Abstract. The purpose of this paper is to connect the proof theory and the model theory of a family of propositional logics weaker than Heyting’s. This family includes systems analogous to the Lambek calculus of syntactic categories, systems of relevant logic, systems related to $BCK$ algebras, and, finally, Johansson’s and Heyting’s logic. First, sequent-systems are given for these logics, and cut-elimination results are proved. In these sequent-systems the rules for the logical operations are never changed: all changes are made in the structural rules. Next, Hilbert-style formulations are given for these logics, and algebraic completeness results are demonstrated with respect to residuated lattice-ordered groupoids. Finally, model structures related to relevant model structures (of Urquhart, Fine, Routley, Meyer, and Maksimova) are given for our logics. These model structures are based on groupoids parallel to the sequent-systems. This paper lays the ground for a kind of correspondence theory for axioms of logics with implication weaker than Heyting’s, a correspondence theory analogous to the correspondence theory for modal axioms of normal modal logics.

The first part of the paper, which follows, contains the first two sections, which deal with sequent-systems and Hilbert-formulations. The second part, due to appear in the next issue of this journal, will contain the third section, which deals with groupoid models.

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**Introduction**

The first impression left by models for the better known systems of relevant logic, $R$ and $E$ (see [8], [15], [21], [22], [23], [24]), was that these models are much less intuitive and much more irregular than Kripke models for intuitionistic or modal logic. However, this impression might be due not so much to the type of models used, but more to the systems these models were used for. The systems $R$ and $E$ are “composite” systems. In the negationless fragment of $R$, which is a subsystem of Heyting’s logic, distribution of conjunction over disjunction does not arise from the ordinary assumptions about conjunction and disjunction, but is added as an extra assumption. Matters are further complicated by having in $R$ a nonintuitionistic negation, which is idempotent and satisfies de Morgan’s Laws. The system $E$ adds to all this considerations about modality, which make its implication a kind of strict implication.

If we do not try to model composite systems with them, models of the type used for relevant logic will prove easier to handle. They will become quite regular, and perhaps even more intuitive. Moreover, they will be suitable not only for relevant propositional logics, but for a whole family of propositional logics included in Heyting’s logic. We shall present here this family of propositional logics. For these logics we shall give models called groupoid models, which are related more to the models of [24] and [8], than to those of [21]. Groupoid models will be rather easy to handle. But their main interest should be in the connection they have with proof theory. The logics in our family have simple and clean sequent-formulations, which are parallel with the groupoid models. This contrasts with the rather problematic position of sequent-formulations for $R$ and $E$. 
The weakest logic in our family will be called $L$. The logic $L$ has the most general sequent-system of the type we shall consider, and it is sound and complete with respect to all groupoid models. It is in a position analogous to the position of the weakest normal modal logic $K$, which is sound and complete with respect to all Kripke modal models. All extensions of $L$ will be obtained by moves in the sequent framework which have their exact parallel in the model-theoretic framework. This foreshadows a correspondence theory between syntax and semantics, analogous to the correspondence theory which exists for normal modal logics. In our family of extensions of $L$ we shall find, among others, systems related to the Lambek calculus of syntactic categories (see [13], [14], [7], [4]), relevant systems without distribution and with intuitionistic negation, systems related to $BCK$ algebras (see [11], [17]), Johansson's propositional calculus, and finally Heyting's propositional calculus.

In the first part of our work we shall consider the sequent-systems for $L$ and its extensions. These sequent-systems have the peculiarity that the comma joining formulae in sequents is taken as a binary operation (cf. [14], [2]). Next, a canonical set of rules is given for the logical operators. By moving up from $L$ to stronger systems, these rules are never changed. What changes are structural assumptions, involving essentially properties of the comma (cf. [6]). We shall give cut-elimination and decidability results for a number of our sequent-systems.

In the second part of our work we shall give Hilbert-style formulations of $L$ and its extensions. In that part we shall also present some simple algebraic completeness results with respect to algebras which we shall call urlogs.

Finally, in the third part we shall present groupoid models for $L$ and its extensions. Groupoid models will be based on groupoids whose binary operation will mirror properties of the comma on the left of the turnstile, in our sequent-systems. For example, if the comma is associative, the groupoid operation will be associative. Completeness proofs with respect to classes of groupoid models will be as elementary as the soundness proofs: we never use the usual infinitary argument, of the Lindenbaum Lemma type, connected with Henkin-style completeness proofs*.

This work is written as a chapter in an intermediate course on propositional calculi. We presuppose for it a certain familiarity with nonclassical propositional calculi, which the reader may have acquired by studying, for example, parts of [1]. A basic understanding of Gentzen-style proof theory, not going beyond [10], is also presupposed. Finally, we assume the reader has some knowledge of Kripke semantics for intuitionistic propositional logic: a familiarity with a few ideas of, for example, [5] (Chapter 5) will suffice.

* I suppose that propositional logics recently investigated in [25] belong to the family of logics investigated here. They should correspond to logics where the comma is not associative.
Otherwise, this work is rather self-contained. It includes a number of elementary results, many of which are probably known to experienced workers in the field, in one form or another. However, these results do not seem to have ever been collected together. We have thought it useful to have them all in one place, because we are not looking for striking single results, but surveying a "landscape". This way we hope to show that behind the intricacies of some results in the literature there is a simple and easily understandable picture.

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1. Sequent-systems

1.1. The language GP. Let P be the language of the propositional calculus with:

- denumerably many propositional variables,
- the binary connectives $\rightarrow$, $\circ$, $\land$, and $\lor$,
- the propositional constants $\top$ and $\bot$,
- the left and right parenthesis.

We shall use $p, p_1, \ldots$, as schematic letters for propositional variables, and $A, B, C, D, A_1, \ldots$, as schematic letters for formulae of P. In the metalanguage we shall use $\rightarrow, \leftrightarrow, iff, &$, or, $\lor$ and $\exists$ with the usual meaning they have in classical logic.

Starting from P we construct the sequent-language GP as follows. The vocabulary of GP is the vocabulary of P plus:

- $\emptyset$, the comma, and the turnstile $\vdash$.

