]
b y $\text{SMC}^c$ Coherence; so in $\text{SA}$ we have
\[
\nu_{C \land B, \bot} \circ (1_C \land \nu_B') = d_{C, B, \bot, \bot}.
\]
We use this equation, together with the naturality of $\hat{\Sigma}$ and $\text{SMC}^c$ Coherence, to verify for $E$ being $A \land (A \to \bot)$
\[
\hat{\Sigma}_{A, C \land B, \bot} \circ (1_E \to \bot) \to \nu_{C \land B, \bot} = d_{C, B, \bot, \bot} \circ (1_E \to \bot, \bot) ,
\]
from which $(d \Delta)$ follows.

For $(\bigvee \Sigma') \land (\bigvee \Delta')$ it is enough to verify the following:
\[
\hat{\Sigma}_{A, A, \bot} \circ d_{A, A \to \bot, A, \bot} \circ (1_A \land (\nu_A' \to 1_A)) \circ \hat{\Delta}_{A, A}
\]
\[
= 1_A \circ \nu_A' \to A \to \bot \circ (1_A \land (\nu_A' \to 1_A)) \circ \hat{\Delta}_{A, A},
\]
by $\text{SMC}^c$ Coherence,
\[
= 1_A, \quad \text{by $(\nu \nu)$ and $\text{SMC}^c$ Coherence.}
\]

For $(\hat{\Sigma}' \hat{\Delta}')$ it is enough to verify the following:
\[
\hat{\Sigma}_{A, A \to \bot, \bot} \circ (\hat{\delta}_{A, A \to \bot, A, \bot} \circ d_{A \to \bot, A, A \to \bot, \bot} \circ (1_A \land (\nu_A' \to 1_A))) \circ \hat{\Delta}_{A, A \to \bot} = 1_A \to \bot,
\]
which is done by using the equation $(\nu_{A \land A}^{-})$, the naturality of $s$, the first $(\nu \nu)$ equation and $\text{SMC}^c$ Coherence. With that we have finished verifying the $\text{PN}$ equations in $\text{SA}$.

It remains to deal with the equations introduced in the preceding section. It is trivial to verify in $\text{SA}$ the equations $(\hat{\delta} \to \text{nat})$ and $(\hat{\delta} \hat{\delta})$, while for $(\hat{\delta} \hat{\delta})$ we apply the naturality of $\nu^{-}$ and $\text{SMC}^c$ Coherence. For $(d \hat{\delta})$ we rely again on $\text{SMC}^c$ Coherence, which also delivers readily
\[
\hat{\delta}_{C \land B} = d_{C, B, \bot} \circ (1_C \land \delta_B^{'})
\]
§3.5. The category $\text{PN}^\neg \rightarrow \bot$

—an equation that, in the presence of $(\delta \delta)$, amounts to $(d \delta)$. It is trivial to verify the equations $(\pi \pi)$, and with that we have finished verifying all the equations of $\text{SA}'$ in $\text{SA}$.

§3.5. The category $\text{PN}^\neg \rightarrow \bot$

As an auxiliary for the proof of the isomorphism of the categories $\text{SA}$ and $\text{SA}'$, which will be completed in §3.7, we introduce a category intermediary between $\text{PN}^\neg$ and $\text{SA}'$ equivalent with $\text{PN}^\neg$, which we call $\text{PN}^\neg \rightarrow \bot$. As a consequence of the equivalence of $\text{PN}^\neg \rightarrow \bot$ with $\text{PN}^\neg$, we will obtain a coherence result for $\text{PN}^\neg \rightarrow \bot$ with respect to $\text{Br}$, and this will enable us to shorten very considerably calculations in $\text{SA}'$ in the next two sections.

The objects of the category $\text{PN}^\neg \rightarrow \bot$ are all the formulae of the language $L_{\top, \bot, \neg, \land, \lor}$ generated by $P$ in which $\top$ does not occur and in which $\bot$ occurs only in subformulae of the form $A \rightarrow \bot$. The arrow terms of $\text{PN}^\neg \rightarrow \bot$ are defined as those of $\text{PN}^\neg$ (in the extended language of formulae), save that among the primitive arrow terms we also have

$$\pi^\neg_{A,B} : A \rightarrow B \vdash \neg A \lor B, \quad \pi^\neg_{A,B} : \neg A \lor B \vdash A \rightarrow B,$$

where $B$ cannot be $\bot$, since $\neg A \lor \bot$ is not an object of $\text{PN}^\neg \rightarrow \bot$.

$$\lambda^\neg_A : \neg A \vdash A \rightarrow \bot, \quad \lambda^\neg_A : A \rightarrow \bot \vdash \neg A.$$

The equations of $\text{PN}^\neg \rightarrow \bot$ are defined as those of $\text{PN}^\neg$ plus the equations $(\pi \pi)$ of §3.3 and $(\lambda \lambda)$ of §3.2. All the equations of $\text{PN}^\neg \rightarrow \bot$ will hold in $\text{SA}'$ once $\lambda^\neg_A$ and $\lambda^\neg_A$ are defined in $\text{SA}'$ as in the next section. This is an important fact for the applications we will make of the coherence of $\text{PN}^\neg \rightarrow \bot$ in the next two sections.

We will now show that $\text{PN}^\neg$ and $\text{PN}^\neg \rightarrow \bot$ are equivalent categories. From $\text{PN}^\neg$ to $\text{PN}^\neg \rightarrow \bot$ we have a functor $I$ such that $IA$ is $A$ for every object $A$ of $\text{PN}^\neg \rightarrow \bot$, and $If$ is $f$ for every arrow term $f$ of $\text{PN}^\neg$. From $\text{PN}^\neg \rightarrow \bot$ to $\text{PN}^\neg$ we have a functor $H$ defined inductively as follows. On objects we have

$$Hp = p, \quad \text{for } p \text{ a letter},$$

$$H\neg A = \neg HA,$$
On arrow terms we have

\[ H(\alpha; A_1, \ldots, A_n) = \alpha; H(A_1), \ldots, H(A_n), \]

for \( \alpha; A_1, \ldots, A_n \) a primitive arrow term different from \( \pi_{\rightarrow A, B}, \pi_{\leftarrow A, B}, \lambda_{\rightarrow A} \) and \( \lambda_{\leftarrow A} \),

\[ H(\pi\rightarrow A, B) = H(\pi\leftarrow A, B) = 1, \]
\[ H(\lambda\rightarrow A) = H(\lambda\leftarrow A) = 1, \]
\[ H(f; \xi; g) = H(f; \xi; H(g)), \quad \text{for } \xi \in \{ \circ, \land, \lor \}. \]

It is clear that for every object \( A \) of \( \mathbf{PN}^- \) we have that \( HI_A = HA = A \).

For every object \( A \) of \( \mathbf{PN}^- \) we have \( IH_A = HA \), and we define an arrow \( h_A; HA \vdash A \) of \( \mathbf{PN}_- \), which is a member of a natural isomorphism of \( \mathbf{PN}_- \) (natural in \( A \)), with inverse \( h_{A}^{-1}; A \vdash HA \). The arrows \( h_A \) and \( h_A^{-1} \) are defined inductively as follows:

\[ h_p = h^{-1}_p = 1_p, \quad \text{for } p \text{ a letter}, \]
\[ h_{A;B} = h_A; \xi; h_B, \quad h_{A;B}^{-1} = h_A^{-1}; \xi; h_B^{-1}, \]

for \( \xi \in \{ \land, \lor \}, \)

\[ h_{A;\to} = \neg h_A^{-1}, \quad h_{A;\to}^{-1} = \neg h_A, \]

where the operation \( \neg \) on arrows is defined as in §2.8,

\[ h_{A;\to B} = \pi_{A;B}^{-1} \circ (\neg h_B^{-1} \lor h_B), \quad h_{A;\to B}^{-1} = (\neg h_A \lor h_B^{-1}) \circ \pi_{A;B}, \]

if \( B \) is not \( \bot \),

\[ h_{A;\to \bot} = \lambda_A^{-1} \circ \neg h_A^{-1}, \quad h_{A;\to \bot}^{-1} = \neg h_A \circ \lambda_A. \]

For \( f; A \vdash B \) we prove that we have

\[ f \circ h_A = h_B \circ Hf \]

in \( \mathbf{PN}_- \) by induction on the complexity of the arrow term \( f \) of \( \mathbf{PN}_- \). In this induction we rely on various bifunctorial, naturality and isomorphism
§3.6. **SA in SA′**

We rely on \((\Delta \ \text{dinet})\) and \((\Sigma \ \text{dinet})\) of §2.8, in addition to \((\Delta \ \text{nat})\) and \((\Sigma \ \text{nat})\) of §2.2, when \(f\) is \(\Delta_{B,A}\) and \(\Sigma_{B,A}\). This establishes that \(h\) is a natural isomorphism, and it follows that the categories \(\mathbf{PN}_{\perp}^{-}\) and \(\mathbf{PN}^{-}\) are equivalent via the functors \(H\) and \(I\).

Let \(G_{\alpha}\) be an identity arrow of \(Br\) when \(\alpha\) is \(\pi_{\overline{A,B}}, \pi_{\overline{A,B}}, \lambda_{\overline{A}}\) or \(\lambda_{\overline{A}}\).

With other clauses for \(G\) being as for \(\mathbf{PN}_{\perp}^{-}\), we obtain a functor \(G\) from \(\mathbf{PN}_{\perp}^{-}\) to \(Br\). Note that \(G_{A} = G_{HA}\).

For every arrow \(f\) of \(\mathbf{PN}_{\perp}^{-}\), we have that \(Gf = GHf\). Then we can prove the following (cf. the proof at the end of §2.7).

**\(\mathbf{PN}_{\perp}^{-}\) Coherence.** The functor \(G\) from \(\mathbf{PN}_{\perp}^{-}\) to \(Br\) is faithful.

**Proof.** Suppose that for \(f\) and \(g\) arrows of \(\mathbf{PN}_{\perp}^{-}\) of the same type we have \(Gf = Gg\). Then \(GHf = GHg\), and \(Hf = Hg\) in \(\mathbf{PN}^{-}\) by \(\mathbf{PN}_{\perp}^{-}\) Coherence. It follows that \(f = g\) in \(\mathbf{PN}_{\perp}^{-}\) by the equivalence of \(\mathbf{PN}_{\perp}^{-}\) with \(\mathbf{PN}^{-}\). \(\square\)

Note that in \(\mathbf{PN}_{\perp}^{-}\) we can define \(\nu_{\overline{A,B}} : (A \lor B) \vdash (A \rightarrow \bot) \rightarrow B\) and \(\nu_{\overline{A,B}} : (A \rightarrow \bot) \rightarrow B \vdash A \lor B\) as in \(\mathbf{SA}'\) in the next section, and it is easy to see that the equations \((\nu \nu)\) of §3.2 hold for these defined arrows.

This section is opposite to §3.4. We will define in it the \(\mathbf{SA}\) structure in \(\mathbf{SA}'\), and then we will show that the equations of \(\mathbf{SA}\) hold in \(\mathbf{SA}'\) for this defined structure.

First, we have the following definitions in \(\mathbf{SA}'\):

\[
\varepsilon_{A,B} = \text{df} \ \hat{\Sigma}_{A,B} \cdot d_{A,\overline{A,B}} \cdot (1_A \land \overline{\pi_{\overline{A,B}}}) : A \land (A \rightarrow B) \vdash B,
\]

\[
\eta_{A,B} = \text{df} \ \overline{\pi_{\overline{A,B}}} \cdot d_{\overline{A,B}} \cdot \hat{\Sigma}_{A,B} : B \vdash (A \land B),
\]

\[
A \rightarrow f = \text{df} \ \overline{\pi_{\overline{A,C}}} \cdot (1_A \lor f) \cdot \overline{\pi_{\overline{A,B}}} : A \rightarrow B \vdash A \rightarrow C, \quad \text{for } f : B \vdash C.
\]

With that we have defined what was missing to obtain the \(\mathbf{SMC}\) structure in \(\mathbf{SA}'\). (We already have in \(\mathbf{SA}'\) the arrows \(1_A, \hat{b}_{\overline{A,B,C}}, \hat{b}_{\overline{A,B,C}}, \hat{c}_{A,B}, \hat{d}_{\overline{A}}\) and \(\hat{\delta}_{\overline{A}}\), and the operations on arrows \(\circ\) and \(\land\).)
CHAPTER 3. STAR-AUTONOMOUS CATEGORIES

For the remainder of the SA structure we have the following definitions in SA':

\[
\begin{align*}
\lambda_A^\perp &= df \pi_{A,\perp} \circ \delta_{\neg A} : \neg A \vdash A \to \perp, \\
\lambda_A &= df \delta_{\neg A} \circ \pi_{A,\perp} : A \to \perp \vdash \neg A, \\
\nu_A &= df n_A \circ \lambda_A \circ \lambda_A^\perp : (A \to \perp) \to \perp \vdash A, \\
\nu_{A,B} &= df \pi_{A,\perp,B} \circ ((\neg \lambda_A \circ n_A^\perp) \lor 1_B) : A \lor B \vdash (A \to \perp) \to B, \\
\nu_{A,B} &= df ((n_A \circ \lambda_A^\perp) \lor 1_B) \circ \pi_{A,\perp,B} : (A \to \perp) \to B \vdash A \lor B,
\end{align*}
\]

where the operation \( \to \) on arrows is defined as in §2.8, the arrows \( \lambda_A^\perp, \lambda_A \) and \( \lambda_A^\perp \) on the right-hand sides are defined as above, while \( n_A^\perp : \neg \neg A \vdash A \) and \( n_A : A \vdash \neg \neg A \) are defined as in §2.6. With that we have finished defining what was missing to obtain the SA structure in SA'.

It is not difficult to show with the help of \((\wedge \Sigma) \text{ dinat})\) and \(\text{PN}^{-} \text{ Coherence}\) that for \(g : A \vdash D\) and \(f : B \vdash C\) in SA' we have the equation

\[
g \to f = \pi_{A,C} \circ (\neg g \lor f) \circ \pi_{D,B}
\]

where \(g \to f\) on the left-hand side is defined in SA' as in §3.1 in terms of \(\varepsilon_{D,C}, \eta_{A,D} \to B\) and the operations \(A \to D\) and \(D \to \), which are themselves defined in SA'.

We verify next that the equations of SA hold for the defined SA structure in SA'. For the equations of SMC of §3.1 we have that \((A \to 1)\) and \((A \to 2)\) are trivial to check, \((\varepsilon \text{ nat})\) and \((\eta \text{ nat})\) follow from various naturality equations, while \((\varepsilon \eta \wedge)\) and \((\varepsilon \eta \to)\) follow from \(\text{PN}^{-} \text{ Coherence}\). For the equations of §3.2 we have that \((\lambda \lambda)\) follow from \((\delta \delta)\) and \((\pi \pi)\) of §3.3, while for \((\nu \nu)\) we use \((\pi \pi)\), \((\lambda \lambda)\), the isomorphism of \(n_A^\perp\) (see §2.6) and \((\neg 2)\) of §2.8.

It remains only to verify \((\nu \nu)\) of §3.2. We will do that after some preparation, which will enable us to apply the \(\text{PN}^{-}_{\perp} \text{ Coherence}\) of the preceding section.

We have the following equation in SA':

\[
(1 \to \varepsilon) \quad 1_A \to \varepsilon_{B,\perp} = \lambda_A^\perp \circ \Delta_{\neg A} \circ (1_{\neg A} \lor (1_B \land \lambda_B^\perp)) \circ \pi_{A,B \land (B \to \perp)}
\]
§3.7. The isomorphism of \textit{SA} and \textit{SA}'

where \( \varepsilon_{B,\bot} \) and \( \to \) on the left-hand side \( 1_A \to \varepsilon_{B,\bot} \), which is equal to \( A \to \varepsilon_{B,\bot} \), are defined in \textit{SA}'', and so are \( \lambda^A_A \) and \( \lambda^B_B \) on the right-hand side. By definitions and isomorphism equations, for \( g \) being

\[
\delta_{\to A} \circ (1_{\to A} \lor \delta_{\to B}) = (1_{\to A} \lor (d_{B,\bot} \land \varepsilon_{B,\bot} \circ (1_{B} \land \delta_{B}))),
\]

in \textit{SA'} we have

\[
1_A \to \varepsilon_{B,\bot} = \lambda^A_A * g * (1_{\to A} \lor (1_B \land \lambda^A_B)) * \pi_{A,B}(B \to \bot).
\]

Next we have

\[
g = \delta_{\to A} \circ (\tilde{\Delta}_{B,\bot} \lor 1_{\bot}) * \tilde{\delta}_{B,A,B \land \bot} * (1_{\to A} \lor \delta_{B \land \bot}),
\]

by \( (\tilde{\delta}_{B} \lor \nu) \) and \( (\tilde{\delta}_{B}) \),

which establishes the equation \( (1 \to \varepsilon) \).

We can now verify \( (\nu \nu) \) by establishing in \textit{SA}' the equation

\[
\nu_{A \to \bot} = \lambda^A_A * g * \eta_{A \to \bot}.
\]

The left-hand side \( \nu_{A \to \bot} \) of this equation is equal to

\[
(1_{A \to \bot} \to \varepsilon_{A,\bot}) * (1_{A \to \bot} \to \hat{c}_{A \to \bot,A}) * \eta_{A \to \bot,A}
\]

(see §3.2), where we can replace \( 1_{A \to \bot} \to \varepsilon_{A,\bot} \) according to the equation \( (1 \to \varepsilon) \), and then apply \( \text{PN}_{A \to \bot} \) Coherence. With that we have finished verifying all the equations of \textit{SA} in \textit{SA}'.

§3.7. The isomorphism of \textit{SA} and \textit{SA}'

In this section we will show that \textit{SA} and \textit{SA}' are isomorphic categories. We have a functor \( F \) from \textit{SA}' to \textit{SA} that is identity on objects and that maps every arrow of \textit{SA}' to the homonymous arrow in the defined \textit{SA}' structure of \textit{SA}. For example,

\[
F \tilde{b}_{A,B,C} = \tilde{b}_{A,B,C}.
\]
where the $\tilde{b}_{A,B,C}$ on the right-hand side is defined as in §3.4. We define analogously a functor $F'$ from $\mathbf{SA}$ to $\mathbf{SA}'$ which is identity on objects and which maps every arrow of $\mathbf{SA}$ to the homonymous arrow in the defined $\mathbf{SA}$ structure of $\mathbf{SA}'$. That $F$ and $F'$ are indeed functors follows from what was established in §3.4 concerning the equations of $\mathbf{SA}'$ in $\mathbf{SA}$, and in the preceding section concerning the equations of $\mathbf{SA}$ in $\mathbf{SA}'$.

It is trivial that $F'F \alpha = \alpha$ and $FF' \alpha = \alpha$ are both $\alpha$. We will show next that $F'F \xi = \xi$ in $\mathbf{SA}'$, and $FF' \xi = \xi$ in $\mathbf{SA}$, from which it will follow that $\mathbf{SA}$ and $\mathbf{SA}'$ are isomorphic categories.

To verify $F'F \xi = \xi$ in $\mathbf{SA}'$, we have to verify this equation for $\xi$ being $\vee b \rightarrow A,B,C$, $\vee b \leftarrow A,B,C$, $\vee c A,B$, $\land \Delta B,A$, $\lor \Sigma B,A$, $\lor \delta \rightarrow A$, $\lor \delta \leftarrow A$, $\pi \rightarrow A,B$ and $\pi \leftarrow A,B$, and we also have to verify that in $\mathbf{SA}'$ we have $(F'F \xi) F F' \xi = F'F \xi F'F \xi$ for $\xi$ being $\vee$. It is trivial that in $\mathbf{SA}'$ the equation $F'F \xi = \xi$ holds for $\xi$ being $1_A$, $\tilde{b}_{A,B,C}$, $\tilde{b}_{A,B,C}$, $\tilde{c}_{A,B}$, $\Delta_{B,A}$, $\Sigma_{B,A}$, $\tilde{\delta} \rightarrow A$, $\tilde{\delta} \leftarrow A$, $\tilde{\pi}_{A,B}$ and $\tilde{\pi}_{A,B}$, and we also need to verify that $\mathbf{PN} \searrow \swarrow \coherence$.
§3.7. The isomorphism of $\mathbf{SA}$ and $\mathbf{SA}'$

$$F'F\check{b}_{A,B,C} = F'F\check{b}_{A,B,C} \ast \check{b}_{A,B,C} \ast \check{b}_{A,B,C} = F'F\check{b}_{A,B,C} \ast F'F\check{b}_{A,B,C} \ast \check{b}_{A,B,C} = \check{b}_{A,B,C}.$$ 

by the functoriality of $F'$ and $F$, though $F'F\check{b}_{A,B,C} = \check{b}_{A,B,C}$ can also be verified directly with the help of $\mathbf{PN}_{\bot}$ Coherence (this is not such a short verification).