Next we give the following recursive definition of G-terms:

DEFINITION 1. Every formula of P is a G-term, and $\emptyset$ is a G-term. If X and Y are G-terms, then $(X, Y)$ is a G-term.
We shall use $X$, $Y$, $Z$, $X_1$, ..., as schematic letters for $G$-terms. A *subterm* of a $G$-term is specified by the following natural recursive definition:

**Definition 2.** The $G$-term $X$ is a subterm of itself. If $(Y, Z)$ is a subterm of $X$, then $Y$ and $Z$ are subterms of $X$.

Let $X[Y]$ stand for a $G$-term in which $Y$ occurs as a subterm, and let $X[Z]$ be the result of substituting $Z$ for a single $Y$ in $X[Y]$. It will always be clear from the context which $X[Y]$ we have started from in order to obtain $X[Z]$. Next we have the following definition:

**Definition 3.** If $X$ and $Y$ are $G$-terms, then $X \vdash Y$ is a *sequent*.

In order to formulate our systems in $GP$ we shall use the following abbreviation. If $S_0$, $S_1$, ..., $S_n$ are sequents, then the *double-line rule*:

$$
\begin{array}{c}
S_1 \ldots S_n \\
\hline
S_0
\end{array}
$$

will be an abbreviation for the following list of rules:

$$
\begin{array}{c}
S_1 \ldots S_n S_0 \\
\hline
S_0, S_1, \ldots, S_n
\end{array}
$$

If $(R)$ is the name of the double-line rule above, $(R)\downarrow$ is the name of the first rule in the list, and $(R)\uparrow$ designates any of the remaining rules in the list. We shall use an analogous abbreviation with systems in $P$, where $S_0$, $S_1$, ..., $S_n$ are formulae of $P$.

In general, we shall omit the outermost parentheses of $G$-terms.

### 1.2. The system $GL$.

Now we formulate our basic sequent-system, which we shall call $GL$. The system $GL$ is formulated in $GP$. It has the following axiom-schema:

(Id) $A \vdash A$

and the following rules:

**Structural rule:**

(Cut) $X_1 \vdash Y_1[A]$ $X_2[A] \vdash Y_2$ $X_2[X_1] \vdash Y_1[Y_2]$

**Operational rules:**

($\rightarrow$) $X, A \vdash Y[B]$ $X \vdash Y[A \rightarrow B]$

($\circ$) $X[A, B] \vdash Y$ $X[A \circ B] \vdash Y$
It is easy to check that if $X \vdash Y$ is a theorem of $GL$, then $Y$ is either a formula of $P$ or $\emptyset$. We shall call this property of $GL$ the single-conclusion property, and sequents $X \vdash Y$ such that $Y$ is either a formula of $P$ or $\emptyset$ will be called single-conclusion sequents. Because of the single-conclusion property, we could have given (Cut) for $GL$ in the form:

$$\frac{X_1 \vdash A \quad X_2 \vdash A \vdash Y}{X_2[X_1] \vdash Y}$$

and analogously for $(\to)$, $(\wedge)$ and $(\bot)$.

1.3. Additional structural rules. We shall consider extensions of $GL$ with some (possibly all) of the following structural rules:

(Assoc) \[
X[Z_1, (Z_2, Z_3)] \vdash Y \\
\frac{X[(Z_1, Z_2), Z_3] \vdash Y}{X[Z_1, (Z_2, Z_3)] \vdash Y}
\]

(Perm) \[
X[Z_1, Z_2] \vdash Y \\
\frac{X[Z_2, Z_1] \vdash Y}{X[Z_1, Z_2] \vdash Y}
\]

(Contr) \[
X[Z, Z] \vdash Y \\
\frac{X[Z] \vdash Y}{X[Z] \vdash Y}
\]

(Expans) \[
X[Z] \vdash Y \\
\frac{X[Z, Z] \vdash Y}{X[Z] \vdash Y}
\]

(Thin) \[
X[\emptyset] \vdash Y \\
\frac{X[Z] \vdash Y}{X[\emptyset] \vdash Y}
\]

(Thin R). \[
Y \vdash X[\emptyset] \\
\frac{Y \vdash X[\emptyset]}{X[\emptyset] \vdash Y}
\]

(Intr $\emptyset$) \[
X[Z] \vdash Y \\
\frac{X[Z] \vdash Y;}{X[\emptyset, Z] \vdash Y}
\]

(Intr $\emptyset$) \[
X[Z] \vdash Y \\
\frac{X[Z] \vdash Y;}{X[\emptyset, Z] \vdash Y}
\]

(Elim $\emptyset$) \[
X[\emptyset, Z] \vdash Y \\
\frac{X[\emptyset, Z] \vdash Y;}{X[Z] \vdash Y}
\]

(Elim $\emptyset$) \[
X[Z, \emptyset] \vdash Y \\
\frac{X[Z, \emptyset] \vdash Y;}{X[Z] \vdash Y}
\]
By \((\text{Contr} \emptyset)\) and \((\text{Expans} \emptyset)\) we shall designate respectively \((\text{Contr})\) and \((\text{Expans})\) where \(Z\) is \(\emptyset\).

It is clear that in the presence of \((\text{Elim} \emptyset)\) (or \((\text{Elim} \emptyset r)\)) the effect of \((\text{Contr})\) could be obtained with:

\[
\begin{align*}
(\text{Contr } l) & \quad \frac{X[Z, Z] \vdash Y}{X[Z, Z] \vdash Y} \quad \text{(or (Contr } r) \quad \frac{X[Z, Z] \vdash Y}{X[Z, \emptyset] \vdash Y}.
\end{align*}
\]

Analogously, in the presence of \((\text{Intr} \emptyset)\) (or \((\text{Intr} \emptyset r)\)) we could use:

\[
\begin{align*}
(\text{Expans } l) & \quad \frac{X[Z, Z] \vdash Y}{X[Z, Z] \vdash Y} \quad \text{(or (Expans } r) \quad \frac{X[Z, Z] \vdash Y}{X[Z, \emptyset] \vdash Y}.
\end{align*}
\]

to get the effect of \((\text{Expans})\).

We could also envisage a form of Contraction where the \(Z\) omitted is not necessarily adjacent to the other \(Z\), as in the following rules:

\[
\begin{align*}
(\text{Can } l) & \quad \frac{X[Z_1[Z], Z_2[Z]] \vdash Y}{X[Z_1[\emptyset], Z_2[Z]] \vdash Y}; \quad (\text{Can } r) \quad \frac{X[Z_1[Z], Z_2[\emptyset]] \vdash Y}{X[Z_1[Z], Z_2[Z]] \vdash Y}.
\end{align*}
\]

The converse rules, corresponding to Expansion, will be called \((\text{Dupl } l)\) and \((\text{Dupl } r)\) respectively. In the presence of \((\text{Assoc})\), \((\text{Perm})\) and \((\text{Contr } /)\) (or \((\text{Expans } /)\)), the rules \((\text{Can } l)\) and \((\text{Can } r)\) (or \((\text{Dupl } l)\) and \((\text{Dupl } r)\)) are derivable.

It is also clear that with \((\text{Intr} \emptyset l)\), \((\text{Intr} \emptyset r)\) and \((\text{Thin})\) the following rules are derivable:

\[
\begin{align*}
(\text{Thin } l) & \quad \frac{X[Z] \vdash Y}{X[Z, Z_1] \vdash Y} \quad \text{; (Thin } r) \quad \frac{X[Z] \vdash Y}{X[Z, Z_1] \vdash Y}.
\end{align*}
\]

Next, it is easy to see that with \((\text{Contr})\) and \((\text{Thin})\) the rules \((\text{Elim} \emptyset l)\) and \((\text{Elim} \emptyset r)\) are derivable. Also, in the presence of \((\text{Perm})\), only one rule in the pair \((\text{Intr} \emptyset l)\) and \((\text{Intr} \emptyset r)\) (or \((\text{Elim} \emptyset l)\) and \((\text{Elim} \emptyset r)\)) gives the effect of the whole pair. Finally, it is easy to show that with \((\text{Cut})\), the rule \((\text{Expans})\), where \(Z\) is \(A\), is interreplaceable with the rule:

\[
(\text{Mingle}) \quad \frac{X_1 \vdash A \quad X_2 \vdash A}{X_1, X_2 \vdash A}.
\]

(Starting from \(A \vdash A\), by \((\text{Expans})\) we obtain \(A, A \vdash A\), which with two applications of \((\text{Cut})\) yields \((\text{Mingle})\). Conversely, starting from \(A \vdash A\) and \(A \vdash A\), by \((\text{Mingle})\) we obtain \(A, A \vdash A\), which with \((\text{Cut})\) yields \((\text{Expans})\) where \(Z\) is \(A\).)

Let \(\tilde{t}(Z)\) be the formula of \(P\) obtained from the \(G\)-term \(Z\) by replacing every comma by \(\circ\), and every \(\emptyset\) by \(\top\). Then it is easy to see that with \((\circ)\) and \((\top)\) we have the double-line rule:

\[
(i) \quad \frac{X[Z] \vdash Y}{X[\tilde{t}(Z)] \vdash Y}.
\]
Hence, in the presence of (o) and (\(\top\)), the double-line rule (Assoc) could have been given with the following instance of (Assoc):

\[
\frac{X[A_1, (A_2, A_3)] \vdash Y}{X[(A_1, A_2), A_3] \vdash Y}
\]

Analogously, in the remaining structural rules, we could replace \(Z, Z_1\) and \(Z_2\) by \(A, A_1\) and \(A_2\) respectively. With (\(\tilde{t}\)) and (Cut) we can also show that (Assoc) could have been given with the following instance of (Assoc):

\[
\frac{Z_1, (Z_2, Z_3) \vdash Y}{(Z_1, Z_2), Z_3 \vdash Y}
\]

To obtain (Assoc) from this double-line rule we first prove \((Z_1, Z_2), Z_3 \vdash \tilde{t}(Z_1, (Z_2, Z_3))\) and \(Z_1, (Z_2, Z_3) \vdash \tilde{t}((Z_1, Z_2), Z_3)\), and then we apply (\(\tilde{t}\)) and (Cut). We can proceed analogously with the remaining structural rules (with the exception of (Thin R), where we need (\(\bot\)) and (Cut)).

It is easy to check that none of the structural rules considered in this section can spoil the single-conclusion property.

1.4. The system \(GL\). Now we consider a system equivalent to \(GL\), called \(GE\), where the operational rules are not given by double-line rules, but by pairs of rules (or axioms) for introducing the connectives and constants on the left and right of the turnstile. The axioms of \(GL\) are of the form:

(Idp) \(p \vdash p\);

its only structural rule is (Cut) in the form:

\[
\frac{X_1 \vdash A \quad X_2[A] \vdash Y}{X_2[X_1] \vdash Y}
\]

and its operational rules and axioms are:

\((\to L)\) \(X_1 \vdash A \quad X_2[B] \vdash Y\); \(X_2[A \to B, X_1] \vdash Y\);

\((\to R)\) \(X, A \vdash B\); \(X \vdash A \to B\);

\((\circ L)\) \(X[A, B] \vdash Y\); \(X[A \circ B] \vdash Y\);

\((\circ R)\) \(X_1 \vdash A \quad X_2 \vdash B\); \(X_1, X_2 \vdash A \circ B\);

\((\land L)\) \(X[A] \vdash Y\); \(X[B] \vdash Y\); \(X[A \land B] \vdash Y\);

\((\land R)\) \(X \vdash A \quad X \vdash B\); \(X \vdash A \land B\);

\((\lor L)\) \(X[A] \vdash Y\); \(X[B] \vdash Y\); \(X[A \lor B] \vdash Y\);

\((\lor R)\) \(X \vdash A \quad X \vdash B\); \(X \vdash A \lor B\).
To establish that in $GL$ and $GL'$ we have the same theorems, let us first note that by an easy induction on the complexity of $A$ we can show that every instance of $A \vdash A$ is provable in $GL$. Next let us note that:

with $(\rightarrow)\uparrow$ or $(\rightarrow L)$ we can prove $A \rightarrow B, A \vdash B$;
with $(\circ)\uparrow$ or $(\circ R)$ we can prove $A, B \vdash A \circ B$;
with $(\wedge)\uparrow$ or $(\wedge L)$ we can prove $A \wedge B \vdash A$ and $A \wedge B \vdash B$;
with $(\vee)\uparrow$ or $(\vee R)$ we can prove $A \vdash A \vee B$ and $B \vdash A \vee B$;
with $(\bot)\uparrow$ or $(\bot R)$ we can prove $\bot \vdash \bot$;
with $(\top)\uparrow$ or $(\top L)$ we can prove $\top \vdash \top$.

Then the equivalence of $GL$ and $GL'$ follows by straightforward applications of (Cut).