For $F'F\check{\Delta}_{B,A} = \check{\Delta}_{B,A}$ we have a verification very much analogous to the verification of $(\varepsilon \rightarrow \varepsilon)$ in the preceding section. In this verification we establish that in $\mathbf{SA}'$ we have

$$F'F\check{\Delta}_{B,A} = (1_A \land \pi_{B,B}) \ast \hat{\Delta}_{B,A},$$

and then, by using the equations $(f \rightarrow g)$ and $(\check{\Delta})$, together with $\mathbf{PN}_{\bot}$ Coherence, we obtain $F'F\check{\Delta}_{B,A} = \hat{\Delta}_{B,A}$. For $F'F\check{\Sigma}_{B,A} = \check{\Sigma}_{B,A}$ we use $(1 \rightarrow \varepsilon)$, $(f \rightarrow 1)$ and $(\check{\Delta})$, together with $\mathbf{PN}_{\bot}$ Coherence.

For $F'F\check{\delta}_A = \check{\delta}_A$ we use

$$\lambda_{A \rightarrow \bot} = \check{\delta}_A \ast \pi_{A \rightarrow \bot, \bot},$$

which holds in $\mathbf{SA}'$ by definition (see the preceding section), and then we apply $(\check{\delta} \leftarrow \text{nat})$, $(\sim 2)$ of §2.8, and isomorphism equations. We obtain $F'F\check{\delta}_A = \check{\delta}_A$ from $F'F\check{\delta}_A = \check{\delta}_A$ (see the verification of $F'F\check{b}_{A,B,C} = \check{b}_{A,B,C}$ above).

It remains to derive $(F'F \lor)$ in $\mathbf{SA}'$. For this rather straightforward derivation we use $(\sim 2)$ and $(\check{\delta} \leftarrow \text{nat})$, together with isomorphism and bifunctorial equations.

To verify $FF'f = f$ in $\mathbf{SA}$, we have to verify this equation for $f$ being $\varepsilon_{A,B}$, $\eta_{A,B}$, $\nu_A$, $\lambda_A$, $\nu_{A,B}$, $\nu_{A,B}$, and we also have to verify that in $\mathbf{SA}$ we have

$$FF'(A \rightarrow g) = FF'A \rightarrow FF'g.$$ 

For that we rely on lengthy, but also rather straightforward, derivations, in which we apply various bifunctorial, naturality, dinaturality and isomorphism equations. We also use the equations $(\nu_{A \rightarrow \bot, \bot})$ and $(\nu_{A \rightarrow \bot})$ of §3.4,
and we apply $\text{SMC}^c$ Coherence of §3.1. It is trivial that in $\text{SA}$ the equation $FF'f = f$ holds for $f$ being $1_A$, $\check{\delta}_A$, $\check{\delta}_{A,B}$, $\check{\delta}_{A,B,C}$, $\check{\gamma}_{A,B}$, $\check{\gamma}_{A,B,C}$, $\check{\delta}_{A}$ and $\check{\delta}_{A,C}$. The equations obtained from $(F'F\xi)$ for $\xi \in \{\lor, \land\}$ by replacing $F'F$ by $FF'$ hold trivially too. With that we have finished establishing that $\text{SA}$ and $\text{SA'}$ are isomorphic categories.

§3.8. The categories $\text{SA}_s$ and $\text{SA}'_s$

The objects of the category $\text{SA}_s$ are the formulae of the propositional language $\mathcal{L}_{\top, \bot, \land, \lor}$, which are the formulae of the propositional language $\mathcal{L}_{\top, \bot, \neg, \land, \lor, \rightarrow}$ of §3.2 in which $\neg$ and $\lor$ do not occur. The arrow terms of $\text{SA}_s$ are defined as those of $\text{SA}$ save that we omit in the definition the primitive arrow terms $\lambda_{A}$, $\lambda_{A}'$, $v_{A,B}$ and $v_{A,B}'$. The equations of $\text{SA}_s$ are defined as those of $\text{SA}$ save that we omit the equations (1) and (2). This means that to the equations assumed for $\text{SMC}$ we add only the equation (3). The category $\text{SA}_s$ is the free star-autonomous category in the sense of [2] (Section 2) generated by $\mathcal{P}$.

We will establish that $\text{SA}$ and $\text{SA}_s$ are equivalent categories. From $\text{SA}_s$ to $\text{SA}$ we have a functor $I$ such that $IA$ is $A$ for every object $A$ of $\text{SA}_s$, and $If$ is $f$ for every arrow term $f$ of $\text{SA}_s$.

From $\text{SA}$ to $\text{SA}_s$ we have a functor $H$ defined inductively as follows.

On objects we have

$$HA = A,$$  
for $A$ a letter, or $\top$, or $\bot$,

$$H\neg A = HA \rightarrow \bot,$$

$$H(A \xi B) = HA \xi HB,$$  
for $\xi \in \{\land, \rightarrow\}$,

$$H(A \lor B) = (HA \rightarrow \bot) \rightarrow HB.$$  

On arrow terms we have

$$H\alpha_{A_1,\ldots,A_n} = \alpha_{HA_1,\ldots,HA_n},$$

for $\alpha_{A_1,\ldots,A_n}$ a primitive arrow term different from $\lambda_{A}$, $\lambda_{A}'$, $v_{A,B}$ and $v_{A,B}'$,

$$H\lambda_{A} = H\lambda_{A}' = 1_{HA \rightarrow \bot},$$

$$Hv_{A,B} = Hv_{A,B}' = 1_{(HA \rightarrow \bot) \rightarrow HB},$$

for $\alpha_{A_1,\ldots,A_n}$ a primitive arrow term different from $\lambda_{A}$, $\lambda_{A}'$, $v_{A,B}$ and $v_{A,B}'$.
§3.8. The categories $\mathbf{SA}_s$ and $\mathbf{SA}'_s$ 

$H(f \xi g) = Hf \xi Hg,$ for $\xi \in \{\&\lor\}$,

$H(A \rightarrow f) = HA \rightarrow Hf.$

It is clear that for every object $A$ of $\mathbf{SA}_s$ we have that $HIA = HA = A$. For every object $A$ of $\mathbf{SA}$ we have $IHA = HA$, and we define an arrow $h_A: HA \vdash A$ of $\mathbf{SA}$, which is a member of a natural isomorphism of $\mathbf{SA}$ (natural in $A$), with inverse $h_A^{-1}: A \vdash HA$. The arrows $h_A$ and $h_A^{-1}$ are defined inductively as follows:

$h_A = h_A^{-1} = 1_A$, for $A$ a letter, or $\top$, or $\bot$

$h_{\neg A} = \lambda \neg A \cdot (h_A^{-1} \rightarrow 1_{\bot}),$

$h_{A \land B} = h_A \land h_B,$

$h_{A \lor B} = \nu_{A,B}^{-1} \cdot ((h_A \rightarrow 1_{\bot}) \rightarrow h_B),$

$h_{A \rightarrow B} = h_A^{-1} \rightarrow h_B,$

$h_{\neg \neg A} = \lambda^\neg\neg A \cdot (h_A \rightarrow 1_{\bot}) \rightarrow \lambda \neg A,$

$h_{A \land B}^{-1} = h_A^{-1} \land h_B^{-1},$

$h_{A \lor B}^{-1} = ((h_A^{-1} \rightarrow 1_{\bot}) \rightarrow h_B^{-1}) \cdot \nu_{A,B},$

$h_{A \rightarrow B}^{-1} = h_A \rightarrow h_B^{-1}$.

For $f: A \vdash B$ we prove that we have

$f \circ h_A = h_B \circ Hf$

in $\mathbf{SA}$ by induction on the complexity of the arrow term $f$ of $\mathbf{SA}$. In this induction we rely on various bifunctorial, naturality, dinaturality and isomorphism equations. This establishes that $h$ is a natural isomorphism, and it follows that the categories $\mathbf{SA}$ and $\mathbf{SA}_s$ are equivalent via the functors $H$ and $I$.

From this equivalence we can deduce that $\mathbf{SA}_s$ is isomorphic to a full subcategory of $\mathbf{SA}$. For every object $A$ of $\mathbf{SA}_s$ we have that $HA = A$ and $h_A = 1_A$. So, for $A$ and $B$ objects of $\mathbf{SA}_s$ and $f: A \vdash B$ an arrow term of $\mathbf{SA}$, there is an arrow term $Hf: A \vdash B$ of $\mathbf{SA}_s$ such that in $\mathbf{SA}$ we have $f = Hf$, because $h_A = 1_A$ and $h_B = 1_B$.

The objects of the category $\mathbf{SA}'_s$ are the formulae of the propositional language $\mathcal{L}_{\top,\bot,\neg,\land,\lor}$ generated by $\mathcal{P}$, which are the formulae of the propositional language $\mathcal{L}_{\top,\bot,\neg,\land,\lor,\rightarrow}$ of §3.2 in which $\rightarrow$ does not occur. The arrow terms of $\mathbf{SA}'_s$ are defined as those of $\mathbf{SA}'$ save that we omit in the definition the primitive arrow terms $\pi_{A,B}$ and $\pi^{-1}_{A,B}$. The equations of $\mathbf{SA}'_s$ are defined as those of $\mathbf{SA}'$ save that we omit the equations ($\pi\pi$).

With the definitions
\[ \begin{align*}
\tau_B^L & = df \, \hat{\delta}_{\neg B \vee B} \circ \hat{\Delta}_{B, \top} : \top \vdash \neg B \vee B, \\
\gamma_B^R & = df \, \hat{\Sigma}_{B, \bot} \circ \hat{\delta}_{B \wedge \neg B} : B \wedge \neg B \vdash \bot,
\end{align*} \]
in \( \mathbf{SA}'_s \), on the one hand, and
\[ \begin{align*}
\hat{\Delta}_{B,A} & = df \, (1_A \wedge \tau_B^L) \circ \hat{\delta}_{A} : A \vdash A \wedge (\neg B \vee B), \\
\hat{\Sigma}_{B,A} & = df \, \hat{\delta}_{A} \circ (\gamma_B^R \vee 1_A) : (B \wedge \neg B) \vee A \vdash A,
\end{align*} \]
on the other hand, it can easily be established that \( \mathbf{SA}'_s \) is isomorphic to the free symmetric linearly (alias weakly) distributive category with negation in the sense of [11] (Section 4, Definition 4.3) generated by \( \mathcal{P} \).

We will establish that \( \mathbf{SA}' \) and \( \mathbf{SA}'_s \) are equivalent categories. From \( \mathbf{SA}'_s \) to \( \mathbf{SA}' \) we have a functor \( I \) such that \( IA \) is \( A \) for every object \( A \) of \( \mathbf{SA}'_s \), and \( If \) is \( f \) for every arrow term \( f \) of \( \mathbf{SA}'_s \).

From \( \mathbf{SA}' \) to \( \mathbf{SA}'_s \) we have a functor \( H \) defined inductively as follows.

On objects we have
\[ \begin{align*}
HA & = A, \quad \text{for } A \text{ a letter, or } \top, \text{ or } \bot, \\
\neg A & = \neg HA, \\
H(A \xi B) & = HA \xi HB, \quad \text{for } \xi \in \{ \wedge, \vee \} \\
H(A \rightarrow B) & = \neg HA \vee HB.
\end{align*} \]

On arrow terms we have
\[ H\alpha_{A_1, \ldots, A_n} = \alpha_{HA_1, \ldots, HA_n} \]
for \( \alpha_{A_1, \ldots, A_n} \) a primitive arrow term different from \( \pi_{A,B}^{-} \) and \( \pi_{A,B}^{\top} \),
\[ \begin{align*}
H\pi_{A,B}^{-} & = H\pi_{A,B}^{\top} = 1_{\neg HA \vee HB}, \\
H(f \xi g) & = Hf \xi Hg, \quad \text{for } \xi \in \{ \wedge, \vee \}.
\end{align*} \]

It is clear that for every object \( A \) of \( \mathbf{SA}'_s \) we have that \( IHA = HA = A \).
For every object \( A \) of \( \mathbf{SA}' \) we have \( IHA = HA \), and we define an arrow \( h_A : HA \vdash A \) of \( \mathbf{SA}' \), which is a member of a natural isomorphism of \( \mathbf{SA}' \) (natural in \( A \)), with inverse \( h_A^{-1} : A \vdash HA \). The arrows \( h_A \) and \( h_A^{-1} \) are defined inductively as follows:
§3.8. The categories $\text{SA}_s$ and $\text{SA}_s'$

\[ h_A = h_A^{-1} = 1_A, \quad \text{for } A \text{ a letter, or } \top, \text{ or } \bot \]
\[ h_{\neg A} = \neg h_A^{-1}, \quad h_{\neg A} = \neg h_A, \]
\[ h_{A} \xi B = h_A \xi h_B, \quad h_{A}^{-1} \xi B = h_A^{-1} \xi h_B^{-1}, \quad \text{for } \xi \in \{\wedge, \vee\}, \]
\[ h_{A \to B} = \pi_{A,B}^\rightarrow (\neg h_A^{-1} \lor h_B), \quad h_{A \to B}^{-1} = (\neg h_A \lor h_B^{-1}) \circ \pi_{A,B}^\rightarrow. \]

We check as before that $h$ is indeed a natural isomorphism, which establishes that the categories $\text{SA}'$ and $\text{SA}'_s$ are equivalent via the functors $H$ and $I$. As we established that $\text{SA}_s$ is isomorphic to a full subcategory of $\text{SA}$, so we establish that $\text{SA}'_s$ is isomorphic to a full subcategory of $\text{SA}'$.

By combining the equivalences of $\text{SA}$ with $\text{SA}_s$ and of $\text{SA}'$ with $\text{SA}'_s$ and the isomorphism of $\text{SA}$ with $\text{SA}'$, established in the preceding section, we obtain that $\text{SA}_s$ and $\text{SA}'_s$ are equivalent categories. This is presumably what was meant in [11] (Section 4, Theorem 4.5) by saying that the notion of symmetric linearly distributive category with negation and the notion of star-autonomous category “coincide”.

To establish the equivalence of $\text{SA}_s$ and $\text{SA}'_s$ directly, without proceeding via $\text{SA}$ and $\text{SA}'$ as we did, is possible, but this cannot be easier than what we did (as a matter of fact, this seems to us much more tangled). One cannot escape that way all the calculations we made in verifying the isomorphism of $\text{SA}$ and $\text{SA}'$. These calculations must be made at least implicitly. We were able to shorten them via $\text{SMC}_c^\circ$ Coherence of §3.1 and our $\text{PN}_{\neg, \bot}$ Coherence of §3.5, which is based on $\text{PN}^\circ$ Coherence.
Chapter 4

Proof-Net and
Star-Autonomous Categories

In this chapter we prove that the free proof-net category $\mathbf{PN}^-$ is isomorphic to a full subcategory of the free proof-net category with units $\mathbf{SA}'_s$, and hence also to full subcategories of the categories $\mathbf{SA}'$ and $\mathbf{SA}$ of the previous chapter. All these categories are freely generated by the same set of objects. The proof is based on a Gentzen sequent formulation of $\mathbf{SA}'_s$, a cut-elimination theorem for this formulation, and a key technical lemma (Lemma 3 of §4.3). The proof of the cut-elimination is facilitated very much by relying on coherence for proof-net categories.

As a corollary, we obtain a coherence theorem with respect to $Br$ for the category $\mathbf{SA}'$, which is the full subcategory of $\mathbf{SA}$ whose objects are isomorphic either to objects in which units do not occur, i.e. to objects of $\mathbf{PN}^-$, or to one of the units. The restriction on the objects of $\mathbf{SA}$ brought by this coherence theorem for $\mathbf{SA}'$ is of the same kind as the proviso concerning the unit object that Kelly and Mac Lane had in their coherence theorem for symmetric monoidal closed categories of [32] (see the end of §3.1). This restricted coherence of star-autonomous categories is a very useful tool for deciding whether a diagram of arrows commutes in these categories.
§4.1. The Gentzenization of $\mathbf{SA}_A'$

We will now define a new language of arrow terms to denote the arrows of the category $\mathbf{SA}_A'$ of §3.8. We call these arrow terms Gentzen terms, and we prove for Gentzen terms a result analogous to Gentzen’s cut-elimination theorem, which we will use to prove that the category $\mathbf{PN}^-$ is isomorphic to a full subcategory of $\mathbf{SA}_A'$.

As the arrow terms of $\mathbf{SA}_A'$, Gentzen terms will be defined inductively starting from primitive Gentzen terms. As primitive Gentzen terms we have $1_A : A \vdash A$, for $A$ being a letter, or $\top$, or $\bot$. To define the operations on Gentzen terms, called Gentzen operations, which are mostly partial operations, we need some preparation.

We define inductively a notion that for $\xi \in {\land, \lor}$ we call a $\xi$-context:

- $\Box$ is a $\xi$-context;
- if $Z$ is a $\xi$-context and $A$ an object of $\mathbf{SA}_A'$, then $Z \land A$ and $A \land Z$ are $\xi$-contexts.

A $\xi$-context is called proper when it is not $\Box$.

Next we define inductively what it means for a $\xi$-context $Z$ to be applied to an object $B$ of $\mathbf{SA}_A'$, which we write $Z(B)$, or to an arrow term $f$ of $\mathbf{SA}_A'$, which we write $Z(f)$:

$$
\begin{align*}
\Box(B) &= B, \\
(Z \land A)(B) &= Z(B) \land A, \\
(A \land Z)(B) &= A \land Z(B); \\
\Box(f) &= f, \\
(Z \land A)(f) &= Z(f) \land 1_A, \\
(A \land Z)(f) &= 1_A \land Z(f).
\end{align*}
$$

We use $X$, perhaps with indices, as a variable for $\land$-contexts, and $Y$, perhaps with indices, as a variable for $\lor$-contexts.

Then we have the Gentzen operation $\hat{B}_X^-$, which involves types specified by

$$
\begin{align*}
f : X(A \land (B \land C)) \vdash D \\
\hat{B}_X^- f : X((A \land B) \land C) \vdash D
\end{align*}
$$

This is read “if $f$ is a Gentzen term, then $\hat{B}_X^- f$ is a Gentzen term”, all that of the required types. We use this rule notation for operations also in the
future. The Gentzen term $\hat{B}_X f$ denotes the arrow of $\mathbf{SA}'_s$ named on the right-hand side of the $=_{dn}$ sign below:

$$\hat{B}_X f =_{dn} f \circ X(\hat{b}^{-}_{A,B,C}).$$

We also have the following Gentzen operation:

$$f : D \vdash Y(A \lor (B \lor C))$$

$$\hat{B}_X f =_{dn} Y(\hat{b}^{-}_{A,B,C}) \circ f : D \vdash Y((A \lor B) \lor C)$$

and the following four analogous Gentzen operations, where the types can be easily guessed:

$$\hat{B}_X f =_{dn} f \circ X(\hat{b}^{-}_{A,B,C}),$$

$$\hat{C}_X f =_{dn} f \circ X(\hat{c}_{A,B}),$$

$$\hat{B}_Y f =_{dn} Y(\hat{c}_{A,B}) \circ f,$$

We also have the Gentzen operations in the following list:

$$f : A \vdash B$$

$$\top \nu f =_{dn} f \circ \hat{\sigma}_A : \top \land A \vdash B$$

$$g : \top \land A \vdash B$$

$$\top \nu g =_{dn} g \circ \hat{\sigma}_A : A \vdash B$$

$$f : B \vdash A$$

$$\bot \eta f =_{dn} \delta_A \circ f : B \vdash A \lor \bot$$

$$g : B \vdash A \lor \bot$$

$$\bot \eta g =_{dn} \delta_A \circ g : B \vdash A$$

for $\hat{c}'_{D,C,B,A} =_{df} (\hat{c}_{C,D} \lor 1_{B \lor A}) \circ \hat{b}^{-}_{C,A,D,B,A} \circ ((d_{C,D,B} \circ \hat{c}_{D \lor B,C}) \lor 1_A) \circ d_{D \lor B,C,A} : (D \lor B) \land (C \lor A) \vdash (D \lor C) \lor (B \lor A)$,

$$f_1 : B_1 \vdash A_1 \lor C_1$$

$$f_2 : B_2 \vdash A_2 \lor C_2$$

$$\land(f_1, f_2) =_{dn} \hat{c}'_{A_1,A_2,C_1,C_2} : (f_1 \land f_2) : B_1 \land B_2 \vdash (A_1 \land A_2) \lor (C_1 \lor C_2)$$

for $\hat{e}'_{A,B,C,D} =_{df} d_{A,C,B \lor D} \circ (1_A \land (\hat{c}_{C,B \lor D} \circ d_{B,D,C})) \circ \hat{b}^{-}_{A,B,D,C} \circ (1_{A,B} \land \hat{c}_{D,C}) : (A \land B) \land (C \lor D) \vdash (A \land C) \lor (B \land D)$,

$$f_1 : C_1 \land A_1 \vdash B_1$$

$$f_2 : C_2 \land A_2 \vdash B_2$$

$$\lor(f_1, f_2) =_{dn} (f_1 \lor f_2) \circ \hat{e}'_{C_1,C_2,A_1,A_2} : (C_1 \lor C_2) \lor (A_1 \lor A_2) \vdash B_1 \lor B_2$$
(see [22], Section 7.6, for \(e'^{i}\) and \(e'^{j}\)),

\[
f : B \vdash A \lor C
\]

\[
\neg f = d_n \Sigma'_{A,C} \circ d_{\neg A,A,C} \circ \hat{e}_{A \lor C,\neg A} \circ (f \land 1_{\neg A}) : B \land \neg A \vdash C
\]

\[
f : C \land A \vdash B
\]

\[
\neg R f = d_n (1_{\neg A} \lor f) \circ \hat{e}_{\neg A,C \land A} \circ d_{C,A,\neg A} \circ \hat{\Delta}'_{A,C} : C \vdash \neg A \lor B
\]

To define the remaining Gentzen operations, we need some preparation.

For every proper \(\land\)-context \(X\) we define inductively as follows an object \(E_X\) of \(\mathsf{SA}'\):

\[
E_{\square \land B} = E_{B \land \square} = B,
\]

\[
E_{X \land B} = E_X \land B, \quad \text{for } X \text{ proper},
\]

\[
E_{B \land X} = B \land E_X, \quad \text{for } X \text{ proper}.
\]

For every proper \(\land\)-context \(X\) and every object \(A\) of \(\mathsf{SA}'\) we define inductively as follows an arrow term \(\hat{\tau}_{X,A} : E_X \land A \vdash X(A)\) of \(\mathsf{SA}'\):

\[
\hat{\tau}_{B \land \square,A} = \hat{1}_{B \land A} : B \land A \vdash B \land A,
\]

\[
\hat{\tau}_{B \land X,A} = \hat{1}_{B \land \hat{\tau}_{X,A}} \circ \hat{b}_{E_X, A} : (B \land E_X) \land A \vdash B \land X(A), \quad \text{for } X \text{ proper},
\]

\[
\hat{\tau}_{\square \land B,A} = \hat{e}_{B,A} : B \land A \vdash A \land B,
\]

\[
\hat{\tau}_{X \land B,A} = \hat{1}_{E_X, \hat{\tau}_{X,A}} \circ \hat{b}_{E_X, A} \circ (1_{E_X} \land \hat{e}_{B,A}) \circ \hat{b}_{E_X, B,A} : (E_X \land B) \land A \vdash X(A) \land B, \quad \text{for } X \text{ proper}.
\]

For every proper \(\lor\)-context \(Y\) we define inductively as follows an object \(D_Y\) of \(\mathsf{SA}'\):

\[
D_{\square \lor B} = D_{B \lor \square} = B,
\]

\[
D_{Y \lor B} = D_Y \lor B, \quad \text{for } Y \text{ proper},
\]

\[
D_{B \lor Y} = B \lor D_Y, \quad \text{for } Y \text{ proper}.
\]

For every proper \(\lor\)-context \(Y\) and every object \(A\) of \(\mathsf{SA}'\) we define inductively as follows an arrow term \(\hat{\tau}_{Y,A} : Y(A) \vdash A \lor D_Y\) of \(\mathsf{SA}'\)
§4.1. The Gentzenization of \( \text{SA}'_s \)

\[
\hat{\tau}_{\Box \lor B,A} = \text{df} \ 1_{A \lor B} : A \lor B \vdash A \lor B,
\]

\[
\hat{\tau}_{Y \lor B,A} = \text{df} \ \tilde{b}_{A,D_Y,Y}^Y \cdot (\hat{\tau}_{Y,A} \lor 1_B) : Y(A) \lor B \vdash A \lor (D_Y \lor B),
\]

for \( Y \) proper,

\[
\hat{\tau}_{B \lor \Box,A} = \text{df} \ \hat{c}_{A,B} : B \lor A \vdash A \lor B,
\]

\[
\hat{\tau}_{B \lor Y,A} = \text{df} \ \tilde{b}_{A,B,D_Y}^Y \cdot (\hat{c}_{A,B} \lor 1_{D_Y}) \cdot \tilde{b}_{B,A,D_Y}^Y : B \lor Y(A) \vdash A \lor (B \lor D_Y), \text{ for } Y \text{ proper.}
\]

For \( f : A \vdash B \), the following equations hold in \( \text{SA}'_s \):

\[
(\hat{\tau} \ \text{nat}) \quad X(f) \times \hat{\tau}_{X,A} = \hat{\tau}_{X,B} = (1_{E_X} \land f),
\]

\[
(\hat{\tau} \ \text{nat}) \quad (f \lor 1_{D_Y}) \times \hat{\tau}_{Y,A} = \hat{\tau}_{Y,B} = Y(f);
\]

they are proved by applying naturality equations.

It is clear that for \( \xi \in \{\land, \lor\} \) and \( \hat{\tau}_{X,A} : A_1 \vdash A_2 \) there is an arrow term \( \hat{\tau}^{-1}_{X,A} : A_2 \vdash A_1 \) of \( \text{SA}'_s \), which is a “mirror image” of \( \hat{\tau}_{X,A} \), such that in \( \text{SA}'_s \) we have

\[
\hat{\tau}^{-1}_{X,A} \times \hat{\tau}_{X,A} = 1_{A_1}, \quad \hat{\tau}_{X,A} \times \hat{\tau}^{-1}_{X,A} = 1_{A_2}.
\]

For example, with

\[
\hat{\tau}_{F \land ((C \land \Box) \lor B),A} = \langle 1_F \land (\tilde{b}_{C,A,B} \times (1_C \land \hat{c}_{B,A}) \times \tilde{b}_{C,B,A}) \rangle \times \tilde{b}_{F,C \land B,A}
\]

we have

\[
\hat{\tau}^{-1}_{F \land ((C \land \Box) \lor B),A} = \tilde{b}_{F,C \land B,A} \times (1_F \land (\tilde{b}_{C,B,A} \times (1_C \land \hat{c}_{B,A}) \times \tilde{b}_{C,A,B})).
\]

Officially, \( \hat{\tau}^{-1}_{X,A} \) is defined inductively as \( \hat{\tau}_{X,A} \), in a dual manner.

Next, we introduce the following abbreviation:

\[
d_{X,A,Y} = \text{df} \ \tilde{\tau}^{-1}_{Y,X(A)} \times (\hat{\tau}_{X,A} \lor 1_{D_Y}) \times \tilde{d}_{E_X,A,D_Y} \times (1_{E_X} \land \tilde{\tau}_{Y,A}) \times \tilde{\tau}^{-1}_{X,Y(A)} : X(Y(A)) \vdash Y(X(A)).
\]

When \( X \) or \( Y \) is \( \Box \), then we assume that \( d_{X,A,Y} \) stands for \( 1_{X(Y(A))} \), which is of type \( X(Y(A)) \vdash Y(X(A)) \), i.e. \( Y(A) \vdash Y(A) \) or \( X(A) \vdash X(A) \).

We can finally define the remaining Gentzen operations, which are all of the following form:
\[
g: B \vdash Y(A) \\
f: X(A) \vdash C
\]

\[\text{cut}_{X,Y}(f,g) = \delta_{X,Y} Y(f) \circ \delta_{X,A,Y} X(g) : X(B) \vdash Y(C)\]

This concludes the definition of Gentzen operations. The set of Gentzen terms is the smallest set containing primitive Gentzen terms and closed under the Gentzen operations above.

It is easy to infer from DS Coherence of §2.3 that the following equations hold in \(\mathsf{SA}'\):

\[
\begin{align*}
(d \wedge X) & \quad d_{A \wedge X,C,Y} = d_{A \wedge X(C),Y} \circ (1_A \wedge d_{X,C,Y}), \\
(dX \wedge) & \quad d_{X \wedge A,C,Y} = d_{X \wedge A,C(X),Y} \circ (d_{X,C,Y} \wedge 1_A), \\
(d \vee Y) & \quad d_{X,C,A \vee Y} = (1_A \vee d_{X,C,Y}) \circ d_{X,Y(A),A \vee C}, \\
(dY \vee) & \quad d_{X,C,Y \vee A} = (d_{X,C,Y} \vee 1_A) \circ d_{X,Y(C),A \vee C}.
\end{align*}
\]

The equation \((d \wedge X)\) is analogous to the equation \((d \wedge)\) of §2.1, while \((d \vee Y)\) is analogous to \((d \vee)\) of §2.1.

We can then prove the following.

**Gentzenization Lemma.** Every arrow of \(\mathsf{SA}'\) is denoted by a Gentzen term.

**Proof.** We first show by induction on the complexity of \(A\) that for every \(A\) the arrow \(1_A: A \vdash A\) is denoted by a Gentzen term. For \(A\) being a letter, or \(\top\), or \(\bot\), this is trivial. For the induction step we use the following equations of \(\mathsf{SA}'\):

\[
\begin{align*}
\wedge & \quad \bot \wedge \bot \hat{\otimes} (\bot \wedge f_1, \bot \wedge f_2) = f_1 \wedge f_2, \\
\vee & \quad \top \vee \bot \hat{\otimes} (\top \vee f_1, \top \vee f_2) = f_1 \vee f_2.
\end{align*}
\]

For \((\wedge)\) we use

\[
e'_{A_1,A_2,\bot,\bot} = (1_{A_1 \wedge A_2} \vee \delta_{\bot}) \circ \hat{\delta}_{A_1 \wedge A_2} \circ (\hat{\delta}_{A_1} \wedge \hat{\delta}_{A_2}),
\]

which follows essentially from \((b \delta)\) and \((d \hat{\delta})\) of §3.3 (we may apply here the Symmetric Bimonoidal Coherence of [22], Section 6.4, which reduces to
§4.2. Cut elimination in $\mathbf{SA'}_s$

Mac Lane’s symmetric monoidal coherence of [37]; see [38], Section VII.7, and [22], Section 5.3. We proceed analogously for ($\lor$).

We also have for the induction step the following equations of $\mathbf{SA'}_s$:

\[
\begin{align*}
\bot & \to \neg R \bot \land C \to \neg R \bot = \top = \top \lor \bot \\
\bot & \to \neg L \bot \lor C \to \neg L \bot = \bot = \bot \lor \bot .
\end{align*}
\]

for which we use ($d\delta$), ($\hat{b}\Sigma'$) and ($\hat{\Sigma}'\Delta'$), among other obvious equations.

The Gentzen term that denotes $\bot$ is written $\bot$.

Next we have the following in $\mathbf{SA'}_s$:

\[
\begin{align*}
\land B & \to \bot \bot \land (A \land B \land C) = \bot \land B \land C = \land B \land C, \\
\lor B & \to \bot \bot \lor (A \lor B \lor C) = \bot \lor B \lor C = \lor B \lor C, \\
\land B & \to \bot \bot \land (A \land B \land C) = \bot \land B \land C = \land B \land C, \\
\lor B & \to \bot \bot \lor (A \lor B \lor C) = \bot \lor B \lor C = \lor B \lor C.
\end{align*}
\]

(For the equations involving $\hat{\Delta}_B,A$ and $\hat{\Sigma}_B,A$ we rely on ($d\hat{\delta}$) and ($d\hat{\delta}$) of §3.3, and on the stem-increasing equations of §2.5.)

For composition we have the following equation of $\mathbf{SA'}_s$:

\[cut_{A\land B}(f,g) = f \circ g,\]

and for the operations $\land$ and $\lor$ on arrows we have the equations ($\land$) and ($\lor$) above.

§4.2. Cut elimination in $\mathbf{SA'}_s$

For the proof of the Cut-Elimination Theorem below we will introduce analogues of Gentzen’s notions of rank and degree. We need some preliminary definitions to define these notions.
CHAPTER 4. PROOF-NET AND STAR-AUTONOMOUS...

For \( \xi \in \{\land, \lor\} \), we define first by induction the notion of \( \xi \)-superficial subformula of a formula of \( \mathcal{L}_{\land, \lor, \neg, \land, \lor} \):

- if \( A \) is of the form \( p, \bot, A_1 \lor A_2 \), or \( \neg A' \), then \( A \) is a \( \land \)-superficial subformula of \( A \);
- if \( A \) is of the form \( p, \top, A_1 \land A_2 \), or \( \neg A' \), then \( A \) is a \( \lor \)-superficial subformula of \( A \);
- if \( A \) is a \( \xi \)-superficial subformula of \( B \), then \( A \) is a \( \xi \)-superficial subformula of \( B \) and \( C \). 

Consider a Gentzen term \( f \) of the form

\[
\land(f_1, f_2): B_1 \land B_2 \vdash (A_1 \land A_2) \lor (C_1 \lor C_2).
\]

The \( \lor \)-superficial subformula \( A_1 \land A_2 \) that is the left disjunct of the target of \( f \) is called the leaf of \( f \). All the other \( \lor \)-superficial subformulas of the target of \( f \), which are subformulas of \( C_1 \) or \( C_2 \), and all the \( \land \)-superficial subformulas of the source of \( f \), which are subformulas of \( B_1 \) or \( B_2 \), are called lower parameters of \( f \).

To every lower parameter \( x \) of \( f \), there corresponds unambiguously a subformula \( y \) in the target or the source of either \( f_1 : B_1 \vdash A_1 \lor C_1 \) or \( f_2 : B_2 \vdash A_2 \lor C_2 \), which we call the upper parameter of \( f \) corresponding to \( x \). The lower parameter \( x \) is a \( \land \)-superficial subformula of the source of \( f \) iff the corresponding upper parameter \( y \) is a \( \land \)-superficial subformula of the source of either \( f_1 \) or \( f_2 \) (it cannot be in both), and analogously for parameters that are \( \lor \)-superficial subformulas of targets. If \( y \) is in the type of \( f_1 \), then \( f_1 \) is called the subterm of \( f \) for the upper parameter \( y \), and analogously for \( f_2 \).

For example, if \( f \) is

\[
\land(1_{p \lor q}, \bot \vdash 1_r): (p \lor q) \land r \vdash (p \land r) \lor (q \lor \bot),
\]

then \( p \land r \) in the target is the leaf of \( f \), while \( q \) in the target of \( f \) and \( p \lor q \) and \( r \) in the source of \( f \) are lower parameters of \( f \). To the lower parameter \( q \) of \( f \) corresponds the upper parameter of \( f \) that is the occurrence of \( q \) in the target of the subterm \( 1_{p \lor q} : p \lor q \vdash p \lor q \) for this upper parameter; to
the lower parameter \( p \lor q \) of \( f \) corresponds the upper parameter of \( f \) that
is the source of the subterm \( 1_{p \lor q} \) for this upper parameter; and to the lower
parameter \( r \) of \( f \) corresponds the upper parameter of \( f \) that is the source
of the subterm \( \bot \leftarrow 1_r : r \vdash r \lor \bot \) for this upper parameter. Note that the
subformula \( \bot \) in the target of \( f \) is not a \( \lor \)-superficial subformula of this
target, and hence is not a lower parameter of \( f \).

If the Gentzen term \( f \) is of the form

\[
\lor(f_1, f_2) : (C_1 \land C_2) \land (A_1 \lor A_2) \vdash B_1 \lor B_2,
\]

then the \( \land \)-superficial subformula \( A_1 \lor A_2 \) that is the right conjunct of the
source of \( f \) is the leaf of \( f \), while all the other \( \land \)-superficial subformulae of
the source of \( f \) and the \( \lor \)-superficial subformulae of the target of \( f \) are the
lower parameters of \( f \). The upper parameters of \( f \) corresponding to these
lower parameters, and the subterms of \( f \) for these upper parameters, are
declared analogously to what we had in the previous case.

The leaf of \( \lnot f : B \land \lnot A \vdash C \) is the \( \land \)-superficial subformula \( \lnot A \) that
is the right conjunct of its source, while the leaf of \( \lnot f : C \vdash \lnot A \lor B \) is the
\( \lor \)-superficial subformula \( \lnot A \) that is the left disjunct of its target. In
both cases, the remaining \( \land \)-superficial subformulae of the source or the
remaining \( \lor \)-superficial subformulae of the target are lower parameters, to
which correspond analogously to what we had before, upper parameters
in the source or target of the subterm \( f \) for these upper parameters.

If our Gentzen term is of the form

\[
\land B_X f, \lor B_Y f, \land B_X f, \lor C_Y f, \lor C_Y f, \top \leftarrow f, \top \rightarrow f, \bot \leftarrow f, \bot \rightarrow f, \text{ or cut}_{X,Y}(f, g),
\]

then it has no leaves, and all the \( \land \)-superficial subformulae of its source
and all the \( \lor \)-superficial subformulae of its target are lower parameters, to
which upper parameters correspond in an obvious manner.

Finally, the Gentzen term \( 1_p : p \vdash p \) has two leaves, which are its source
\( p \) and its target \( p \). There are no parameters of \( 1_p \), neither lower nor upper.
The Gentzen term \( 1_\top : \top \vdash \top \) has as its leaf the target \( \top \), and no parameters
(the source \( \top \) of \( 1_\top \) is not a \( \land \)-superficial subformula of itself). The Gentzen
term \( 1_\bot : \bot \vdash \bot \) has as its leaf the source \( \bot \), and no parameters (the target
\( \bot \) of \( 1_\bot \) is not a \( \lor \)-superficial subformula of itself).
Let $x$ be a $\land$-superficial subformula of the source of a Gentzen term $f$ or a $\lor$-superficial subformula of the target of $f$. Then the cluster of $x$ in $f$ is a sequence of occurrences of formulae defined inductively as follows:

if $x$ is a leaf of $f$, then the cluster of $x$ in $f$ is $x$,

if $x$ is not a leaf of $f$, then $x$ is a lower parameter of $f$, and for $y_1$ being the upper parameter of $f$ corresponding to $x$, take the cluster $y_1 \ldots y_n$, where $n \geq 1$, of $y_1$ in the proper subterm $f'$ of $f$ that is the subterm of $f$ for the upper parameter $y_1$ (the sequence $y_1 \ldots y_n$ is already defined, by the induction hypothesis); the cluster of $x$ in $f$ is the sequence $xy_1 \ldots y_n$.

All occurrences of formulae in a cluster are $\xi$-superficial subformulae for $\xi$ being one of $\land$ and $\lor$. If $\xi$ is $\land$, then the cluster is a source cluster, and if $\xi$ is $\lor$, then it is a target cluster.

A cut is a Gentzen term of the form $\text{cut}_{X,Y}(f, g)$. For $g: B \vdash Y(A)$ and $f: X(A) \vdash C$ let the formula $A$ be called the cut formula of the cut $\text{cut}_{X,Y}(f, g)$. Let $x$ be the displayed occurrence of $A$ in the source $X(A)$ of $f$, and let $s$ be the length of the cluster of $x$ in $f$ (we write $s$ because we have here a source cluster). Let $y$ be the displayed occurrence of $A$ in the target $Y(A)$ of $g$, and let $t$ be the length of the cluster of $y$ in $g$ (we write $t$ because we have here a target cluster).