1.5. Cut-elimination in $GL$. The point of considering $GL'$ is to show that (Cut) can be eliminated from it, i.e. that every theorem of $GL'$ can be proved in $GL'$ without using (Cut). This cut-elimination can be proved as follows.

Let the degree of an application of (Cut) be the number of occurrences of $\rightarrow, \circ, \wedge, \vee, \top$ and $\bot$ in the cut-formula $A$, i.e., the formula $A$ eliminated by applying (Cut). Then we shall prove the following lemma:

**Lemma 1.** For every proof of $X \vdash Y$ in $GL$, with a single application of (Cut), there is a proof of $X \vdash Y$ in $GL'$, with no application of (Cut), or with one or two applications of (Cut) of smaller degree.

Since no degree is negative, Lemma 1 establishes that (Cut) can be eliminated (we eliminate applications of (Cut) in proofs starting from the top).

**Proof of Lemma 1.** (1) If the left or right premise of (Cut) is $A \vdash A$, then the conclusion of (Cut) is identical with the remaining premise, which is proved without (Cut).

(2) If the left premise of (Cut) is $\bot \vdash \top$, then in the proof of the right premise $X_2[\top] \vdash Y$ replace every $\top$ on the left of the turnstile by $\bot$. This way we obtain a (Cut)-free proof of $X_2[\bot] \vdash Y$, which was the conclusion of our (Cut).

(3) If the right premise of (Cut) is $\bot \vdash \bot$, then in the proof of the left premise $X_1 \vdash \bot$ replace every $\bot$ on the right of the turnstile by $\bot$. This way we obtain a (Cut)-free proof of $X_1 \vdash \bot$, which was the conclusion of our (Cut).

(4) If the left or right premise of (Cut) is obtained by an operational rule of $GL'$, such that the connective or constant introduced is not the main connective or constant of the cut-formula $A$, then we must work through all possible cases.
in order to show that these operational rules need never precede (Cut) immediately in a proof. For example, the proofs on the left will be replaced by the proofs on the right:

\[(\rightarrow L)\]

\[
\begin{align*}
X_1' & \vdash B \\
X_1'[B \rightarrow C, X_1'] & \vdash A \\
X_2 & \vdash [X_1'[B \rightarrow C, X_1']] \vdash A;
\end{align*}
\]

\[
\begin{align*}
X_1' & \vdash A \\
X_2 & \vdash [X_1'[B \rightarrow C, X_1']] \vdash Y
\end{align*}
\]

(\text{Cut})

\[(\rightarrow R)\]

\[
\begin{align*}
X_2 & \vdash [A, B \rightarrow C] \\
X_1 & \vdash A
\end{align*}
\]

\[
\begin{align*}
X_2 & \vdash [A, B \rightarrow C] \\
X_1 & \vdash A
\end{align*}
\]

(\text{Cut})

(5) If both premises of (Cut) are obtained by introducing the main connective of the cut-formula \(A\), then we must work through all possible cases in order to show that in each case we need not apply the operational rules in question: instead we shall have one or two applications of (Cut) of smaller degree. For example, the proof on the left will be replaced by the proof on the right:

\[(\rightarrow L)\]

\[
\begin{align*}
X_1' & \vdash B \\
X_1'[B \rightarrow C, X_1'] & \vdash A \\
X_2 & \vdash [X_1'[B \rightarrow C, X_1']] \vdash Y
\end{align*}
\]

(\text{Cut})

\[
\begin{align*}
X_1' & \vdash A \\
X_2 & \vdash [X_1'[B \rightarrow C, X_1']] \vdash Y
\end{align*}
\]

(\rightarrow L)

\[(\rightarrow R)\]

\[
\begin{align*}
X_2 & \vdash [A, B \rightarrow C] \\
X_1 & \vdash A
\end{align*}
\]

(\text{Cut})

\[
\begin{align*}
X_2 & \vdash [A, B \rightarrow C] \\
X_1 & \vdash A
\end{align*}
\]

(\rightarrow R)

and both applications of (Cut) on the right are of smaller degree than the application of (Cut) on the left. This exhausts all possible cases. q.e.d.

A consequence of this cut-elimination is that \(GL\), and hence also \(GL\), are \textit{decidable} systems. In order to verify whether \(X \vdash Y\) is provable in \(GL\), we attempt to construct a proof of it in \(GL\) working from the bottom up, without using (Cut). Every upward step eliminates an occurrence of a connective or of a constant, and there is only a finite number of ways to make this step. It follows easily that the total number of proofs that can be attempted is finite. The sequent \(X \vdash Y\) is provable iff one of the attempted proofs is successful.

Another consequence of cut-elimination is the \textit{subformula property} for \(GL\), i.e. if \(X \vdash Y\) is provable in \(GL\), there is a proof of \(X \vdash Y\) in \(GL\) such that all formulae of \(P\) occurring in this proof are subformulae of those formulae of \(P\) which occur in \(X \vdash Y\).

\textbf{1.6. Cut-elimination with additional structural rules.} Let us now consider the extensions of \(GL\) with some (possibly all) of the structural rules (Assoc), (Perm), (Contr), (Thin), (Intr \(\emptyset\)), (Intr \(\emptyset\) \(r\)), (Elim \(\emptyset\)), (Elim \(\emptyset\) \(r\)), (Contr \(\emptyset\)) and (Expans \(\emptyset\)) (note that (Expans) and (Thin \(R\)) are not in this list). In order to show that in any of these extensions (Cut) is eliminable, it is enough to prove analogues of Lemma 1. To prove these analogues, we shall
now show, as in case (4) of the proof of Lemma 1, that none of our additional structural rules need ever precede (Cut) immediately in a proof; the rest is proved as in Lemma 1.

If we have the proof on the left, we shall replace it by the proof on the right:

\[
\frac{X_1 \vdash A \quad X_2[A] \vdash Y}{X_2[X_1] \vdash Y;} \quad \text{(Cut)} \quad \frac{X_1 \vdash A \quad X_2[A] \vdash Y}{X_2[X_1] \vdash Y.} \quad \text{(*)}
\]

where (*) is any of the additional structural rules we have envisaged.

If we have the proof on the left, we shall replace it by the proof on the right:

\[
\frac{X_1 \vdash A \quad X_2[A] \vdash Y}{X_2[X_1] \vdash Y;} \quad \text{(Cut)} \quad \frac{X_1 \vdash A \quad X_2[A] \vdash Y}{X_2[X_1] \vdash Y.} \quad \text{(*)}
\]

where (*) is an application of (Assoc), (Perm), (Intr $\emptyset$), (Intr $\emptyset r$), (Elim $\emptyset$), (Elim $\emptyset r$), (Contr $\emptyset$), (Expans $\emptyset$), or an application of:

\[
\frac{X[Z, Z] \vdash Y}{X[Z] \vdash Y} \quad \text{(Contr)} \quad \frac{X[\emptyset] \vdash Y}{X[Z] \vdash Y} \quad \text{(Thin)}
\]

where the cut-formula $A$ is not a subterm of $Z$.

If we have the proof below, where the cut-formula $A$ is a subterm of the $Z$ involved in (Contr):

\[
\frac{X_1 \vdash A \quad X_2[Z[A], Z[A]] \vdash Y}{X_2[Z[X_1], Z[X_1]] \vdash Y} \quad \text{(Contr)} \quad \frac{X_1 \vdash A \quad X_2[Z[A], Z[A]] \vdash Y}{X_2[Z[X_1], Z[X_1]] \vdash Y} \quad \text{(Cut)}
\]

we shall replace it by the following proof:

\[
\frac{X_1 \vdash A \quad X_2[Z[A], Z[A]] \vdash Y}{X_2[Z[X_1], Z[X_1]] \vdash Y} \quad \text{(Cut)} \quad \frac{X_1 \vdash A \quad X_2[Z[A], Z[A]] \vdash Y}{X_2[Z[X_1], Z[X_1]] \vdash Y} \quad \text{(Cut)}
\]

\[
\frac{X_2[Z[X_1]] \vdash Y.}{X_2[Z[X_1]] \vdash Y.} \quad \text{(Contr)}
\]

In the analogous case with (Thin), the proof on the left will be replaced by the proof on the right:

\[
\frac{X_1 \vdash A \quad X_2[Z[A]] \vdash Y}{X_2[Z[X_1]] \vdash Y.} \quad \text{(Thin)} \quad \frac{X_2[\emptyset] \vdash Y}{X_2[Z[X_1]] \vdash Y.} \quad \text{(Thin)}
\]

This exhausts all possible cases and shows that none of the additional structural rules we have envisaged need ever precede (Cut) immediately in a proof.

We had to exclude (Expans) from the considerations above because of
problematic cases like the following:

$$
\begin{align*}
X_1 \vdash A & \quad X_2 \vdash Z[A] \\
\frac{X_2 \vdash Z[A], Z[A] \vdash Y}{X_2[Z[X_1], Z[A]] \vdash Y} \quad \text{(Expans)}
\end{align*}
$$

If now we apply (Cut) to $X_1 \vdash A$ and $X_2[Z[A]] \vdash Y$, we obtain $X_2[Z[X_1]] \vdash Y$, and it seems we can pass from that to $X_2[Z[X_1], Z[A]] \vdash Y$ only with the help of (Intr $\lor$) and (Thin). But in the presence of (Intr $\lor$) and (Thin), the rule (Expans) is derivable, and hence need not be considered as an independent rule.

Indeed, (Cut) is not eliminable from GL plus (Expans), as witness the sequent $p_1, p_2 \vdash p_1 \lor p_2$, which is provable in this system only with the help of (Cut).

With (Thin $R$), which was also excluded from the considerations above, we have the following problematic case:

$$
\begin{align*}
\frac{X_1 \vdash \emptyset}{X_1 \vdash A} & \quad X_2[A] \vdash Y \\
\frac{X_2[A] \vdash Y}{X_2[X_1] \vdash Y} & \quad \text{(Thin $R$)}
\end{align*}
$$

This proof can be replaced by:

$$
\begin{align*}
X_1 \vdash \emptyset & \quad X_1 \vdash Y \\
\frac{X_1 \vdash Y}{X_2[X_1] \vdash Y} & \quad \text{eventually (Thin $R$)}
\end{align*}
$$

so that in the presence of (Intr $\emptyset$), (Intr $\lor$) and (Thin), we can also eliminate (Cut) from extensions of GL with (Thin $R$).

The rule (Cut) is not eliminable from GL plus (Thin $R$), as witness the sequent $\perp, p_1 \vdash p_2$, which is provable in this system only with the help of (Cut) (to prove this sequent apply (Cut) to $\perp \vdash p_1 \rightarrow p_2$ and $p_1 \rightarrow p_2, p_1 \models p_2$).

From the considerations above it is easy to infer in what cases we obtain cut-elimination with the remaining additional structural rules of Section 1.3.

As in Section 1.5, we can try using cut-elimination to prove decidability and the subformula property. Whereas the subformula property follows easily for all our extensions where (Cut) is eliminable, with decidability the situation is not so clear, and we would have to consider the matter case by case. Since decidability is not our main concern here, we shall just make the following remarks.

The decidability of the extension of GL with (Assoc), or of the extension of GL with (Assoc) and (Perm), is obtained by a simple adaptation of the proof of decidability for GL. Of course, this also means that the equivalent extensions of GL (which we shall consider in Section 2.5) are decidable. Let $GL'$ be the extension of GL with (Assoc), (Perm), (Contr), (Thin) and (Intr $\emptyset$), and let $GH'$
be the extension of $GJ'$ with $(\text{Thin} R)$. It is not difficult to recognize in $GJ'$ and $GH'$, and in the equivalent systems $GJ$ and $GH$ extending $GL$, sequent-formulations of respectively the Johansson and Heyting propositional calculus. Techniques very well known from [10] would show the decidability of $GJ'$ and $GH'$.

1.7. The system $GC$. We have remarked at the end of Section 1.2 that $GL$ could be formulated with a modified presentation of $(\text{Cut})$, $(\rightarrow)$, $(\wedge)$ and $(\perp)$, which takes into account the single-conclusion property. We have not introduced $GL$ in this form in order to be able to pass easily from $GL$ to the system $GC$, which corresponds to the classical propositional calculus. The system $GC$ does not have the single-conclusion property: it is obtained by extending $GL$ with $(\text{Assoc})$, $(\text{Perm})$, $(\text{Contr})$, $(\text{Thin})$, $(\text{Intr} \emptyset l)$, and the rules obtained from these by putting $G$-terms which are on the left of the turnstile to the right, and vice versa. If $(\ldots)$ is the rule we start from, let $(\ldots R)$ be the rule obtained by so permuting left and right $G$-term (as in Section 1.3 $(\text{Thin} R)$ is obtained from $(\text{Thin})$ where $Z$ is $A$).