Depending on the form of $A$, we define a number $r$, which we call the rank of the cut $\text{cut}_{X,Y}(f, g)$. If the cut formula $A$ is of the form $p$ or $\neg A'$, then

$$r = \min(s, t) - 1, \quad \text{if } A = p,$$

$$r = s + t - 2, \quad \text{if } A = \neg A'.$$

(As a matter of fact, when $A$ is $p$, we could stipulate that $r$ is either $s + t - 2$, as when it is $\neg A'$, or $s - 1$, or $t - 1$, but the computation of rank we have introduced makes the cut-elimination procedure run faster, and does not complicate the proof.)

If the cut formula $A$ is of the form $\top$ or $A_1 \land A_2$, then $r = t - 1$. If, finally, the cut formula $A$ is of the form $\bot$ or $A_1 \lor A_2$, then $r = s - 1$.

We define the degree $d$ of a cut as the number of occurrences of $\land$, $\lor$ and $\neg$ in its cut formula. The complexity of a cut is the ordered pair $(d, r)$, where
§4.2. Cut elimination in $\mathbf{SA}'_s$

d is its degree and $r$ its rank. The complexities of cuts are lexicographically ordered (i.e., $(d_1, r_1) < (d_2, r_2)$ iff $d_1 < d_2$, or $d_1 = d_2$ and $r_1 < r_2$).

A Gentzen term is called cut-free when no subterm of it is a cut. A cut $\text{cut}_{X,Y}(f, g)$ is topmost when $f$ and $g$ are cut-free. (Since in the proof below, we compute the rank only for topmost cuts, our definition of cluster can be shortened a little bit by not considering the parameters of cuts; but this is not a substantial shortening.)

We can then prove the following.

**Cut-Elimination Theorem.** For every Gentzen term $h$ there is a cut-free Gentzen term $h'$ such that $h = h'$ in $\mathbf{SA}'_s$.

**Proof.** It suffices to prove the theorem when $h$ is a topmost cut. We proceed by induction on the complexity $(d, r)$ of this topmost cut.

Suppose $r = 0$ and $d = 0$. Then $h$ can be of one of the following forms:

- $\text{cut}_{X,\Box}(f, 1_A)$ for $A$ being $p$ or $\top$,
- $\text{cut}_{\Box,Y}(1_A, g)$ for $A$ being $p$ or $\bot$,

and we have in $\mathbf{SA}'_s$

\[
\text{cut}_{X,\Box}(f, 1_A) = f,
\]
\[
\text{cut}_{\Box,Y}(1_A, g) = g.
\]

This settles the basis of the induction.

Suppose $r = 0$ and $d > 0$. Then the cut formula must be of the form $A_1 \land A_2$ or $A_1 \lor A_2$ or $\neg A'$. In the first case, for $f : X(A_1 \land A_2) \vdash D$, $g_1 : B_1 \vdash A_1 \lor C_1$ and $g_2 : B_2 \vdash A_2 \lor C_2$ we have the equation

\[
\text{cut}_{X,\Box^\lor(C_1 \lor C_2)}(f, \land(g_1, g_2)) = \text{cut}_{X,\Box^\lor C_2}(\text{cut}_{X,\Box^\lor C_1}(f, g_1), g_2)
\]

where $X'(C)$ is $X(C \land A_2)$ and $X''(C)$ is $X(B_1 \land C)$. To prove this equation we apply naturality equations and $\text{DS Coherence}$.

The complexity of the topmost cut $\text{cut}_{X,\Box^\lor C_1}(f, g_1)$ is $(d', r')$ with $d' < d$, and we can apply the induction hypothesis to obtain a cut-free Gentzen term $f'$ equal to it in $\mathbf{SA}'_s$. The complexity of the topmost cut $\text{cut}_{X,\Box^\lor C_2}(f', g_2)$ is $(d'', r'')$ with $d'' < d$, and we can again apply the induction hypothesis.
In case the cut formula is \( A_1 \lor A_2 \), we have an analogous equation, for which we use again DS Coherence, and we reason analogously, applying the induction hypothesis twice.

In case the cut formula is \( \neg A' \), for \( f : D \land A' \vdash E \) and \( g : B \vdash A' \lor C \) we have the equation

\[
cut_{B \land \Box \lor E}(\neg^\land g, \neg^R f) = \hat{C} \Box \cut_{D \land \Box \lor C}(f, g),
\]

which holds by naturality equations and \( \text{PN}^- \) Coherence. Then we apply the induction hypothesis to the topmost cut on the right-hand side, which has a smaller degree.

Suppose now \( r > 0 \). If \( r \) was computed as \( s - 1 \), or as \( s + t - 2 \), where \( s > 1 \), then we may apply equations of \( \text{SA}' \) of the following form

\[
(*) \quad \cut_{X \land Y}(\gamma f', g) = \gamma_1 \cdots \gamma_n \cut_{X' \land Y}(f', g)
\]

for \( \gamma, \gamma_1, \ldots, \gamma_n \) unary Gentzen operations. If \( (d, r) \) is the complexity of the topmost cut \( \cut_{X \land Y}(\gamma f', g) \), then the complexity of the topmost cut \( \cut_{X' \land Y}(f', g) \) is \( (d, r - 1) \), and so we may apply to it the induction hypothesis.

If \( \gamma \) is a unary Gentzen operation different from \( \top \rightarrow, \top \leftarrow, \bot \leftarrow \) and \( \bot \rightarrow \), then so are \( \gamma_1, \ldots, \gamma_n \), and to prove \((*)\) we apply naturality equations and \( \text{PN}^- \) Coherence (sometimes DS Coherence suffices, depending on \( \gamma \)). We have analogous equations involving binary Gentzen operations, which are proved analogously, relying on DS Coherence (cf. [22], Section 11.2, Case (6), where on p. 251, in the second line \( \land^R(f, \cut(g, h)) \) should be replaced by \( \land^R(g, \cut(f, h)) \), and in the third line \( \cut(g, h) \) should be replaced by \( \cut(f, h) \)).

If \( \gamma \) in \((*)\) is \( \top \rightarrow \), then \( n = 1 \) and \( \gamma_1 \) is \( \top \rightarrow \). To prove \((*)\), we then apply essentially the equation

\[
Y(\hat{\gamma} X(A)) \ast d_{T \land X, A, Y} = d_{X, A, Y} \ast \hat{\gamma} X(Y(A))
\]

which we obtain with the help of \( (d \land X) \) of the preceding section, \( (d \hat{\gamma}) \) of \( \S 3.3 \), and \( (\hat{\gamma} \ \text{nat}) \) of the preceding section (as a matter of fact, we may apply here the Symmetric Bimonoidal Coherence of [22], Section 6.4). We proceed analogously if \( \gamma \) is \( \top \leftarrow \).
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If \( \gamma \) in (*) is \( \bot \rightarrow \) or \( \bot \leftarrow \), then we apply essentially Mac Lane’s symmetric monoidal coherence of [37] (see also [38], Section VII.7, and [22], Section 5.3).

If \( r \) was computed as \( t - 1 \), or as \( s + t - 2 \), where \( t > 1 \), then we proceed in a dual manner. Instead of (*), we have equations of \( \text{SA}^\prime \) of the following form:

\[
\text{cut}_{X,Y}(f, \gamma g') = \gamma_1 \ldots \gamma_n \text{cut}_{X,Y}(f, g').
\]

This concludes the proof of the theorem.

\( \sqcup \)

§4.3. \text{SA} c Coherence

There is a functor \( G \) from the category \( \text{SA}^\prime \) of §3.3 to \( \text{Br} \), which is defined as the functor \( G \) from \( \text{PN} \) to \( \text{Br} \) (see §2.3) with the additional clauses that say that \( G\alpha \) is an identity arrow of \( \text{Br} \) for \( \alpha \) being \( \delta_{A}^{-}, \delta_{A}^{+}, \pi_{A,B}^{-}, \pi_{A,B}^{+} \) and \( \pi_{A,B}^{-}, \pi_{A,B}^{+} \), where \( \xi \in \{ \wedge, \vee \} \). There is analogously a functor \( G \) from \( \text{SA}^\prime \) to \( \text{Br} \), which is defined as \( G \) from \( \text{SMC} \) to \( \text{Br} \) save that we do not have the clauses for \( \pi_{A,B}^{+} \) and \( \pi_{A,B}^{-} \).

It follows from the existence of these functors and \( \text{PN}^\sim \) Coherence that \( \text{PN}^\sim \) is isomorphic to subcategories of \( \text{SA}^\prime \) and \( \text{SA}^\prime \) (cf. [22], Section 14.4).

Our purpose in this section is to prove the following theorem.


**Conservativeness Theorem.** If \( A \) and \( B \) are objects of \( \text{PN}^\sim \), then for every arrow \( f: A \vdash B \) of \( \text{SA}^\prime \) there is an arrow term \( f': A \vdash B \) of \( \text{PN}^\sim \) such that \( f = f' \) in \( \text{SA}^\prime \).

This theorem implies that \( \text{PN}^\sim \) is isomorphic to a full subcategory of \( \text{SA}^\prime \), from which, according to what we established in §§3.7-8, we can conclude that \( \text{PN}^\sim \) is isomorphic to a full subcategory of \( \text{SA}^\prime \), and of \( \text{SA} \) too. In these isomorphisms every object of \( \text{PN}^\sim \) is mapped to itself, and so every object of \( \text{PN}^\sim \) in \( \text{SA}^\prime \); \( \text{SA}^\prime \) or \( \text{SA} \) is in the image of \( \text{PN}^\sim \).

Let the functor \( G \) from \( \text{SA} \) to \( \text{Br} \) be defined as \( G \) from \( \text{SMC} \) to \( \text{Br} \) (see §3.1) with the additional clauses that say that \( G\alpha \) is an identity arrow of \( \text{Br} \) for \( \alpha \) being \( \nu_{A}^{-}, \lambda_{A}, \nu_{A,B}^{+}, \nu_{A,B}^{-} \) and \( \nu_{A,B}^{-}, \nu_{A,B}^{+} \). One can easily check that this functor \( G \) restricted to the subcategory of \( \text{SA} \) isomorphic to \( \text{PN}^\sim \) satisfies all the clauses of the definition of the functor \( G \) from \( \text{PN}^\sim \) to \( \text{Br} \) (see §2.3).
Let $\text{SA}'$ be the full subcategory of $\text{SA}$ whose objects are all the objects $A$ of $\text{SA}$ such that there is an isomorphism of type $A \vdash A'$ of $\text{SA}$ for $A'$ an object of $\text{PN}^-$. (The category $\text{SA}'$ is a replete subcategory of $\text{SA}$; cf. the end of §3.1.) Then we can restrict the functor $G$ from $\text{SA}$ to $\text{Br}$ to a functor $G$ from $\text{SA}'$ to $\text{Br}$, for which we can prove the following, relying on the Conservativeness Theorem.

$\text{SA}'$ Coherence. The functor $G$ from $\text{SA}'$ to $\text{Br}$ is faithful.

Proof. Suppose $A$ and $B$ are objects of $\text{SA}'$, and let $j_A : A \vdash A'$ and $j_B : B \vdash B'$ be isomorphisms of $\text{SA}$ for $A'$ and $B'$ objects of $\text{PN}^-$. Suppose that $f_1, f_2 : A \vdash B$ are arrows of $\text{SA}$, i.e. of $\text{SA}'$, such that $Gf_1 = Gf_2$.

Since $\text{PN}^-$ is isomorphic to a full subcategory of $\text{SA}$ such that every object of $\text{PN}^-$ in $\text{SA}$ is in the image of $\text{PN}^-$, we have in $\text{SA}$ that

$$j_B \circ f_i \circ j_A^{-1} = f_i'$$

for $i \in \{1, 2\}$ and $f_i'$ an arrow term of $\text{PN}^-$. It follows that $Gf_1' = Gf_2'$, and, according to what we said immediately after the definition of the functor $G$ from $\text{SA}$ to $\text{Br}$, by $\text{PN}^-$ Coherence we have that $f_1' = f_2'$ in $\text{PN}^-$, and hence also in $\text{SA}$. So $f_1 = f_2$ in $\text{SA}$.

The category $\text{SA}'$ is a category equivalent to $\text{PN}^-$, and its coherence is a consequence of $\text{PN}^-$ Coherence. We can find full subcategories of $\text{SA}'$, some of which are full subcategories of $\text{SA}$, too, that are not only equivalent, but also isomorphic to $\text{PN}^-$. 

Let $\text{SA}^c$ be the full subcategory of $\text{SA}$ whose objects are all the objects $A$ of $\text{SA}$ such that there is an isomorphism of type $A \vdash A'$ of $\text{SA}$ for $A'$ being either an object of $\text{PN}^-$, or $\top$, or $\bot$. (The category $\text{SA}^c$ is as $\text{SA}'$ a replete subcategory of $\text{SA}$.) Then we can restrict the functor $G$ from $\text{SA}$ to $\text{Br}$ to a functor $G$ from $\text{SA}^c$ to $\text{Br}$, for which we can prove the following, relying on the Conservativeness Theorem and on $\text{SA}'$ Coherence.

$\text{SA}^c$ Coherence. The functor $G$ from $\text{SA}^c$ to $\text{Br}$ is faithful.

Proof. There is no arrow of type $\top \vdash \bot$ in $\text{SA}$. (Otherwise, classical propositional logic would be inconsistent.) There is also no arrow of type
§4.3. **SAc Coherence**

\(\bot \vdash \top\) in \(\text{SA}\). If \(f: \bot \vdash \top\) were such an arrow, then we would have in \(\text{SA}\) the arrow

\[
((\hat{\delta}^-_p : (1_p \land f)) \lor 1_q) = d_{p,\bot,q} : (1_p \land \hat{\delta}^-_q) : p \land q \vdash p \lor q.
\]

Hence, by the Conservativeness Theorem, there would be an arrow term \(f': p \land q \vdash p \lor q\) of \(\text{PN}^\neg\), and that such an \(f'\) does not exist can be shown by appealing to the connectedness condition of proof nets (see §7.1).

Suppose \(A\) and \(B\) are objects of \(\text{SAc}\); so \(A\) and \(B\) are isomorphic in \(\text{SA}\) to respectively \(A'\) and \(B'\), each of which is either an object of \(\text{PN}^\neg\), or \(\top\), or \(\bot\). Suppose that \(f_1, f_2: A \vdash B\) are arrows of \(\text{SA}\), i.e. of \(\text{SAc}\), such that \(Gf_1 = Gf_2\).

As we have seen above, it is excluded that one of \(A'\) and \(B'\) is \(\top\) while the other is \(\bot\). If \(A'\) and \(B'\) are objects of \(\text{PN}^\neg\), then we apply \(\text{SAc Coherence}\).

Let \(\text{SA}_{+p}\) be \(\text{SA}\) generated by \(P \cup \{p\}\) for a letter \(p\) foreign to \(P\), and hence also to \(A\) and \(B\). Let \(\text{SC}_{+p}\) be the \(\text{SC}\) subcategory of \(\text{SA}_{+p}\). In the remaining cases, if either \(A'\) or \(B'\) is \(\top\), then \(G(f_1 \land 1_p) = G(f_2 \land 1_p)\). It is easy to see that \(f_1 \land 1_p, f_2 \land 1_p: A \land p \vdash B \land p\) are arrows of \(\text{SC}_{+p}\), and so \(f_1 \land 1_p = f_2 \land 1_p\) in \(\text{SA}_{+p}\) by \(\text{SC}\) Coherence applied to \(\text{SC}_{+p}\). Then in \(\text{SA}\) generated by \(P\) we have \(f_1 \land 1_\top = f_2 \land 1_\top\) (we just substitute \(\top\) for \(p\) in the derivation of \(f_1 \land 1_p = f_2 \land 1_p\) in \(\text{SA}_{+p}\)), and so we have in \(\text{SA}\)

\[
f_1 = f_1 \circ \hat{\delta}^-_A \circ \hat{\delta}^-_A, \quad \text{by } (\hat{\delta} \hat{\delta}),
\]

\[
= \hat{\delta}^-_B \circ (f_1 \land 1_\top) \circ \hat{\delta}^-_A, \quad \text{by } (\hat{\delta} \text{ nat}),
\]

\[
= \hat{\delta}^-_B \circ (f_2 \land 1_\top) \circ \hat{\delta}^-_A
\]

\[= f_2.\]

If either \(A'\) or \(B'\) in the remaining cases is \(\bot\), then \(G(f_1 \lor 1_p) = G(f_2 \lor 1_p)\), and we proceed analogously.

Both \(\text{SC}\) Coherence and \(\text{SC}\) Coherence are analogous to Kelly’s and Mac Lane’s \(\text{SMC}\) Coherence (see the end §3.1); for \(\text{SC}\) Coherence the analogy is complete.

Note that many computations of equality of arrows in Chapter 3, which we could not settle previously by \(\text{SMC}\) Coherence or \(\text{PN}^\neg\) Coherence...
alone, are now settled by simple applications of $\text{SA}^c$ Coherence. As a matter of fact, $\text{SA}^c$ Coherence suffices. (Of course, we used these computations to establish $\text{SA}^c$ Coherence, and we can judge now only retrospectively that they are dispensable in the presence of this coherence.) With $\text{SA}^c$ Coherence we have found a powerful tool to establish equality of arrows in a considerable fragment of $\text{SA}$, and also of $\text{SA}_s$. This covers the $\{\land, \rightarrow\}$ fragment, the $\{\neg, \land, \lor\}$ fragment, and also other fragments of star-autonomous categories involving $\top$ and $\bot$ at some particular places.

Coherence with respect to $Br$ for the whole of $\text{SA}$ or $\text{SA}_s$ presumably does not hold. According to [6] (Sections 4.2, 2.3), in $\text{SA}_s$ we do not have

$$\nu^{\top}_{\top \rightarrow \top} \circ (\nu^{\top}_{\top \rightarrow 1} \circ (A \rightarrow \top \rightarrow \top) = 1_{(A \rightarrow \top) \rightarrow \top},$$

(cf. [32], [43]), nor

$$\check{e}_{\bot \bot} = 1_{\bot \land \bot}, \quad \check{e}_{\top \top} = 1_{\top \lor \top},$$

while if $f = g$ is one of these equations we have $Gf = Gg$. (The claim made in [6], Section 2.3, that the category of sets with functions is a linearly distributive category is not correct.)

The remainder of this section is devoted to the proof of the Conservativeness Theorem. This will be accomplished with the help of several lemmata, for whose formulation we introduce the following terminology.

An object of $\text{SA}_s'$, i.e. a formula of $L_{\top, \bot, \neg, \land, \lor}$, is constant-free when neither $\top$ nor $\bot$ occurs in it. In other words, the constant-free objects of $\text{SA}_s'$ are the objects of $\text{PN}^\top$.

An object of $\text{SA}_s'$ is called literate when at least one letter occurs in it; otherwise, it is letterless. Every constant-free formula is literate (but not conversely).

For $\xi \in \{\land, \lor\}$, we define inductively when a formula of $L_{\land, \bot, \neg, \land, \lor}$ is $\xi$-nice:

- $\top$ is $\land$-nice and $\bot$ is $\lor$-nice;
- constant-free objects of $\text{SA}_s'$ are $\xi$-nice;
- if $A$ and $B$ are $\xi$-nice, then $A \land B$ is $\xi$-nice.

For a $\xi$-nice formula $A$ we define inductively an arrow term $\hat{\rho}_A: A \vdash A'$ of $\text{SA}_s'$ such that $A'$ is constant-free if $A$ is literate, $A'$ is $\top$ if $A$ is letterless and $\land$-nice, and $A'$ is $\bot$ if $A$ is letterless and $\lor$-nice:
§4.3. \textbf{SA}′ Coherence

\begin{align*}
p_\top & = 1_\top, & p_\bot & = 1_\bot, & p_A & = 1_A, & \text{for } A \text{ constant-free,} \\
p_{A \sqcup B} & = p_A \land p_B, & \text{for } A \text{ and } B \text{ literate,} \\
p_{A \sqcap B} & = \delta_A \land (p_A \lor p_B), & \text{for } B \text{ letterless,} \\
p_{A \sigma B} & = \delta_B \land (p_A \lor p_B), & \text{for } A \text{ letterless.}
\end{align*}

It is clear that $\hat{p}_A$ is an isomorphism of $\text{SA}_F$, with inverse $\hat{p}_A^{-1} : A \vdash A$.