It is easy to show that with $(\text{Id})$, $(\text{Cut})$, $(\vee)$, $(\text{Contr} R)$, $(\text{Thin} R)$, $(\text{Intr} \emptyset l R)$ and $(\text{Intr} \emptyset r R)$ (this last rule is derivable in $GC$) we have the double-line rule:

$$
\frac{X \vdash Y[A, B]}{X \vdash Y[A \vee B]}
$$

With this double-line rule and $(\perp)$ we can derive:

$$(\tilde{t}) \quad \frac{X \vdash Y[Z]}{X \vdash Y[\tilde{t}(Z)]}$$

where $\tilde{t}(Z)$ is the formula of $P$ obtained from the subterm $Z$ by replacing every comma by $\vee$, and every $\emptyset$ by $\perp$. With $(\tilde{t})$ we have that $(\text{Assoc} R)$ and $(\text{Perm} R)$ can be obtained from:

$$A_1 \vee (A_2 \vee A_3) \vdash (A_1 \vee A_2) \vee A_3$$

$$(A_1 \vee A_2) \vee A_3 \vdash A_1 \vee (A_2 \vee A_3)$$

$$A_1 \vee A_2 \vdash A_2 \vee A_1$$

which are provable in $GL$.

It is also easy to show that with $(\text{Id})$, $(\text{Cut})$, $(\wedge)$, $(\text{Contr})$, $(\text{Thin})$, $(\text{Intr} \emptyset l)$ and $(\text{Intr} \emptyset r)$ we have the double-line rule:

$$
\frac{X[A, B] \vdash Y}{X[A \wedge B] \vdash Y}.
$$

Hence, in $GJ$, $GH$, and $GC$, the connectives $\circ$ and $\wedge$ are equivalent.
2. Hilbert-formulations

2.1. The system $L$. The system $L$ is formulated in the language $P$ of Section 1.1. It has the following axiom-schemata and rules:

\[
\begin{align*}
(1) \quad & A \to A \\
(2) \quad & \frac{A \to B \quad B \to C}{A \to C} \\
(3) \quad & \frac{A \to A_1 \quad B \to B_1}{(A \to B) \to (A \to B_1)} \\
(4) \quad & ((A \to B) \circ A) \to B \\
(5) \quad & A \to (B \to (A \circ B)) \\
(6) \quad & \frac{A \to A_1 \quad B \to B_1}{(A \circ B) \to (A_1 \circ B_1)} \\
(7) \quad & (A \land B) \to A \\
(8) \quad & (A \land B) \to B \\
(9) \quad & \frac{C \to A \quad C \to B}{C \to (A \land B)} \\
(10) \quad & A \to (A \lor B) \\
(11) \quad & B \to (A \lor B) \\
(12) \quad & \frac{A \to D \quad B \to D}{(A \lor B) \to D} \\
(13) \quad & \frac{C \to (A \to D) \quad C \to (B \to D)}{C \to ((A \lor B) \to D)} \\
\end{align*}
\]

It is easy to check that every theorem of $L$ is of the form $A \to B$. Note that $\top$ and $\bot$ are idle in $L$, i.e. there are no specific assumptions involving $\top$ and $\bot$. So, $\top$ and $\bot$ behave in $L$ as arbitrary propositional variables, and $L$ in $P$ without $\top$ and $\bot$ is essentially the same system as $L$. If we define $\neg A$ as $A \to \bot$, the negation of $L$ will be of the same type as the "purely implicational negation" of the Johansson propositional calculus (see Section 2.10).

We shall now survey some theorems and derived rules of $L$. First, it is easy to show that (3) is replaceable in $L$ by the rules:

\[
\begin{align*}
(3') \quad & \frac{B \to B_1}{(A_1 \to B) \to (A_1 \to B_1)}; \quad (3'') \quad & \frac{A \to A_1}{(A \to B_1) \to (A \to B_1)}.
\end{align*}
\]

It is also easy to show that (3), (4) and (5) are replaceable in $L$ by the double-line rule:

\[
\begin{align*}
(14) \quad & \frac{C \circ A \to B}{C \to (A \to B)}.
\end{align*}
\]
Next, we easily obtain that the following rules are derivable in $L$:

$$
(15) \quad \frac{A \rightarrow A_1 \quad B \rightarrow B_1}{(A \land B) \rightarrow (A_1 \land B_1)}
$$

$$
(16) \quad \frac{A \rightarrow A_1 \quad B \rightarrow B_1}{(A \lor B) \rightarrow (A_1 \lor B_1)}
$$

These two rules together with (3) and (6) guarantee that $L$ is closed under the following Rule of Replacement:

$$
(17) \quad \frac{A \equiv B}{C \equiv C'_B}
$$

where $A \equiv B$ is an abbreviation for $A \rightarrow B$ and $B \rightarrow A$, and $C'_B$ is obtained from $C$ by substituting zero or more occurrences of $A$ by $B$.

It is easy to check that in $L$ we have the following theorems:

$$
(18) \quad ((A \land B) \land C) \equiv (A \land (B \land C))
$$

$$
(19) \quad (A \land B) \equiv (B \land A)
$$

$$
(20) \quad A \equiv (A \land A)
$$

$$
(21) \quad ((A \lor B) \land A) \equiv A
$$

$$
(22) \quad ((A \lor B) \lor C) \equiv (A \lor (B \lor C))
$$

$$
(23) \quad (A \lor B) \equiv (B \lor A)
$$

$$
(24) \quad A \equiv (A \lor A)
$$

$$
(25) \quad ((A \lor B) \lor A) \equiv A
$$

$$
(26) \quad (C \circ (A \lor B)) \equiv ((C \circ A) \lor (C \circ B))
$$

$$
(27) \quad ((A \lor B) \circ C) \equiv ((A \circ C) \lor (B \circ C)).
$$

A factor of a formula $A$ of $P$ is defined recursively as follows: $A$ is a factor of $A$; if $B \circ C$ is a factor of $A$, then $B$ and $C$ are factors of $A$. Let $A[B]$ stand for a formula of $P$ in which $B$ occurs as a factor, and let $A[C]$ be the result of substituting $C$ for a single $B$ in $A[B]$ (it will always be clear from the context which $A[B]$ we have started from in order to obtain $A[C]$). Then with the help of (26) and (27), it is not difficult to show by induction on the complexity of $C$ that (12) and (13) can be replaced in $L$ by:

$$
(28) \quad \frac{C[A] \rightarrow D \quad C[B] \rightarrow D}{C[A \lor B] \rightarrow D}.
$$