We can then prove the following lemma.

\textbf{Lemma 1.} Let $f : A \vdash B$ be a \(\xi_{C,D,E}\)-term for $C$, $D$ and $E$ literate $\xi$-nice formulae. Then there is \(\hat{f}_{C,D,E}: A' \vdash B'$ for $C'$, $D'$ and $E'$ constant-free such that

\[ \hat{p}_B \circ f = f' \circ \hat{p}_A. \]

\textbf{Proof.} We proceed by induction on the complexity of $f$. If $f$ is \(\xi_{C,D,E}\), then we have that

\[ \hat{p}_{(C \sqcup D) \xi E} = (\hat{p}_C \land \hat{p}_D) \xi \hat{p}_E, \]

and we apply \(\xi^{-} \text{nat}\). For the induction step, suppose $f$ is $g \xi 1_F : G \xi F \vdash H \xi F$ (we proceed analogously when $f$ is $1_F \xi g$). Then we have two cases.

If $F$ is literate, then $\hat{p}_{H \xi F} = \hat{p}_H \xi \hat{p}_F$, and we just apply bifunctorial equations and the induction hypothesis.

If $F$ is letterless, then for $\zeta \in \{\top, \bot\}$ we have

\[ \hat{p}_{H \xi F} \circ (g \xi 1_F) = \delta_H \circ (\hat{p}_H \xi \hat{p}_F) \circ (g \xi 1_F) = \delta_H \circ (g' \xi 1_\zeta) \circ (\hat{p}_G \xi \hat{p}_F), \]

by bifunctorial equations and the induction hypothesis. Then we apply \(\delta^{-} \text{nat}\) to obtain $g' \circ \hat{p}_{G \xi F}$.

We have analogous lemmata, which we call also \textit{Lemma 1}, when $f$ is a \(\delta_{C,D,E}\)-term or a \(\xi_{C,D}\)-term. We also have the following.
Lemma 2. Let $f : A \vdash B$ be a $\xi_{C,D} \rightarrow E$-term, $\xi_{C,D} \leftarrow E$-term or $\xi_{C,D}$-term for $C$ or $D$ or $E$ being a letterless $\xi$-nice formula, or let $f$ be a $\sigma_F \rightarrow E$-term, $\sigma_F \leftarrow E$-term or $\sigma_F$-term for $F$ being $\xi$-nice. Then

\[ \hat{\rho}_B \circ f = \hat{\rho}_A. \]

Proof. We proceed either by induction, applying essentially the following equations of monoidal categories:

\[
\begin{align*}
\hat{\xi}_{C,D,\zeta} &= \hat{\xi}_{C,D} \circ (1_C \xi \hat{\delta}_D), \\
\hat{\xi}_{C,\zeta,E} &= \hat{\xi}_C \xi \hat{\delta}_E, \\
\hat{\xi}_{\zeta,D,E} &= (\hat{\delta}_D \xi 1_E) \circ \hat{\delta}_{DE},
\end{align*}
\]

where $\zeta$ is $\top$ if $\xi$ is $\land$, and $\bot$ if $\xi$ is $\lor$,

\[
\hat{c}_{\top,A} = \hat{\delta}_{A} \circ \hat{\delta}_{A}, \quad \hat{c}_{\bot,A} = \hat{\delta}_{A} \circ \hat{\delta}_{A},
\]

or we infer that under the conditions of the lemma $A^r$ and $B^r$ must be equal, and then we apply essentially Mac Lane’s symmetric monoidal coherence of [37] (see also [38], Section VII.7, and [22], Section 5.3). \( \dashv \)

We prove next the key lemma of this section, whose corollary is the Conservativeness Theorem (we just instantiate statement (1) of this lemma).

Lemma 3. Let $f : A \vdash B$ be an arrow of $\mathbf{SA}'_s$ such that $A$ is $\land$-nice and $B$ is $\lor$-nice.

1. If both $A$ and $B$ are literate, then there is an arrow term $f^r : A^r \vdash B^r$ of $\mathbf{PN}^-$ such that in $\mathbf{SA}'_s$ we have

\[ \hat{\rho}_B \circ f \circ \hat{\rho}^{-1}_A = f^r. \]

2. If $A$ is letterless and $B$ is literate, then for every constant-free $C$ there is an arrow term $f^r : C \vdash C \land B^r$ of $\mathbf{PN}^-$ such that in $\mathbf{SA}'_s$ we have

\[ (1_C \land (\hat{\rho}_B \circ f \circ \hat{\rho}^{-1}_A)) \circ \hat{\delta}_{C} = f^r. \]
§4.3. **SA′ Coherence**

(3) If \( A \) is literate and \( B \) is letterless, then for every constant-free \( C \) there is an arrow term \( f^\ast : A^\ast \vee C \vdash C \) of \( \text{PN}^\ast \) such that in \( \text{SA}'_s \) we have

\[
\delta_C^\ast = \left( \hat{\rho}_B^1 \circ f = \hat{\rho}_A^{-1} \right) \vee 1_C = f^\ast.
\]

Before we start the proof of this lemma, note that it is impossible that \( A \) and \( B \) be both letterless. Otherwise, we would have an arrow of the type \( \top \vdash \bot \) in \( \text{SA}'_s \).

**Proof of Lemma 3.** By the Gentzenization Lemma and the Cut-Elimination Theorem of the preceding two sections, we may suppose that \( f \) is a cut-free Gentzen term. Then we proceed by induction on the complexity of \( f \).

In the basis, we have that \( f \) can only be \( 1_p : p \vdash p \). It cannot be \( 1_\top : \top \vdash \top \) or \( 1_\bot : \bot \vdash \bot \), because \( \top \) is not \( \vee \)-nice and \( \bot \) is not \( \wedge \)-nice.

Then \( f^\ast \) is also \( 1_p \), and statement (1) of the lemma is satisfied, since \( \hat{\rho}_p = \hat{\rho}_p^{-1} = 1_p \).

Suppose \( f \) is of the form \( Sf_1 \) for \( f_1 : A_1 \vdash B_1 \) and \( S \) being \( \hat{B}_X^\ast, \hat{B}_X^\right, \hat{B}_Y^\ast, \hat{B}_Y\), \( \hat{C}_X, \hat{C}_Y, \top, \top\), \( \bot \), or \( \bot \). Then, by the induction hypothesis, we have either statement (1), or (2), or (3), of the lemma for \( f \) replaced by \( f_1 \), and \( A \) and \( B \) replaced by \( A_1 \) and \( B_1 \).

Suppose (1) is the case for \( f_1 \). Then by Lemmata 1 and 2 above we have that

\[
\hat{\rho}_B \circ Sf_1 = \hat{\rho}_A^{-1}
\]

is equal in \( \text{SA}'_s \) to one of the following arrow terms of \( \text{SA}'_s \):

\[
\hat{\rho}_B \circ f_1 = \hat{\rho}_A^{-1} \circ g^r, \quad \text{for } B \text{ being } B_1,
\]

\[
\hat{\rho}_B \circ f_1 = \hat{\rho}_A^{-1}, \quad \text{for } A \text{ being } A_1,
\]

\[
\hat{\rho}_B \circ f_1 = \hat{\rho}_A^{-1},
\]

where \( g^r \) is a \( b_{C,D,E} \)-term, or \( b_{C,D,E} \)-term, or \( c_{C,D} \)-term, with \( \xi \) being \( \wedge \) in the first arrow term and \( \vee \) in the second arrow term. In either case, by the induction hypothesis, we infer (1) for \( f \).

If (2) is the case for \( f_1 \), then by Lemmata 1 and 2 we have that
(1_C \land (\hat{\rho}_B \circ Sf_1 \circ \hat{\rho}_A^{-1})) \circ \hat{\delta}_C^{-1}

is equal in \textsc{Sa}_s to one of the following arrow terms of \textsc{Sa}_s:

\begin{align*}
(1_C \land g'^*) &\circ (1_C \land (\hat{\rho}_B \circ f_1 \circ \hat{\rho}_A^{-1})) \circ \hat{\delta}_C^{-1}, \\
(1_C \land (\hat{\rho}_B \circ f_1 \circ \hat{\rho}_A^{-1})) &\circ \hat{\delta}_C^{-1},
\end{align*}

where \( g' \) is as above with \( \xi \) being \( \lor \). In either case, we apply the induction hypothesis, and infer (2) for \( f \). We proceed analogously if (3) is the case for \( f_1 \).

Suppose that for \( f_i \): \( B_i \vdash A_i \lor C_i \), where \( i \in \{1, 2\} \), we have that \( f \) is

\( \land (f_1, f_2) : B_1 \land B_2 \vdash (A_1 \land A_2) \lor (C_1 \lor C_2) \).

Here \( A_1 \) and \( A_2 \) must be constant-free; otherwise, the target of \( f \) would not be \( \lor \)-nice. Let \( g \) be

\[ \hat{\rho}_{(A_1 \land A_2) \lor (C_1 \lor C_2)} \circ \hat{\varepsilon}_{A_1, A_2, C_1, C_2} \]

Depending on whether \( C_i \) is literate or letterless, we have the following equations in \( \textsc{Sa}_s \):

if \( C_1 \) and \( C_2 \) are both literate, then

\begin{enumerate}
\item \[ g = \hat{\varepsilon}_{A_1, A_2, C_1, C_2} \circ (\hat{\rho}_{A_1 \lor C_1} \land \hat{\rho}_{A_2 \lor C_2}); \]
\end{enumerate}

if \( C_1 \) is literate and \( C_2 \) is letterless, then

\begin{enumerate}
\item \[ g = (\hat{\varepsilon}_{A_2, A_1} \lor 1_C) \circ d_{A_2, A_1, C_1} \circ \hat{\varepsilon}_{A_1 \lor C_1, A_2} \circ (\hat{\rho}_{A_1 \lor C_1} \land \hat{\rho}_{A_2 \lor C_2}); \]
    by applying essentially \((\hat{b} \hat{\delta})\) and \((\hat{d} \hat{\delta})\);
\end{enumerate}

if \( C_1 \) is letterless and \( C_2 \) is literate, then

\begin{enumerate}
\item \[ g = d_{A_1, A_2, C_2} \circ (\hat{\varepsilon}_{A_1 \lor C_1} \land \hat{\rho}_{A_2 \lor C_2}); \]
    by applying essentially
\[ (1_{A_2 \land A_1} \lor \hat{\delta}_{C_2}) \circ \hat{\varepsilon}_{A_2 \land A_1, \bot, C_2} = \hat{\delta}_{A_2 \land A_1 \lor C_2} \]
(c.f. the second equation displayed in the proof of Lemma 2) and \((d \hat{\delta})\);
if $C_1$ and $C_2$ are both letterless, then

$$(IV) \quad g = \hat{\rho}_{A_1 \lor C_1} \land \hat{\rho}_{A_2 \lor C_2},$$

by applying essentially $(b\delta)$ and $(d\delta)$.

Suppose statement (1) of the lemma holds for both $f_1$ and $f_2$. Then $B_1$ and $B_2$ are literate, and we have

$$\hat{\rho}_{(A_1 \land A_2) \lor (C_1 \lor C_2)} = \land (f_1, f_2) \cdot \hat{\rho}_{B_1 \land B_2} = g \cdot (f_1 \land f_2) \cdot (\hat{\rho}_{B_1} \land \hat{\rho}_{B_2})$$

$$= h \cdot (f_1^t \land f_2^t)$$

for $h$ a $\mathbf{PN}^-\text{-term}$; here we apply one of $(1)$-$(4)$ and the induction hypothesis. So $(1)$ holds for $f$.

Suppose $(1)$ holds for $f_1$ and $(2)$ holds for $f_2$. Then $B_1$ is literate and $B_2$ is letterless, and we have

$$\hat{\rho}_{(A_1 \land A_2) \lor (C_1 \lor C_2)} = \land (f_1, f_2) \cdot \hat{\rho}_{B_1 \land B_2} = g \cdot (f_1 \land f_2) \cdot (\hat{\rho}_{B_1} \land \hat{\rho}_{B_2})$$

$$= h \cdot (f_1^t \land 1_{(A_2 \lor C_2)^r}) \cdot f_2^t$$

for $h$ a $\mathbf{PN}^-\text{-term}$; here we apply again one of $(1)$-$(4)$ and the induction hypothesis, which for $f_2$ yields

$$(1)_{B_1^t} \land (\hat{\rho}_{A_2 \lor C_2} \cdot f_2 \cdot \hat{\rho}_{B_2}^t) = \hat{\delta}_{B_2^t} = f_2^t.$$

So $(1)$ holds for $f$.

If $(2)$ holds for $f_1$ and $(1)$ holds for $f_2$, then we proceed analogously to what we had in the previous case. Here, the induction hypothesis for $f_2$ yields

$$(\hat{\rho}_{A_1 \lor C_1} \cdot f_1 \cdot \hat{\rho}_{B_1}^t) \land 1_{B_2^t} = \hat{\delta}_{C_2^t} = f_1^t.$$

Suppose $(2)$ holds for both $f_1$ and $f_2$. Then both $B_1$ and $B_2$ are letterless, and we have

$$(1) \land (\hat{\rho}_{(A_1 \lor A_2) \lor (C_1 \lor C_2)} = \land (f_1, f_2) \cdot \hat{\rho}_{B_1 \land B_2}) \cdot \hat{\delta}_{C}^t$$

$$= (1) \land (g \cdot (f_1 \land f_2) \cdot (\hat{\rho}_{B_1} \land \hat{\rho}_{B_2}) \cdot \hat{\delta}_{C}^t)$$

$$= (1) \land h \cdot (1) \land (\hat{\rho}_{(A_1 \lor C_1)^r} \land (\hat{\rho}_{A_2 \lor C_2} \cdot f_2 \cdot \hat{\rho}_{B_2}^t) \cdot \hat{\delta}_{(A_1 \lor C_1)^r}) \cdot$$

$$\cdot (1) \land (\hat{\rho}_{A_1 \lor C_1} \cdot f_1 \cdot \hat{\rho}_{B_1}^t) \cdot \hat{\delta}_{C}^t$$

$$= (1) \land h \cdot (1) \land f_2^t \cdot f_1^t,$$
for $h$ a $\text{PN}^\neg$-term; here we apply again one of (I)-(IV) and the induction hypothesis. So (2) holds for $f$. Since statement (3) of the lemma cannot hold for $f_1$ and $f_2$, this exhausts all possible cases when $f$ is $\wedge(f_1, f_2)$.

If $f$ is $\vee(f_1, f_2)$, then we proceed in a manner dual to the case when $f$ is $\wedge(f_1, f_2)$, relying on statement (3) of the lemma in places where previously we relied on statement (2).

Suppose that for $f_1 : C \wedge A \vdash B$ we have that $f$ is $\neg^R f_1 : C \vdash \neg A \vee B$. Here $A$ must be constant-free; otherwise the target of $f$ would not be $\vee$-nice. We derive first the following equations of $\text{SA}_f^\prime$:

\begin{align*}
(V) & \quad \check{\cdot}_{A,C,A} \cdot d_{C,A,A} \cdot \check{\Delta}_{A,C} \cdot \hat{\rho}_C^{-1} = \\
& \quad (1 \wedge (\hat{\rho}_C^{-1} \wedge 1_A)) \cdot \check{\cdot}_{A,C,A} \cdot d_{C,A,A} \cdot \check{\Delta}_{A,C},

(VI) & \quad (1_D \wedge (\check{\cdot}_{A,T,A} \cdot d_{T,A,A} \cdot \check{\Delta}_{A,T})) \cdot \check{\hat{\delta}}_D = \\
& \quad (1_D \wedge (1_A \vee \hat{\sigma}_A)) \cdot \check{\Delta}_{A,D},
\end{align*}

by applying naturality equations for (V), and essentially $(b \check{\Delta}^\prime)$ and $(d \check{\hat{\sigma}})$ for (VI).

We have to consider four cases:

(i) both $B$ and $C$ are literate,
(ii) $B$ is literate and $C$ is letterless,
(iii) $B$ is letterless and $C$ is literate,
(iv) both $B$ and $C$ are letterless.

In case (i), we obtain easily by using (V) and the induction hypothesis that statement (1) of the lemma holds for $f$.

In case (ii), for $D$ constant-free we have

\begin{align*}
(1_D \wedge (\check{\rho}_{\neg B} \cdot \check{\hat{\rho}}_1 \cdot \hat{\rho}_C^{-1})) \cdot \check{\hat{\delta}}_D = \\
(1_D \wedge (1_A \vee (\check{\rho}_B \cdot \check{\hat{\rho}}_1 \cdot \hat{\rho}_C^{-1} \wedge 1_A)) \cdot \check{\hat{\sigma}}_A)) \cdot \check{\Delta}_{A,D},
\end{align*}

by using (V) and (VI), and from that, by applying

\((\hat{\rho}_C^{-1} \wedge 1_A) \cdot \check{\hat{\sigma}}_A = \hat{\rho}_{C \wedge A}^{-1}\)
and the induction hypothesis, we obtain that statement (2) of the lemma holds for $f$.

In case (iii), we obtain that statement (1) of the lemma holds for $f$ by applying (V) and the induction hypothesis, which yields

$$\dot{s}_{-A} \circ ((\dot{\rho}_B \circ f_1 \circ \dot{\rho}_C^{-1}) \lor 1_{-A}) = f_1^r.$$

In case (iv), we obtain that statement (2) of the lemma holds for $f$ by applying (V), (VI) and the induction hypothesis.

The only remaining case is when $f$ is $\neg^L f_1$, and this is settled dually to the previous case, where $f$ is $\neg^R f_1$.\[\Box\]
Chapter 5

Involutive Adjunctions and Proof-Net Categories

One finds in the notion of star-autonomous category the well-known adjunction of symmetric monoidal closed categories involving the tensor (multiplicative conjunction), denoted by $\land$ in §3.1, and exponentiation (linear implication), denoted by $\rightarrow$ in §3.1. The functor $A \land$ is left adjoint to the functor $A \rightarrow$ (see §3.1). In proof-net categories, the functor $A \land$ is left adjoint to the functor $\neg A \lor$, and the functor $\neg A \land$ is left adjoint to the functor $A \lor$ (see the end of §2.8).

There is also in proof-net categories, and hence also implicitly in star-autonomous categories, something generalizing the notion of adjunction, which involves dissociativity $d_{A,B,C} : A \land (B \lor C) \vdash (A \land B) \lor C$, and is perceived on another level, where the objects $A$ and $\neg A$ are conceived as functors. We have remarked in §§2.2-3 that the equations $(\Delta \Delta'), (\Delta', \Delta)$, $(\Delta', \Delta')$ and $(\Delta \Delta)$ are related to the triangular equations of an adjunction.

The goal of this chapter is to show that there is in proof-net categories yet another phenomenon of adjunction. The assumptions of proof-net categories involving only negation are a particular, trivial, case of an adjoint situation that we call an involutive adjunction. The notion of involutive adjunction amounts, in a sense to be made precise, to adjunction where an endofunctor is adjoint to itself, which in [18] is called self-adjunction.
§5.1. Self-adjunctions

To fix notation and terminology, we will rely on the following definition of the notion of adjunction (cf. [38], Section IV.1, and [14], Section 4.1.3).