Finally, it is easy to see that in $L$ we have the following double-line rules:

$$
(29) \quad \frac{A \equiv (A \land B)}{A \rightarrow B};
$$

$$
(30) \quad \frac{(A \lor B) \equiv B}{A \rightarrow B}.
$$

2.2. $L$ and urlogs. First we give the definition of algebras which we shall call uni-residuated lattice-ordered groupoids (abbreviated by urlog), and which are right-residuated lattice-ordered groupoids in the sense of [9] (pp. 189-191):
**Definition 4.** The algebra \(<\mathcal{A}, \setminus, \cdot, \cap, \cup, 1, 0>\) is an urlog if 1 and 0 are elements of \(\mathcal{A}\) (not necessarily distinct), \(\mathcal{A}\) is closed under the binary operations \(\setminus, \cdot, \cap\) and \(\cup\), and for every \(a, b, c \in \mathcal{A}\) we have:

\[
\begin{align*}
(i) \quad & (a \cap b) \cap c = a \cap (b \cap c) \\
(ii) \quad & a \cap b = b \cap a \\
(iii) \quad & a = a \cap a \\
(iv) \quad & (a \cup b) \cap a = a \\
(v) \quad & (a \cup b) \cup c = a \cup (b \cup c) \\
(vi) \quad & a \cup b = b \cup a \\
(vii) \quad & a = a \cup a \\
(viii) \quad & (a \cap b) \cup a = a \\
(ix) \quad & c \cdot (a \cup b) = (c \cdot a) \cup (c \cdot b) \\
(x) \quad & (a \cup b) \cdot c = (a \cdot c) \cup (b \cdot c) \\
(xi) \quad & a \leq b \iff a \cdot b \leq c, \text{ where } a \leq b \text{ is an abbreviation for } a = a \cap b.
\end{align*}
\]

Note that in Definition 4 nothing specific is assumed about 1 and 0, save that they are elements of \(\mathcal{A}\).

It is easy to show that for any urlog we can prove:

\[
\begin{align*}
(xii) \quad & a = a \cap b \iff a \cup b = b \\
(xiii) \quad & \leq \text{ is a partial order on } \mathcal{A} \\
(xiv) \quad & (a \leq a_1 \& b \leq b_1) \implies a_1 \setminus b \leq a_1 \cap b_1 \\
& \implies a \cdot b \leq a_1 \cdot b_1 \\
& \implies a \cap b \leq a_1 \cap b_1 \\
& \implies a \cup b \leq a_1 \cup b_1.
\end{align*}
\]

Next, we introduce the following definition:

**Definition 5.** Let \(V\) be the set of propositional variables of \(P\), and let \(F\) be the set of formulae of \(P\). Given an urlog with domain \(\mathcal{A}\), and a mapping \(v_0: V \rightarrow \mathcal{A}\), we define the valuation \(v: F \rightarrow \mathcal{A}\) by the following recursion:

\[
\begin{align*}
v(p) &= v_0(p) \\
v(A \rightarrow B) &= v(A) \setminus v(B) \\
v(A \circ B) &= v(A) \cdot v(B) \\
v(A \land B) &= v(A) \cap v(B) \\
v(A \lor B) &= v(A) \cup v(B) \\
v(\top) &= 1 \\
v(\bot) &= 0.
\end{align*}
\]

By a straightforward induction on the length of proof of \(A \rightarrow B\) in \(L\), we can prove the following soundness lemma for \(L\):

**Lemma 2.** If \(A \rightarrow B\) is provable in \(L\), then for every urlog and every valuation \(v\) we have \(v(A) \leq v(B)\).

Next, let \(|A| = \{B: A \Rightarrow B\text{ is provable in } L\}\), and let:

\[
\begin{align*}
|A| \setminus |B| &= |A \rightarrow B| \\
|A| \cdot |B| &= |A \circ B| \\
|A| \cap |B| &= |A \land B| \\
|A| \cup |B| &= |A \lor B|.
\end{align*}
\]
The Rule of Replacement (17) guarantees that these equalities define genuine operations on the set $\mathcal{L} = \{|A|: A \in F\}$. The Lindenbaum algebra of $L$ is the algebra defined on $\mathcal{L}$ by these operations, i.e. the algebra $\langle \mathcal{L}, \lor, \land, \top, \bot, \vdash \rangle$. Then we prove the following lemma:

**Lemma 3.** The Lindenbaum algebra of $L$ is an urlog.

**Proof.** The equalities (i)-(x) follow immediately from (18)-(27) of Section 2.1, whereas (xi) is a consequence of (14) and (29). q.e.d.

(Note that in the Lindenbaum algebra of $L$ we have $\top \neq \bot$, since $\top \vdash \bot$ is not provable in $L$: if we identify $\circ$ and $\land$, all theorems of $L$ are tautologies.)

With the help of this lemma we can now prove the following soundness and completeness result:

\[ A \rightarrow B \text{ is provable in } L \text{ iff for every urlog and every valuation } v \text{ we have } v(A) \leq v(B). \]

From left to right we use Lemma 2. For the other direction, suppose that for every urlog and for every $v$ we have $v(A) \leq v(B)$. Then by Lemma 3, for the Lindenbaum algebra of $L$ we have $|A| \leq |B|$, since $v(A) = |A|$ is a valuation from $F$ to $\mathcal{L}$. From $|A| \leq |B|$, by definition we obtain $|A| = |A| \cap |B|$, i.e., $|A| = |A \land B|$. Hence, $A \vdash (A \land B)$ is provable in $L$, from which by (29) we obtain that $A \rightarrow B$ is a theorem of $L$.

We can also prove the following lemma:

**Lemma 4.** The Lindenbaum algebra of $L$ is a free algebra in the class of all urlogs, the set $\mathcal{G} = \{|p|: p \in V\}$ being the set of free generators of this algebra.