An adjunction is a sextuple \((A, B, F, G, \varphi, \gamma)\) where

- \(A\) and \(B\) are categories,
- \(F\) from \(B\) to \(A\) and \(G\) from \(A\) to \(B\) are functors,
- \(\varphi\) is a natural transformation of \(A\) from the composite functor \(FG\) to the identity functor of \(A\), which means that the following equation holds in \(A\) for every arrow \(f: A_1 \to A_2\) of \(A\):
  \[
  (\varphi \text{ nat}) \quad f \circ \varphi_{A_1} = \varphi_{A_2} \circ F G f,
  \]
- \(\gamma\) is a natural transformation of \(B\) from the identity functor of \(B\) to the composite functor \(GF\), which means that the following equation holds in \(B\) for every arrow \(g: B_1 \to B_2\) of \(B\):
  \[
  (\gamma \text{ nat}) \quad GF g \circ \gamma_{B_1} = \gamma_{B_2} \circ g,
  \]
- the following triangular equations hold in \(A\) and \(B\) respectively:
  \[
  (\varphi \gamma F) \quad \varphi_{FB} \circ F \gamma_B = 1_{FB},
  \]
  \[
  (\varphi \gamma G) \quad G \varphi_A \circ \gamma_{GA} = 1_{GA}.
  \]

A self-adjunction is a quadruple \((S, L, \varphi, \gamma)\) where \((S, S, L, L, \varphi, \gamma)\) is an adjunction (this notion is taken over from [18], Section 10). So, in a self-adjunction, \(L\) is an endofunctor, and the equations \((\varphi \text{ nat})\) and \((\gamma \text{ nat})\) become

- \(f \circ \varphi_{A_1} = \varphi_{A_2} \circ LLf,\)
- \(LLf \circ \gamma_{A_1} = \gamma_{A_2} \circ f,\)

while the triangular equations become

- \((\varphi \gamma L) \quad \varphi_{LA} \circ L \gamma_A = L \varphi_A \circ \gamma_{LA} = 1_{LA}.\)

A \(K\)-self-adjunction is a self-adjunction that satisfies the additional equation
5.1. Self-adjunctions

\[(\varphi \gamma K) \quad L(\varphi_A \circ \gamma_A) = \varphi_{LA} \circ \gamma_{LA},\]

and a $\mathcal{J}$-self-adjunction is a self-adjunction that satisfies the additional equation

\[(\varphi \gamma \mathcal{J}) \quad \varphi_A \circ \gamma_A = 1_A\]

(these notions are also from [18], Section 10). It is easy to see that every $\mathcal{J}$-self-adjunction is a $\mathcal{K}$-self-adjunction (the converse need not hold).

A $\mathcal{J}$-self-adjunction that satisfies

\[(\gamma \varphi) \quad \gamma_A \circ \varphi_A = 1_{LLA}\]

is called a trivial self-adjunction. Note that for trivial self-adjunctions it is superfluous to assume the equations $(\gamma \text{ nat})$ and $(\varphi \gamma G)$, or alternatively $(\varphi \text{ nat})$ and $(\varphi \gamma F)$; these equations can be derived from the remaining ones.

The free self-adjunction $(S, L, \varphi, \gamma)$ generated by $\{p\}$ (we call $p$ a letter, as before) is defined as follows. The category $S$ has as objects the formulae of the propositional language generated by $\{p\}$ with a unary connective $L$. We may identify the formulae $p, Lp, LLp, \ldots$ of this language with the natural numbers $0, 1, 2, \ldots$

The arrow terms of $S$ are defined inductively out of the primitive arrow terms

\[1_A: A \vdash A, \quad \varphi_A: LLA \vdash A, \quad \gamma_A: A \vdash LLA,\]

for every object $A$ of $S$, with the help of the operations of composition $\circ$ and the unary operation that assigns to the arrow term $f: A \vdash B$ the arrow term $Lf: LA \vdash LB$. On these arrow terms we impose the equations of self-adjunctions (cf. §2.1). In the set of these equations we have of course all the equations $f = f$, and this set is closed under symmetry and transitivity of equality, under the rule $(cong \xi)$ for $\xi$ being $\circ$ (see §2.1), and also under the rule

\[(cong L) \quad \frac{f = g}{Lf = Lg} \]

We define analogously the free $\mathcal{K}$-self-adjunction, the free $\mathcal{J}$-self-adjunction and the free trivial self-adjunction generated by $\{p\}$, just by imposing additional equations.
§5.2. Involutive adjunctions

Consider a category \(\mathcal{A}\) and a contravariant functor \(\neg\) from \(\mathcal{A}\) to \(\mathcal{A}\), which means that for \(f: A \rightarrow B\) in \(\mathcal{A}\) we have \(\neg f: \neg B \rightarrow \neg A\) in \(\mathcal{A}\), and the equations (\(\neg 1\)) and (\(\neg 2\)) of §2.8 are satisfied. The contravariant functor \(\neg\) may be conceived either as a functor from the category \(\mathcal{A}^{op}\) to \(\mathcal{A}\), which we denote by \(\neg\), or as a functor from \(\mathcal{A}\) to \(\mathcal{A}^{op}\), which we denote by \(\neg^{op}\).

Suppose that for every object \(A\) of \(\mathcal{A}\) we have an arrow \(n_A: \neg \neg A \rightarrow A\) of \(\mathcal{A}\). The arrow \(n_A\) becomes the arrow \(n_A^{op}: A \rightarrow \neg \neg A\) in \(\mathcal{A}^{op}\).

We say that \(\langle A, \neg, n^{-}\rangle\) is an \(n^{-}\)-adjunction when

\[
\langle A, \mathcal{A}^{op}, \neg, \neg^{op}, n^{-}, n^{-^{op}} \rangle
\]

is an adjunction. This means that in \(\mathcal{A}\) we have for every \(f: A_1 \rightarrow A_2\) the equation

\[
(n^{- \text{ nat}}) \quad f \circ n_A^{-1} = n_{A_2}^{-} \circ \neg f,
\]

alternatively written \(f \circ n_A^{-1} = n_{A_2}^{-} \circ \neg^{op} f\), which also delivers \((n^{-^{op}} \text{ nat})\) in \(\mathcal{A}^{op}\), and the equation

\[
(n^{- \text{ triag}}) \quad n_{\neg A}^{-} \circ n_A^{-1} = 1_{\neg A},
\]

which delivers both the equation \((\varphi \gamma F),\) i.e. \((n^{-n^{-^{op}} \neg})\), in \(\mathcal{A}\), and the equation \((\varphi \gamma G),\) i.e. \((n^{-n^{-^{op}} -^{op}})\), in \(\mathcal{A}^{op}\).

Suppose now that we have as before a category \(\mathcal{A}\) and a contravariant functor \(\neg\) from \(\mathcal{A}\) to \(\mathcal{A}\), and that for every object \(A\) of \(\mathcal{A}\) we have an arrow \(n_A^{-1}: \neg \neg A \rightarrow A\) of \(\mathcal{A}\). The arrow \(n_A^{-1}\) becomes the arrow \(n_A^{-^{op}}: \neg \neg A \rightarrow A\) in \(\mathcal{A}^{op}\).

We say that \(\langle A, \neg, n^{-}\rangle\) is an \(n^{-}\)-adjunction when

\[
\langle \mathcal{A}^{op}, A, \neg^{op}, \neg, n^{-^{op}}, n^{-} \rangle
\]

is an adjunction. This means that in \(\mathcal{A}\) we have for every \(f: A_1 \rightarrow A_2\) the equation

\[
(n^{- \text{ nat}}) \quad \neg f \circ n_A^{-1} = n_{A_2}^{-} \circ f,
\]

which also delivers \((n^{-^{op}} \text{ nat})\) in \(\mathcal{A}^{op}\), and the equation
§5.2. **Involutive adjunctions**

\[(n^- \text{ triang}) \quad \neg n_A^- \circ n_A^- = 1_{\neg A},\]

which delivers both the equation \((\varphi \gamma F)\), i.e. \((n^- \circ \varphi n^- \circ \varphi)\), in \(A^{op}\), and the equation \((\varphi \gamma G)\), i.e. \((n^- \circ \varphi n^- \circ \varphi)\), in \(A\). Note that what we call \(n^-\)-adjunction is called *self-adjunction* in [40] (Section 3.1; cf. also [39], Section I.8), which should not be confused with our notion of self-adjunction in the preceding section.

We say that \(\langle A, \neg, n^- \rangle\) is an *involutive* adjunction when \(\langle A, \neg, n^- \rangle\) is an \(n^-\)-adjunction and \(\langle A, \neg, n^- \rangle\) is an \(n^-\)-adjunction.

A **\(K\)-involutive** adjunction is an involutive adjunction that satisfies the additional equation

\[(n^- n^- K) \quad \neg(n_A^- \circ n_A^-) = n_A^- \circ n_A^-;\]

and a **\(J\)-involutive** adjunction is an involutive adjunction that satisfies the additional equation

\[(n^- n^- J) \quad n_A^- \circ n_A^- = 1_A.\]

It is easy to see that every \(J\)-involutive adjunction is a \(K\)-involutive adjunction (the converse need not hold).

A \(J\)-involutive adjunction that satisfies

\[(n^- n^-) \quad n_A^- \circ n_A^- = 1_{\neg A}\]

is called a *trivial* involutive adjunction.

Note that for trivial involutive adjunctions it is superfluous to assume the equations \((n^- \text{ nat})\) and \((n^- \text{ triang})\), or alternatively \((n^- \text{ nat})\) and \((n^- \text{ triang})\); these equations can be derived from the remaining ones. In trivial involutive adjunctions we have the equations

\[n_A^- = \neg n_A^-;\]
\[n_A^- = \neg n_A^-;\]

which should be compared with the equations \((\nu_A^- \circ \perp \perp)\) and \((\nu_A^- \circ \perp \perp)\) of §3.4.
The free involutive adjunction \( \langle A, \neg, n^- \rangle \) generated by \( \{ p \} \) is defined as follows. The category \( A \) has as objects the formulae of the propositional language generated by \( \{ p \} \) with a unary connective \( \neg \). We may identify these formulae with the natural numbers.

The arrow terms of \( A \) are defined inductively out of the primitive arrow terms

\[
1_A: A \vdash A, \quad n_A: \neg\neg A \vdash A, \quad n_A^-: A \vdash \neg\neg A,
\]

for every object \( A \) of \( A \), with the help of the operations of composition \( \circ \) and the unary operation that assigns to the arrow term \( f: A \vdash B \) the arrow term \( \neg f: \neg B \vdash \neg A \). On these arrow terms we impose the equations of involutive adjunctions (cf. §2.1 and the preceding section). In the set of these equations we have of course all the equations \( f = f \), and this set is closed under symmetry and transitivity of equality, under the rule \((\text{cong } \xi)\) for \( \xi \) being \( \circ \) (see §2.1), and also under the rule

\[
(\text{cong } \neg) \quad \frac{f = g}{\neg f = \neg g}
\]

We define analogously the free \( K \)-involutive adjunction, the free \( J \)-involutive adjunction and the free trivial involutive adjunction generated by \( \{ p \} \), just by imposing additional equations.

Note that the category of the free involutive adjunction generated by an arbitrary set having more than one letter would be the disjoint union of isomorphic copies of the category \( A \) of the free involutive adjunction generated by \( \{ p \} \). An analogous remark applies to the category of the free self-adjunction generated by an arbitrary set having more than one member: it would be the disjoint union of isomorphic copies of the category \( S \) of the free self-adjunction generated by \( \{ p \} \).

§5.3. Self-adjunctions and involutive adjunctions

We are now going to prove that in the free self-adjunction \( \langle S, L, \varphi, \gamma \rangle \) and the free involutive adjunction \( \langle A, \neg, n^-, n^- \rangle \), both generated by \( \{ p \} \), the categories \( S \) and \( A \) are isomorphic categories.
First, we define \( \neg, n^- \) and \( n^\rightarrow \) in \( \mathcal{S} \) in the following manner. On objects we have that \( \neg \) is \( L \), while for the arrow term \( f : A \vdash B \) of \( \mathcal{S} \) we define the arrow term \( \neg f : \neg B \vdash \neg A \) of \( \mathcal{S} \) inductively as follows:

\[
\neg 1_A = L 1_A = 1_{LA} = 1_{\neg A},
\]

\[
\neg \varphi_A = L \gamma, \quad \neg \gamma_A = L \varphi_A, \quad \neg (f \circ g) = \neg g \circ \neg f,
\]

\[
\neg L f = L \neg f.
\]

That this defines an operation \( \neg \) on the arrows of \( \mathcal{S} \) is shown by verifying that if \( f = g \) in \( \mathcal{S} \), then \( \neg f = \neg g \) in \( \mathcal{S} \); we verify, namely, that the equations of \( \mathcal{S} \) are closed under the rule \( (\text{cong } \neg) \) of the preceding section. This is done by a straightforward induction on the length of the derivation of \( f = g \) in \( \mathcal{S} \). For that we use the fact that for every arrow term \( f \) of \( \mathcal{S} \) the arrow term \( \neg f \) is equal in \( \mathcal{S} \) to an arrow term of the form \( Lf' \).

Finally, we have

\[
n^- = \neg, \quad \neg^- = \neg.
\]

Next, we define \( L, \varphi \) and \( \gamma \) in \( \mathcal{A} \) in the following manner. On objects we have that \( L \) is \( \neg \), while for the arrow term \( f : A \vdash B \) of \( \mathcal{A} \) we define the arrow term \( L f : LA \vdash LB \) of \( \mathcal{A} \) inductively as follows:

\[
L 1_A = \neg 1_A = 1_{\neg A} = 1_{LA},
\]

\[
L n^- = \neg n^-,
\]

\[
L n^\rightarrow = \neg n^\rightarrow,
\]

\[
L (f \circ g) = L f \circ L g,
\]

\[
L \neg f = \neg L f.
\]

That this defines an operation \( L \) on the arrows of \( \mathcal{A} \) is shown by verifying that if \( f = g \) in \( \mathcal{A} \), then \( L f = L g \) in \( \mathcal{A} \); we verify, namely, that the equations of \( \mathcal{A} \) are closed under the rule \( (\text{cong } L) \) of §5.1. This is done by a straightforward induction on the length of the derivation of \( f = g \) in \( \mathcal{A} \). For that we use the fact that for every arrow term \( f \) of \( \mathcal{A} \) the arrow term \( L f \) is equal in \( \mathcal{A} \) to an arrow term of the form \( f' \).
Finally, we have

\[ \varphi_A = df \overset{\sim}{n_A}, \quad \gamma_A = df \overset{\sim}{n_A}. \]

We verify easily by induction on the complexity of the arrow term \( f \) that both in \( S \) and in \( A \) we have the equation

\[ (LL\neg\neg) \quad LLf = \neg\neg f. \]

Next we verify that the equations of involutive adjunctions hold for the defined \( \neg, \overset{\sim}{n} \) and \( \overset{\sim}{n} \) in \( S \). This is done in a straightforward manner by induction on the length of derivation. In the basis of this induction, we use \( (LL\neg\neg), (\varphi \text{ nat}) \) and \( (\gamma \text{ nat}) \) to verify \( (\overset{\sim}{n} \text{ nat}) \) and \( (\overset{\sim}{n} \text{ nat}) \), while the equations \( (\overset{\sim}{n} \text{ triang}) \) and \( (\overset{\sim}{n} \text{ triang}) \) reduce to \( (\varphi \gamma L) \). In the induction step, we rely on the closure of \( S \) under \( (\text{cong } \neg) \), which we established above.

We verify also that the equations of self-adjunctions hold for the defined \( L, \varphi \) and \( \gamma \) in \( A \). This is done again in a straightforward manner by induction on the length of derivation. In the basis of this induction, we use \( (LL\neg\neg), (\overset{\sim}{n} \text{ nat}) \) and \( (\overset{\sim}{n} \text{ nat}) \) to verify \( (\varphi \text{ nat}) \) and \( (\gamma \text{ nat}) \), while the equations \( (\varphi \gamma L) \) reduce to \( (\overset{\sim}{n} \text{ triang}) \) and \( (\overset{\sim}{n} \text{ triang}) \). In the induction step, we rely on the closure of \( A \) under \( (\text{cong } L) \), which we established above.

We have a functor \( F_A \) from \( S \) to \( A \) that maps the object of \( S \) corresponding to the natural number \( n \) to the object of \( A \) corresponding to \( n \), and that maps every arrow of \( S \) to the homonymous arrow in the defined \( S \) structure of \( A \). For example,

\[ F_A \varphi p = \varphi p = n p. \]

We define analogously a functor \( F_S \) from \( A \) to \( S \) (cf. the functors \( F \) and \( F' \) in §3.7). That \( F_A \) and \( F_S \) are indeed functors follows from what we established above.

It is trivial that on objects we have that \( F_SF_AA \) is \( A \), and that \( F_AF_SB \) is \( B \). We show next by induction on the complexity of \( f \) that in \( S \) we have

\[ F_SF_A f = f. \]
When \( f \) is of the form \( Lf' \), we make an auxiliary induction on the complexity of \( f' \), in which we use \((LL\neg\neg)\). We show analogously that in \( A \) we have

\[
F_A F_S g = g.
\]

This concludes the proof that \( S \) and \( A \) are isomorphic categories.

We demonstrate analogously that the categories of, respectively,

- the free \( K \)-self-adjunction and the free \( K \)-involutive adjunction,
- the free \( J \)-self-adjunction and the free \( J \)-involutive adjunction,
- the free trivial self-adjunction and the free trivial involutive adjunction,

all generated by \( \{p\} \), are isomorphic categories.

The interest of considering \( K \) and \( J \) versions of self-adjunctions and involutive adjunctions comes from connections with Temperley-Lieb algebras and the associated geometrical interpretation (see [18] and references therein). Roughly speaking, \( K \) is what we find in Temperley-Lieb algebras, where only the number of circles (which correspond to \( \varphi_A \circ \gamma_A \) or \( n_A^A \circ n_A^A \)) counts, while in \( J \) circles are disregarded. (How these circles arise may be grasped from the first diagram in §2.3, where there is a circle involving 7 and 8.)

The free trivial self-adjunction, and hence also the free trivial involutive adjunction, are preorders; namely, all arrows with the same source and target are equal. This follows from the results of [18] (unabridged version) or [19].

§5.4. Trivial involutive adjunctions and proof-net categories

In every proof-net category we encounter a trivial involutive adjunction, where \( \neg \) is defined as in §2.8, while \( n^- \) and \( n^-^- \) are defined as in §2.6. That all the equations of trivial involutive adjunctions are satisfied with these definitions in proof-net categories is easily verified with what we have in §2.8, naturality equations and \( \text{PN}^-^- \) Coherence. According to what we established in the preceding section, in every proof-net category we have a
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subcategory that is a trivial self-adjunction. This does not mean, however, that in every proof-net category, and in $\mathbf{PN}^-$ in particular, we can define the endofunctor $L$ of the trivial self-adjunction.

The notion of star-autonomous category arises out of the notion of symmetric monoidal closed category by assuming in addition the arrows $\nu_A^\to : (A \to \bot) \to \bot \vdash A$ and the isomorphism equations $(\nu \nu)$ that tie these arrows to the arrows $\nu_{A,\bot}^\to : A \vdash (A \to \bot) \to \bot$ of the symmetric monoidal closed structure (see §3.2). For every symmetric monoidal closed category $\mathcal{A}$ we have that $\langle A, \_ \to \bot, \nu_{\_ \to \bot} \rangle$ is an $n^\bot$-adjunction. With $\nu^\bot$ added together with the equations $(\nu \nu)$, we obtain a trivial involutive adjunction (see §5.2).

A non-equational definition of star-autonomous category is obtained by assuming instead of the arrows $\nu_A^\to$ and the equations $(\nu \nu)$ just that $A$ and $(A \to \bot) \to \bot$ are naturally isomorphic. That $\nu_{A,\bot}^\to$ is an isomorphism follows then from a lemma in [30] (Lemma 1.3; see also [29], Section A1.1, Lemma 1.1.1) and the fact that $\langle A, \_ \to \bot, \nu_{\_ \to \bot} \rangle$ is an $n^\bot$-adjunction. If $i_A : (A \to \bot) \to \bot \vdash A$ is a member of a natural isomorphism, then the inverse of $\nu_{A,\bot}^\to : A \vdash (A \to \bot) \to \bot$ is

$$i_A \circ (((i_A \to 1_\bot) \circ \nu_{A \to \bot}^\to) \to 1_\bot) : (A \to \bot) \to \bot \vdash A.$$  

Since $i$ is a natural isomorphism, we have $i_{(A \to \bot) \to \bot} = (i_A \to 1_\bot) \to 1_\bot$. 
Chapter 6

Coherence of Mix-Proof-Net Categories

In this chapter we add mix arrows of the type \( A \land B \vdash A \lor B \) to proof-net categories, with appropriate conditions that will enable us to prove coherence with respect to \( Br \) for the resulting categories, which we call mix-proof-net categories. The mix arrows, which underly the mix principle of linear logic, were treated extensively in [22] (Chapters 8, 10, 11, 13). The proof of coherence for mix-proof-net categories is an adaptation of the proof of coherence for proof-net categories given in Chapter 2.

§6.1. The category MDS

The category \( \text{MDS} \) is defined as the category \( \text{DS} \) in §2.1 save that we have the additional primitive arrow terms

\[
m_{A,B} : A \land B \vdash A \lor B
\]

for all objects, i.e. for all formulae, \( A \) and \( B \) of \( L_{\land, \lor} \), and we assume the following additional equations:

\[
(m \ \text{nat}) \quad (f \lor g) \circ m_{A,B} = m_{D,E} \circ (f \land g), \quad \text{for } f : A \vdash D \text{ and } g : B \vdash E,
\]

\[
(b \ m) \quad m_{A,B,C} \circ \tilde{b}_{A,B,C}^{-1} = d_{A,B,C} \circ (1_A \land m_{B,C}),
\]

\[
(\tilde{b} \ m) \quad \tilde{b}_{C,B,A}^{-1} \circ m_{C,B \lor A} = (m_{C,B} \lor 1_A) \circ d_{C,B,A},
\]

\[
(cm) \quad m_{B,A} \circ \tilde{c}_{A,B} = \tilde{c}_{B,A} \circ m_{A,B}.
\]
The proof-theoretical principle underlying $m_{A,B}$ is called mix (see [22], Section 8.1, and references therein).

To obtain the functor $G$ from MDS to $Br$, we extend the definition of the functor $G$ from DS to $Br$ (see §2.3) by adding the clause that says that $Gm_{A,B}$ is the identity arrow $1_{GA+GB}$ of $Br$. Then we have the following result of [22] (Section 8.4).

**MDS Coherence.** The functor $G$ from MDS to $Br$ is faithful.

In the remainder of this section we will prove some lemmata concerning MDS, which we will use for the proof of coherence in the next section. For that we need some preliminaries.

For $x$ a particular proper subformula of a formula $A$ of $\mathcal{L}_{\land,\lor}$, and $\xi \in \{\land, \lor\}$, we define $A^{-x}$ inductively as follows:

$$(B \xi x)^{-x} = (x \xi B)^{-x} = B,$$

for $x$ a proper subformula of $C$,

$$(B \xi C)^{-x} = B \xi C^{-x},$$

$$(C \xi B)^{-x} = C^{-x} \xi B.$$

For $i \in \{1, 2\}$, let $A_i$ be a formula of $\mathcal{L}_{\land,\lor}$ with a proper subformula $x_i$, which is an occurrence of a letter $q$, and let $x_i$ be the $n_i$-th occurrence of letter counting from the left. We define the following functions $\mu_i : \mathbb{N} - \{n_i-1\} \rightarrow \mathbb{N}$:

$$\mu_i(n) = \begin{cases} n & \text{if } n < n_i - 1 \\ n-1 & \text{if } n > n_i - 1. \end{cases}$$

The definition of tied occurrence of a letter in an arrow of MDS is analogous to what we had in §2.4. Then we can prove the following.

**Lemma 1.** For every arrow term $f : A_1 \vdash A_2$ of MDS such that $x_1$ and $x_2$ are tied in the arrow $f$, there is an arrow term $f^{-q} : A_1^{-x_1} \vdash A_2^{-x_2}$ of MDS such that the members of $\text{part}(G f^{-q})$ are $\{s(\mu_1(m_1)), t(\mu_2(m_2))\}$ for each $\{s(m_1), t(m_2)\}$ in $\text{part}(G f)$, provided $m_i \neq n_i - 1$.

**Proof.** We proceed by induction on the complexity of the arrow term $f$. If $f$ is a primitive arrow term $\alpha_{B_1,...,B_m}$, then for some $j \in \{1,...,m\}$ we
§6.1. \textit{The category MDS}

have that $x_i$ occurs in a subformula $B_j$ of $A_i$. If $x_i$ is a proper subformula of this subformula $B_j$, then $B_j^{−x_i}$ is defined, and $f^{−q}$ is

\[
\alpha_{B_1, \ldots, B_{j−1}, B_{j+1}, \ldots, B_m}
\]

(note that $B_j^{−x_1}$ and $B_j^{−x_2}$ are the same formula). If $x_i$ is not a proper subformula of the subformula $B_j$, then $d_{B_i,B_j}$ is $m_{B_i,B_j}$ or $f^{−q}$ is $1_{A^{−x_1}}$.

If $f$ is $g \cdot h$, then $f^{−q}$ is $g^{−q} \cdot h^{−q}$, and if $f$ is $g \xi h$ for $\xi \in \{\& , \vee\}$, then $f^{−q}$ is either $g^{−q} \xi h$, or $g \xi h^{−q}$, or $g$ when $h = 1_{x_1}$, or $h$ when $g = 1_{x_1}$.

Note that this lemma does not hold for DS, because we cannot cover $d_{B_i,B_j}$.

Here is an example of the application of Lemma 1. If $f : A_1 \vdash A_2$ is

\[
((m_{q,p} \circ (1_q \& c_{q,p}) \circ c_{q,p} \circ \& 1_p)) \circ d_{q,p,q,p} \circ \& (1_{q \& (p \& q)} \circ (1_{q \& (p \& q)}) \circ p,\]

where $x_1$ is the second (rightmost) occurrence of $q$ in $(p \& (q \& p)) \& q$, while $x_2$ is the second occurrence of $q$ in $(q \& (p \& q)) \& p$, then $f^{−q} : A_1^{−q} \vdash A_2^{−q}$ is

\[
((m_{q,p} \circ (1_q \& 1_p) \circ c_{p,q}) \circ \& 1_p)) \circ d_{p,q,p} \circ \& (1_{p \& (q \& p)} \circ (1_{p \& (q \& p)}) : p \& (q \& p) \vdash (q \& p) \& p,
\]

which is equal to $((m_{q,p} \circ c_{p,q}) \circ (1_p) \circ d_{p,q,p}$. As another example, we have that $((m_{q,p} \circ c_{p,q}) \circ \& 1_p) \circ d_{p,q,p})^{−q}$ is equal to $m_{p,q}$.

We define inductively a notion we call a \textit{context} (analogous up to point to notions introduced in §4.1):

\[ \Box \text{ is a context; } \]

if $Z$ is a context and $A$ a formula of $\mathcal{L}_{\& , \&}$, then $Z \& A$ and $A \& Z$ are contexts for $\xi \in \{\& , \&\}$.

Note that now we have contexts like $p \& (q \& \Box)$, which are neither $\&$-contexts nor $\&$-contexts in the sense of §4.1. We define $Z(B)$ and $Z(f)$ as in §4.1, and we use $X$, $Y$, $Z$, . . . for contexts.

For $f : A \vdash C$ an arrow of MDS, we say that an occurrence $x$ of a formula B as a subformula of A and an occurrence $y$ of the same formula
Let \( f : X(p) \land B \vdash Y(p \land B) \) be an arrow term of \( \text{MDS} \) such that the displayed occurrences of \( p \) in the source and target, and also the displayed occurrences of \( B \), are tied in the arrow \( f \). Then, by successive applications of Lemma 1, for each occurrence of a letter in \( B \), we obtain the arrow term \( f^{-B} : X(p) \vdash Y(p) \) of \( \text{MDS} \), and the displayed occurrences of \( p \) in \( X(p) \) and \( Y(p) \) are tied in the arrow \( f^{-B} \).

Let \( f^1 : X(p \land B) \vdash Y(p \land B) \) be the arrow term of \( \text{MDS} \) obtained from \( f^{-B} \) by replacing the occurrences of \( p \) that correspond to those displayed in \( X(p) \) and \( Y(p) \) by occurrences of \( p \land B \). This replacement is made in the indices of primitive arrow terms that occur in \( f^{-B} \), and it need not involve all the occurrences of \( p \) in these indices. For example, if \( X = \Box \land (q \lor p) \) and \( Y = (q \lor \Box) \lor p \), while \( f^{-B} \) is

\[
((m_{q,p} \circ \hat{c}_{p,q}) \lor 1_p) \circ d_{p,q,p} : p \land (q \lor p) \vdash (q \lor p) \lor p,
\]

then \( f^1 \) is

\[
((m_{q,p \land B} \circ \hat{c}_{p \land B,q}) \lor 1_p) \circ d_{p \land B,q,p} : (p \land B) \land (q \lor (p \land B)) \lor p.
\]

Then we can prove the following.

**Lemma 2.** Let \( f : X(p) \land B \vdash Y(p \land B) \) and \( f^1 : X(p \land B) \vdash Y(p \land B) \) be as above. Then there is an arrow term \( h_X : X(p) \land B \vdash X(p \land B) \) of \( \text{DS} \) such that \( f = f^1 \circ h_X \) in \( \text{MDS} \).

**Proof.** We construct the arrow term \( h_X \) of \( \text{DS} \) by induction on the complexity of the context \( X \). For the basis we have that \( h_\Box = 1_p \land B \). In the induction step we have

\[
h_{Z \land A} = (h_Z \land 1_A) \circ \hat{c}_{Z,A}(p) \land B \circ \hat{b}_{Z,A}(p) \land B, \\
h_{Z \lor A} = (h_Z \lor 1_A) \circ \hat{c}_{Z,A}(p) \land B \circ d_{Z,A}(p) \land B, \\
h_{A \land Z} = (1_A \land h_Z) \circ \hat{b}_{A,Z}(p) \land B, \\
h_{A \lor Z} = (1_A \lor h_Z) \circ d_{A,Z}(p) \land B.
\]

It is easy to see that \( Gf = G(f^1 \circ h_X) \), and then the lemma follows by applying \( \text{MDS} \) Coherence.
6.2. MPN\textsuperscript{−} Coherence

Let $f : Y(B \lor p) \vdash B \lor X(p)$ be an arrow term of $\text{MDS}$ such that the displayed occurrences of $p$ in the source and target, and also the displayed occurrences of $B$, are tied in the arrow $f$. Then, as above by Lemma 1, we obtain the arrow term $f^{−B} : Y(p) \vdash X(p)$ of $\text{MDS}$, and the displayed occurrences of $p$ in $Y(p)$ and $X(p)$ are tied in the arrow $f^{−B}$.

Let $f^\dagger : Y(B \lor p) \vdash X(B \lor p)$ be the arrow term of $\text{MDS}$ obtained from $f^{−B}$ by replacing the occurrences of $p$ that correspond to those displayed in $Y(p)$ and $X(p)$ by occurrences of $B \lor p$ (cf. the example above). Then we can prove the following, analogously to Lemma 2\land.

**Lemma 2\lor.** Let $f : Y(B \lor p) \vdash B \lor X(p)$ and $f^\dagger : Y(B \lor p) \vdash X(B \lor p)$ be as above. Then there is an arrow term $h_X : X(B \lor p) \vdash B \lor X(p)$ of $\text{DS}$ such that $f = h_X \cdot f^\dagger$ in $\text{MDS}$.

§6.2. MPN\textsuperscript{−} Coherence

The category $\text{MPN}\textsuperscript{−}$ is defined as the category $\text{PN}\textsuperscript{−}$ in §2.2 save that we have the additional primitive arrow terms $m_{A,B} : A \land B \vdash A \lor B$ for all objects $A$ and $B$ of $\text{PN}\textsuperscript{−}$, and we assume as additional equations $(m \text{ nat})$, $(\hat{b}m)$, $(\hat{b}m)$ and $(cm)$ of the preceding section. To obtain the functor $G$ from $\text{MPN}\textsuperscript{−}$ to $\text{Br}$, we extend the definition of the functor $G$ from $\text{PN}\textsuperscript{−}$ to $\text{Br}$ by adding the clause that says that $Gm_{A,B}$ is the identity arrow $1_{GA+GB}$ of $\text{Br}$.

A mix-proof-net category is defined as a proof-net category (see §2.2) that has in addition a natural transformation $m$ satisfying the equations $(\hat{b}m)$, $(\hat{b}m)$ and $(cm)$. The category $\text{MPN}\textsuperscript{−}$ is up to isomorphism the free mix-proof-net category generated by $\mathcal{P}$.

The category $\text{MPN}$ is defined as the category $\text{PN}$ in §2.5 save that we have the additional primitive arrow terms $m_{A,B}$ for all objects of $\text{PN}$, and we assume as additional equations $(m \text{ nat})$, $(\hat{b}m)$, $(\hat{b}m)$ and $(cm)$. We can prove that $\text{MPN}\textsuperscript{−}$ and $\text{MPN}$ are equivalent categories as in §2.6. (We have an additional case involving $m_{A,B}$ in the proof of the analogue of the Auxiliary Lemma of §2.6, and similar trivial additions elsewhere; otherwise the proof is quite analogous.)

We have a functor $G$ from $\text{MPN}$ to $\text{Br}$ defined by restricting the defi-
nition of the functor $G$ from $\text{MPN}^{-}$ to $\text{Br}$ (cf. the beginning of §2.7), and we will prove the following.

**MPN Coherence.** The functor $G$ from $\text{MPN}$ to $\text{Br}$ is faithful.

The proof of this coherence proceeds as the proof of $\text{PN}$ Coherence in §2.7. The only difference is in the $\wedge\Sigma$-Permutation and $\lor\Sigma$-Permutation Lemmata of §2.5.

The formulation of the $\wedge\Sigma$-Permutation Lemma is modified by replacing $\text{PN}$ and $\text{DS}^{-p}$ by respectively $\text{MPN}$ and $\text{MDS}^{-p}$, where the category $\text{MDS}^{-p}$ is defined as $\text{MDS}$ save that it is generated not by $\mathcal{P}$, but by $\mathcal{P} \cup \mathcal{P}^{-}$ (cf. §2.5); moreover, we assume that $y_{1}$ and $\neg y_{2}$ occur in $E$ within a subformula of the form $p \land (\neg y_{2} \lor y_{1})$ or $\neg p \land (y_{1} \lor \neg y_{2})$. We modify the proof of this lemma as follows.

If in $E$ we have $p \land (\neg y_{2} \lor y_{1})$, then by the stem-increasing equations of §2.5 we have that the $\wedge\Sigma_{p,B}$-term $g : C \vdash D$ is equal to $f'' \circ \wedge\Delta_{p,C}$ for $f'' : C \land (\neg p \lor p) \vdash D$ an arrow term of $\text{DS}^{-p}$, and so for $f : D \vdash E$ an arrow term of $\text{MDS}^{-p}$ satisfying the conditions of the lemma we have in $\text{MPN}$

$$f \ast g = f \ast f'' \circ \hat{\Delta}_{p,C}.$$  

Then we apply Lemma 2 of the preceding section to

$$f \ast f'' : C \land (\neg p \lor p) \vdash E,$$

where $C$ is $X(p)$, $\neg p \lor p$ is $B$ and $E$ is $Y(p \land (\neg p \lor p))$. So for

$$h_{X} : X(p) \land (\neg p \lor p) \vdash X(p \land (\neg p \lor p))$$

an arrow term of $\text{DS}^{-p}$, and

$$(f \ast f'')^{1} : X(p \land (\neg p \lor p)) \vdash Y(p \land (\neg p \lor p))$$

we have

$$f \ast f'' = (f \ast f'')^{1} \ast h_{X}.$$  

By the $\wedge\Sigma$-Permutation Lemma of §2.5 we have

$$h_{X} \ast \hat{\Delta}_{p,C} = g' \ast f'$$
where $g'$ is the $\Delta_{p,p}$-term $X(\Delta_{p,p})$, and by bifunctorial and naturality equations we have

$$(f' \circ f''\dagger) \Delta_{p,p} = Y(\Delta_{p,p}) \bullet (f' \circ f'' \downarrow -\neg p \neg p).$$

Note that $(f' \circ f''\dagger)$ is obtained from $(f' \circ f'' \downarrow -\neg p \neg p) : X(p) \vdash Y(p)$ by replacement of $p$.

So we have in MPN

$$f \circ g = f' \circ f'' \Delta_{p,c}$$
$$= (f' \circ f''\dagger) \bullet hX \Delta_{p,c}$$
$$= (f' \circ f''\dagger) \bullet X(\Delta_{p,p}) \circ f'$$
$$= Y(\Delta_{p,p}) \bullet f''$$

for $f''$, which is $(f' \circ f'' \downarrow -\neg p \neg p) \circ f'$, an arrow term of MDS.$^-p$.

We proceed analogously if in $E$ we have $\neg p \lor (y_1 \lor \neg y_2)$; instead of $\Delta_{p,p}$ we then have $\Delta_{p,p}$. We have an analogous reformulation of the $\neg \neg\lor$-Permutation Lemma of §2.5, with a proof based on Lemma 2 of the preceding section.

Instead of Lemma 2 of the preceding section, we could have proved, with more difficulty, an analogous lemma where $f$ is of type

$$Z(X_1(p) \land X_2(B)) \vdash Y(p \land B),$$

and $f \dagger$ is of one of the following types:

$$Z(X_1(p) \land B) \land (X_2(B))^{-B} \vdash Y(p \land B),$$
$$Z(X_1(p) \land B) \vdash Y(p \land B).$$

Then in the proof of the $\neg \neg\lor$-Permutation Lemma modified for MPN we would not need to pass from $g$ to $f'' \Delta_{p,c}$ via stem-increasing equations, but this alternative approach is altogether less clear.

Note that we have no analogue of Lemma 2 of §2.4 for MDS. The lack of this lemma, on which we relied in §2.5 for the proof of the $\neg \neg\lor$-Permutation and $\neg \neg\land$-Permutation Lemmata, is tied to the modifications we made for these lemmata with MPN. We have also no analogue of Lemma 4 of §2.4, but the analogue of Lemma 3 of §2.4 does hold.
From **MPN** Coherence and the equivalence of the categories **MPN**⁻ and **MPN** we can then infer the following.

**MPN**⁻ **Coherence.** The functor \( G \) from **MPN**⁻ to \( Br \) is faithful.

If we extend the definition of the category \( SA' \) with the primitive arrow terms \( m_{A,B}: A \land B \vdash A \lor B \), together with the equations \( (m \ n a t), (\hat{b} m) \), \( (\hat{b} m) \) and \( (cm) \), we obtain a star-autonomous category of the mix kind. In this category we have arrows of the types \( \bot \land A \vdash A \) and \( A \vdash A \lor \top \), and also \( \bot \vdash \top \). (Arrows of type \( \bot \vdash \top \) may be used to define arrows of the type of \( m_{A,B} \); see the proof of \( SA' \) Coherence in §4.3.)
Chapter 7

Proof Nets

In this, final, chapter we justify the name we have given to proof-net categories. We show how they are related to a two-sided version of the proof nets of [26], such as have already been considered in the literature. Roughly speaking, the Brauerian split equivalences of §2.3 are the graph core of proof nets. As we have shown previously, we need just this core to prove coherence for proof-net categories (see Chapter 2) and restricted coherence for star-autonomous categories (see §4.3).

We discuss next the usefulness of proof nets in general proof theory. While they may be useful to decide the question whether there is an arrow of a particular kind (of a given type, or of a given type with a given graph), we do not find proof nets very useful to answer the question whether a diagram of arrows commutes. We find that this question, which is one of the central questions of general proof theory, is answered more efficiently by the graph core of proof nets.

§7.1. Proof nets and proof-net categories

The connection between proof-net categories and the proof nets of [26] is the following.

For an object $C$ of the category $\textbf{PN}$ of §2.5, we define inductively the target tree $t(C)$ of $C$ in the following manner:
t(p) and t(¬p) are the one-node tree labelled by respectively p and ¬p;

\[
\begin{array}{c}
t(A) \quad t(B) \\
A \land B
\end{array}
\quad \text{and} \quad
\begin{array}{c}
t(A) \quad t(B) \\
A \lor B
\end{array}
\]

We define the source tree s(C) of C inductively in a dual manner:

s(p) and s(¬p) are the one-node tree labelled by respectively p and ¬p;

\[
\begin{array}{c}
s(A) \quad s(B) \\
s(A \land B)
\end{array}
\quad \text{and} \quad
\begin{array}{c}
s(A) \quad s(B) \\
s(A \lor B)
\end{array}
\]

We have in target trees and source trees edges of two kinds: solid edges, like those in the clauses for t(A \land B) and s(A \lor B), and dotted edges, like those in the clauses for t(A \lor B) and s(A \land B). Nodes are labelled by subformulae of C.

An occurrence x of a letter in an object A of the category \textbf{PN} is called negative when ¬x is a subformula of A; otherwise, the occurrence is positive.

An arrow \( \varphi: GA \vdash GB \) of the category Br of §2.3 is said to respect A and B when every member of the partition part(\( \varphi \)) satisfies the following:

- if it is of the form \( \{m_s, n_t\} \), then the \( m+1 \)-th occurrence of letter in A (counting from the left) and the \( n+1 \)-th occurrence of letter in B are occurrences of the same letter, and they are either both positive or both negative;

- if it is of the form \( \{m_i, n_i\} \) for \( i \in \{s, t\} \), then the \( m+1 \)-th and the \( n+1 \)-th occurrences of letter in A, when \( i \) is s, or in B, when \( i \) is t, are occurrences of the same letter, and one of them is positive while the other is negative.

We call proof structures graphs of the form
where $\varphi : GA \vdash GB$ is an arrow of $Br$ that respects $A$ and $B$. The leaf of $t(B)$ labelled by the $n$-th occurrence of letter in $B$ is identified with $n-1$ in the target $GB$ of $\varphi$ (more precisely, with $(n-1)_i$ in $GB_i$), and analogously with $s(A)$ and the source $GA$ of $\varphi$. So all the nodes of this graph are nodes of the trees $s(A)$ and $t(B)$, while $\varphi$ provides only edges, which are solid.

A proof structure where $\varphi$ is $Gf$ for some arrow $f : A \vdash B$ of $\mathbf{PN}$ is called a proof net. This notion of proof net is a two-sided version of the notion, like notions that may be found in [6], [11] and [41].

A two-sided proof net, such as we have introduced above, is transformed into a one-sided proof net, such as those of [26], in the following manner:

The source tree $s(A)$ is now conceived as being tied to $F\neg A$ (see §2.6), and the semicircles in $Gf$ are called axiom links.

There are one-sided proof nets in [26] which are not obtained in this manner from our two-sided proof nets. For example, proof nets like
These proof nets are not tied to the category $\mathbf{PN}$, but to the arrows of $\mathbf{SA}'$ or $\mathbf{SA}'$ whose source is $\top$ and whose target is an object of $\mathbf{PN}$. We cover only one-sided proof nets corresponding to the sequents $\vdash \neg A, B$, but this is not an essential departure from the format of [26].

Composition, i.e. cut, of our two-sided proof nets is reduced to composition in $Br$ in the following manner:

With one-sided proof nets one has instead

where

is transformed into a strip of semicircles
7.1. Proof nets and proof-net categories

called cut links. So applying cut to one-sided proof nets also reduces to composition in $Br$.

One-sided proof nets serve to answer the question whether a given formula is provable in the multiplicative fragment of linear logic without propositional constants. (This question can also be answered, with apparently not more difficulty, by using standard sequent tools.) Formulated in terms of two-sided proof nets and categories, this is the question whether for a given type $A \vdash B$ there is an arrow of this type. We call this the theoremhood problem. A variant of the theoremhood problem, which we call the graph-theoremhood problem, is the problem whether for a given type $A \vdash B$ and a given arrow $\varphi : GA \vdash GB$ of $Br$ there is an arrow $f : A \vdash B$ such that $Gf = \varphi$. Proof nets may serve to solve also this problem (which is of lesser complexity than the general theoremhood problem) for the categories $PN$ and $PN^\neg$. Both the general and the graph-theoremhood problem can be understood either constructively or nonconstructively, depending on the reading of the quantifier “there is” in the formulation of these problems. When only one $\varphi$ respects $A$ and $B$ (and this is the case when $A \vdash B$ is diversified; i.e., when each letter occurs in it exactly twice), then solving the general theoremhood problem for $A \vdash B$ reduces to solving the graph-theoremhood problem.

Proof nets are connected more remotely with the question whether two arrow terms $f$ and $g$ of the same type $A \vdash B$ stand for the same arrow. To answer this latter question, which makes what we call the commuting problem, we do not need $s(A)$ and $t(B)$. One could say that in that context $s(A)$ and $t(B)$ are irrelevant material. As our $PN$ Coherence shows, it is enough to check whether $Gf$ is equal to $Gg$ to answer this question: $PN$ Coherence solves the commuting problem for the category $PN$.

The theoremhood problem for the category $PN$ is solved by the acyclicity and connectedness conditions of proof nets (see [13]; another condition, equivalent to these two, may be found in [26]). A switching is a graph
obtained from a proof structure by erasing for each pair of dotted edges growing in the same direction out of a common node one of these edges. A proof structure is called \textit{acyclic} when each of its switchings is an acyclic graph, and it is called \textit{connected} when each of its switchings is a connected graph.

It follows from [13] that a proof structure is acyclic and connected iff it is a proof net. Acyclicity implies Lemma 3 of §2.4, while connectedness implies Lemma 4 of §2.4. When we pass from $\mathbf{PN}$ to the category $\mathbf{MPN}$ of §6.2, so that in proof nets $Gf$ arises out of an arrow $f$ of $\mathbf{MPN}$ instead of $\mathbf{PN}$, then connectedness is rejected and acyclicity is kept only. A solution of the theoremhood problem for $\mathbf{PN}$ and $\mathbf{MPN}$ yields a solution of this problem for the categories $\mathbf{PN}^{-}$ and $\mathbf{MPN}^{-}$.

As far as we know, a definition of the category $\mathbf{PN}^{-}$ of §2.2, and of the general notion of proof-net category, has first been given in this study. This is an equational definition. The same applies to the category $\mathbf{PN}$. (It is not clear whether the non-equationally defined star-autonomous categories without units of [35] amount to our proof-net categories; the authors of [27] conjecture that their definition is equivalent to ours.) Related categories with the units $\top$ and $\bot$ have, however, been defined previously. These are either the symmetric linearly distributive categories with negation of [11], or the star-autonomous categories of [1] or [2] (see §3.8 for these two notions). Results that might be interpreted as coherence theorems for these categories with respect to proof nets, instead of the category $\mathbf{Br}$, are stated in [11] and [6]. To these papers should be added as the most recent [24], [25] and [34], which are contemporaneous with our work.

In all these papers the categories envisaged have the units, and coherence in our sense with the units is not forthcoming (cf. §4.3). It is not clear how the results of [24], which are about proof nets with $\top$ and $\bot$, overcome the difficulties brought in by adding $\top$ and $\bot$ to $\mathbf{PN}^{-}$, of which the authors of [6] are aware. Another approach for overcoming these difficulties may be found in [34].

These papers do not state that proof nets bring in irrelevant material for the study of the commuting problem, though this may be gathered from [5] (which we have considered in §1.1). It is not even clear whether these papers are oriented towards solving the commuting problem, rather than
some form of the theoremhood problem, or perhaps another problem.

We were clearly oriented towards solving the commuting problem, and our coherence results with respect to $Br$ do that. Equality of arrows in $Br$ is decidable in an elementary way, and the commuting problem is hence decidable in an elementary way in every category for which we have coherence with respect to $Br$.

Our approach differs also in style from these other papers. We have strived to present proofs as complete as possible. We do rely on previous results, but they may all be found exposed in detail in [22]. We find that proofs in the papers cited above can hardly qualify as complete. Sometimes, as in [5], the equations for the categories are not even stated, and have to be guessed. In [24] and [34], the previous results of [5] and [6] are not taken for granted, but other proofs are supplied.

If it is claimed that the category $\text{PN}^-$, though it has not been previously defined equationally, has been defined by coherence, then we are in the situation that we have described at the end of §1.1. For us, as for Mac Lane, coherence is not built into the definition, but it is a theorem.

We would also not be satisfied with defining the category $\text{PN}^-$ as the full subcategory on the objects of $\text{PN}^-$ of the free symmetric linearly distributive category with negation (i.e. of the category $\text{SA}_4^\prime$ of §3.8). That $\text{PN}^-$ is such a subcategory is for us a theorem, which we prove in §4.3, and is not built into the definition.

§7.2. Proof nets in general proof theory

Natural deduction is sometimes presented as being more practical than sequent systems because it involves less writing, less copying. Sequent systems note explicitly undischarged hypotheses, and keep copying them. Here is, for example, a proof of the dissociativity principle, corresponding to the type of $d_{p,q,r}$, in natural deduction format and in sequent format:

\[
\frac{p}{p \land q} \quad \frac{q \lor r}{(p \land q) \lor r} \quad \frac{(p \land q) \lor r}{(p \land q) \lor r}
\]
The natural deduction proof involves 8 formulae, while the sequent proof involves 17 of them.

This advantage of natural deduction over sequents vanishes when we reach the standpoint of general proof theory, where we are interested in proofs and not in provability (see [15]). From that standpoint, it is not correct to say that a proof in natural deduction is a tree whose nodes are formulae. This is not precise enough. One should not forget about the rules used for building the trees, and in particular about the very important rules for discharging hypotheses in the leaves of the trees. We had such a rule in our example with disjunction elimination, and we noted the discharging with the label 1. It is more correct to say that a proof in natural deduction is the building of a tree. The tree itself provides just an incomplete record of this building, part of the information. Rules, i.e. operations (usually partial), for making proofs are not explicit in the tree, and we are very much interested in these operations. We want to see the operations, and do not want to keep guessing about them. We want to see how proofs are inductively built.

In general proof theory one studies inference rules, i.e. operations for building proofs, as in arithmetic one studies operations on natural numbers. A language for arithmetic in which operations would not be explicitly noted could hardly be suitable.

Complete information about proofs in natural deduction is obtained by introducing codes for proofs, in a notation usually inspired by the lambda calculus (following ideas of Curry and Howard). In our example we have

\[
\begin{align*}
 p \vdash p & \quad q \vdash q \\
 p, q \vdash p \land q & \\
 r \vdash r & \\
 p, q \vdash (p \land q) \lor r & \\
 r \vdash (p \land q) \lor r & \\
 p, q \lor r \vdash (p \land q) \lor r & \\
\end{align*}
\]

The natural deduction proof involves 8 formulae, while the sequent proof involves 17 of them.
where $\delta_{y,z}$ is a ternary partial operation binding $y$ and $z$. The tree of formulae contains now just the record of the types of the subterms of the term that codes the proof. This term, rather than the tree of formulae, stands for the proof.

We can incorporate the information about types in the term itself so that the tree of formulae disappears—or, rather, becomes implicit in the tree of the term:

$$\delta_{y,z}(u_{q\lor r}, \iota^1_p(x_p, y_q), \iota^2_r p \land q z_r).$$

In the same way, complete information about the sequent proof is obtained by coding. Here is a coding of the sequent proof above:

$$\frac{1_p : p \vdash p \quad 1_q : q \vdash q}{\land^R(1_p, 1_q) : p, q \vdash p \land q} \quad 1_r : r \vdash r$$

$$\frac{\iota^{1\land^R}(1_p, 1_q) : p, q \vdash (p \land q) \lor r \quad \iota^{2\land^R}_p 1_r : r \vdash (p \land q) \lor r}{\lor^L(\iota^{1\land^R}(1_p, 1_q), \iota^{2\land^R}_p 1_r) : p, q \lor r \vdash (p \land q) \lor r}$$

and the Gentzen term

$$\lor^L(\iota^{1\land^R}(1_p, 1_q), \iota^{2\land^R}_p 1_r),$$

in which the tree of sequents is implicit, is not more complicated than the term

$$\delta_{y,z}(u_{q\lor r}, \iota^1_p(x_p, y_q), \iota^2_r p \land q z_r)$$

above. (Actually, it is slightly shorter.)

The sequent proof becomes recorded with another Gentzen term when it is modified in the style of linear logic, so that the rules for $\lor$ are "multiplicative", as was the rule for introducing $\land$ on the right-hand side:

$$\frac{1_p : p \vdash p \quad 1_q : q \vdash q}{\land^R(1_p, 1_q) : p, q \vdash p \land q} \quad 1_r : r \vdash r$$

$$\frac{\lor^L(\land^R(1_p, 1_q), 1_r) : p, q \vdash r \vdash p \land q, r}{\lor^R \lor^L(\land^R(1_p, 1_q), 1_r) : p, q \lor r \vdash (p \land q) \lor r}$$

After introducing, multiplicatively, a conjunction on the left-hand side, we obtain a Gentzen term that should be equal to the DS arrow term
When they appeared nearly twenty years ago, proof nets were advertised as a new syntax bringing an economy over the sequent calculus, similar to the economy natural deduction brings. Proof nets were said to involve even less copying—even less “bureaucracy”. As for natural deduction, this advantage vanishes from the standpoint of general proof theory. Unfortunately, proof nets never quite reached that standpoint. For the time being, they are approximately where natural deduction would be if typed lambda terms were not introduced to code derivations in natural deduction. Lambek has established in [36] a clear connection between cartesian closed categories and natural-deduction proofs in the conjunction-implication fragment of intuitionistic logic by proving an equivalence between the category of cartesian closed categories and the category of typed lambda calculuses. Could one obtain such a result without introducing typed lambda terms? We would know that proof nets had reached the standpoint of general proof theory if codes were introduced to record the building of proof nets. Because what corresponds to a proof is not a proof net, but rather the building of a proof net. We want to see the operations for building proofs. If such codes were introduced, then we would see that the advantage over sequents vanishes. For example, here is a one-sided proof net corresponding to the proof of $p, q ∨ r ⊢ (p ∧ q) ∨ r$:

![Proof Net Diagram]

and here is a code recording its building:
§7.2. Proof nets in general proof theory

\[
\begin{align*}
1_p &: \neg p, p & 1_q &: \neg q, q \\
\land p, q (1_p, 1_q) &: \neg p, \neg q, p \land q & 1_r &: \neg r, r \\
\land \neg q, r (\land p, q (1_p, 1_q), 1_r) &: \neg p, \neg q \land \neg r, p \land q, r \\
\lor p, q, r (\land \neg q, r (\land p, q (1_p, 1_q), 1_r)): \neg p, \neg q \land \neg r, (p \land q) \lor r
\end{align*}
\]

As far as length is concerned, the term

\[
\lor p, q, r (\land \neg q, r (\land p, q (1_p, 1_q), 1_r))
\]

bears no advantage over the Gentzen term

\[
\lor^R \lor^L (\land (1_p, 1_q), 1_r),
\]

which we had above. The proof net is a substitute for the tree of sequents, which is of secondary importance in general proof theory. The term coding the building of the proof net is not shorter than the term coding the building of the tree of sequents. In general proof theory, these terms occupy the centre of the stage, and not their types, which are implicit in the terms.

Proof nets do bring something more than just the types. It is as if besides \(A \vdash B\) we were also given an arrow \(\varphi : GA \vdash GB\) of \(Br\) that respects \(A\) and \(B\). From our point of view, proof nets are not really syntax. In addition to the type, they bring something that belongs to the category \(Br\), which is for us a model of our syntactical categories, with respect to which we prove completeness with our coherence theorems. Still, with a proof net we do not yet have an analogue of an arrow term \(f\) of type \(A \vdash B\). Such arrow terms, like our arrow terms of \(PN\), are syntax for us.

In some of the early papers of categorial proof theory, and in particular in the book [44], the types of arrow terms were more prominent than the arrow terms, which were often not mentioned. The readers were left to guess the arrow terms out of the types. This has serious disadvantages if the theory is concerned with commuting diagrams of arrows, i.e. equations between arrow terms that stand for proofs. The theory of proof nets suffered up to now from a similar disadvantage when it was proposed as a tool for recording equations between proofs. Again, we have to guess the syntax.
We have to guess the terms from the proof nets, which now replace the types.

Proponents of proof nets claim that theirs is a syntax with an advantage over ordinary syntax. Terms introduce an order on the application of operations, which is deemed unimportant. The term

\[ \lor_{p,q,r} \land_{\neg q, \neg r} (\land_{p,q}(1_p, 1_q), 1_r), \]

which we had above, and the term

\[ \land_{\neg q, \neg r} \lor_{p,q,r} (\land_{p,q}(1_p, 1_q), 1_r) \]

stand for the same proof net, drawn above, granted that the order of application of operations is unimportant. We can, however, introduce an equivalence relation on terms, which would make equivalent these two terms, and deal with equivalence classes of terms. This seems wiser than to relinquish completely the use of terms, and keep guessing what the terms are. Equations between terms, or equations between equivalence classes of terms, are more easily written down, and better understood, than equations between objects that are not syntactical in the ordinary sense, but are some sort of complicated graphs.

Here we touch upon deep questions. Why is it that we prefer languages made of sequences of symbols, rather than other structures of symbols, such as trees, or graphs of another sort (or simpler collections of symbols like multisets or sets)? This may have to do with the fact that we speak in time, which is one-dimensional, and not in something two-dimensional or three-dimensional. We do write in space, but nevertheless writing keeps to a great extent the one-dimensional organization of speech. Without that one-dimensional organization, written language is more difficult to understand. It is usually easier for pupils to absorb matters written on a blackboard by following them as they are being written down, than to face a blackboard fully filled before the lesson. This is so even when the matter on the blackboard is not one-dimensional, as in geometrical drawings. It is probably not fortuitous that Frege’s two-dimensional notation for formulae, where formulae are drawn as trees, has not become standard in logic. (Although the tree structure implicit in formulae is very important, and Frege was not mistaken to stress it.)
We do not think, however, this is only a psychological or typographical problem. There may be mathematical reasons to prefer ordinary syntax. Variables are essential for the language of mathematics, and there is something in the ease with which we introduce variables for ordinary syntactical objects, and perform substitution for these variables, in a handy and very precise way, which seems to be lost by passing to a nonstandard syntax of more complicated graphs. What are variables for proof nets? As an analogue for that, one finds in the literature some sort of empty boxes with wires going out of them, which are not particularly easy to draw or handle, and for which the rules of substitution are not entirely explicit and precise.

A related matter is that we want syntactical notions to be decidable in an elementary way. The notions of term, formula and derivation in a formal system are decidable in this way in standard logic. The notion of proof net is not decidable in an elementary way. It may require considerable effort to recognize that a proof structure is a proof net. This is yet another disadvantage of proof nets taken as syntax.

For these reasons, we do not believe that proofs, as syntactical objects, should be identified with graphs like proof nets. They could perhaps be identified with the building of these graphs, in which building the order of application of operations would be unimportant, but we want to have these operations recorded nevertheless, and this recording is still achieved in the most secure way by sequences of symbols, i.e. terms.

A disadvantage of proof nets from the point of view of category theory is that with one-sided proof nets we do not have a clear information about the source and the target, but this can be remedied with two-sided proof nets. Still another disadvantage is excessive flexibility, verging on imprecision, when the order of formulae in proof nets is in question. If this order is disregarded, then we lose information about the arrows of $B\!e$ implicit in proof nets, and lose a tool for solving the commuting problem. (This order can be disregarded sometimes, with diversified types, but this has to be justified; cf. [22], Section 3.3.) Sometimes the disadvantage of proof nets consists in bringing in irrelevant material, as we have indicated in the previous section, but this is when proof nets are taken as belonging to the model rather than to syntax.

This does not mean that proof nets cannot be useful for some purposes
in general proof theory. They may serve to solve the graph-theoremhood problem for proof-net categories (see the preceding section). They have merit too for having attracted attention to coherence questions in logic, though they were not alone in doing that (interest in the generality of proofs has as much, if not more, merit; see [15] or [22], Sections 1.3-4), and though coherence may be, and usually is, proved without appealing to them, as this was done in [22] and in this work. (We appealed to proof nets only once, to decide a question of theoremhood in the proof of SA: Coherence in §4.3, but this was not indispensable; we could have used a sequent system, or ordinary model-theoretical tools.)

In some circles at the border of logic and theoretical computer science, the belief is still spread with much enthusiasm that proof nets are an indispensable tool. We do not call into question here the role proof nets may play outside general proof theory. We have examined only to a certain extent what they brought up to now to this particular region of logic.
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