**Proof.** It is clear that $\mathcal{G}$ generates all elements of $\mathcal{L}$ (where $\top$ and $\bot$ are nullary operations). Let $\mathcal{A}$ be the domain of an urlog and let $f: \mathcal{G} \rightarrow \mathcal{A}$. Then let $v_0(p) = f(|p|)$. Starting from $v_0$ we define the valuation $v: F \rightarrow \mathcal{A}$. Next let $h(|A|) = v(A)$. For $h$ to be well-defined we must have that $|A| = |B|$ implies $v(A) = v(B)$; i.e. we must have that if $A \equiv B$ is provable in $L$, then $v(A) = v(B)$, and this follows from Lemma 2. It is easy to see that $h: \mathcal{L} \rightarrow \mathcal{A}$ is a homomorphism. q.e.d.

2.3. GL and L. Let $\tilde{t}(Z)$ be the formula of $P$ obtained from a $G$-term $Z$ by replacing every comma by $\circ$, and every $\emptyset$ by $\top$, as in Section 1.3; and let $\tilde{t}(Z)$ be the formula of $P$ obtained from $Z$ by replacing every comma by $\lor$, and every $\emptyset$ by $\bot$, as in Section 1.7. In sequent-systems with the single-conclusion property, if $X \vdash Y$ is provable, there are no commas occurring in $Y$. Since in the sequel we shall deal only with such sequent systems, $\tilde{t}(Y)$ will be either $Y$ itself if $Y$ is not $\emptyset$, or $\bot$ if $Y$ is $\emptyset$. We shall now prove the following lemma:

**Lemma 5.** If $X \vdash Y$ is provable in $GL$, then $\tilde{t}(X) \rightarrow \tilde{t}(Y)$ is provable in $L$.

**Proof.** By induction on the length of proof of $X \vdash Y$ in $GL$.

If $X \vdash Y$ is $A \vdash A$, then in $L$ we have $A \rightarrow A$. 
If $X_2[X_1] \vdash Y$ is obtained by (Cut) from $X_1 \vdash A$ and $X_2[A] \vdash Y$, then in $L$ from $\bar{t}(X_1) \rightarrow A$, by using (6), we can pass to $\bar{t}(X_2[X_1]) \rightarrow \bar{t}(X_2[A])$, and then with $\bar{t}(X_2[A]) \rightarrow \bar{t}(Y)$ and (2) we obtain $\bar{t}(X_2[X_1]) \rightarrow \bar{t}(Y)$.

If $Y \vdash X$ is obtained by $(\rightarrow)$, we use (14). The case when $X \vdash Y$ is obtained by $(\circ)$ is trivial.

If $X \vdash Y$ is obtained by $(\land)\downarrow$, we use (9), and if it is obtained by $(\land)^\uparrow$, we use (7) (or (8)) and (2). If $X \vdash Y$ is obtained by $(\lor)\downarrow$, we use (28), and if it is obtained by $(\lor)^\uparrow$, we use (10) (or (11)), (6) and (2). The cases when $X \vdash Y$ is obtained by $(\top)$ and $(\bot)$ are trivial. q.e.d.

As an immediate corollary of Lemma 5 we have that if $A \vdash B$ is provable in $GL$, then $A \rightarrow B$ is provable in $L$. The converse of this implication is proved by a straightforward induction on the length of proof of $A \rightarrow B$ in $L$. (In Section 3.5 we shall give another proof of this converse implication.) Hence, we have the following:

(I) $A \vdash B$ is provable in $GL$ iff $A \rightarrow B$ is provable in $L$.

### 2.4. The $(\rightarrow, \circ)$ and implicational fragment of $L$. Let $L_0$ be the system in $P$ without $\land$ and $\lor$, given by the axiom-schemata and rules (1)-(6). Let $\langle \mathcal{S}, \land, \land, \leq, \top, \bot \rangle$ be the Lindenbaum algebra of $L_0$, where $|A| \leq |B|$ iff $A \rightarrow B$ is provable in $L_0$. Right-residuated partially ordered groupoids are algebras $\langle \mathcal{A}, \land, \land, \leq, 1, 0 \rangle$, where $\leq$ is a partial order on $\mathcal{A}$ and (xi) and the first two implications of (xiv) hold (as a matter of fact, the first implication of (xiv) follows from the second implication and (xi)).

It is not difficult to show that $L_0$ is sound and complete, in the sense of (0), with respect to right-residuated partially ordered groupoids, and that the Lindenbaum algebra of $L_0$ is free in the class of all such groupoids. If $GL_0$ is the system in $GP$ without $\land$ and $\lor$, obtained from $GL$ by discarding $(\land)$ and $(\lor)$, then it is easy to demonstrate the analogue of (I) for $GL_0$ and $L_0$.

That $L_0$ is indeed the $(\rightarrow, \circ, \top, \bot)$ fragment of $L$ can be shown with the help of $GL_0$, which is obtained from $GL_0$ as $GL$ is obtained from $GL$. The system $L_0$ in $P$ with only $\rightarrow$ and $\circ$, gives the $(\rightarrow, \circ)$ fragment of $L$.

Suppose $A_1 \rightarrow A_2$ is a purely implicational formula of $P$, and suppose $A_1 \vdash A_2$ is provable in $GL$. It is not difficult to show (using the results of Sections 1.4 and 1.5) that in that case $A_1 \vdash A_2$ is provable in the sequent-system given by:

\[
p \vdash p; \quad X, A \vdash B; \quad C \vdash A; \quad (\ldots(B, B_1), \ldots, B_{n-1}) \vdash B_n; \quad X \rightarrow A \rightarrow B; \quad (\ldots ((A \rightarrow B, C), B_1), \ldots, B_{n-1}) \vdash B_n
\]

from which it easily follows that $A_1$ is the same formula as $A_2$. So, the implicational fragment of $L$ is given by $A \rightarrow A$, i.e. nothing but identities is provable in this implicational logic. Note that nothing but identities follows from (1), (2) and (3).
2.5. Extensions of $L$. We shall now consider a number of extensions of $L$. These extensions are obtained by extending $L$ with some (possibly all) of the axiom-schemata or axioms given in the first column of the table which we shall produce below. (An extension of $L$ inherits from $L$ both its axioms and rules.) In the second column of this table we indicate what we must assume about urlogs in order to obtain a soundness and completeness result like (0) for our extensions. Also the Lindenbaum algebra of a given extension will be free in the class of all corresponding urlogs. Finally, in the third column we indicate with what structural rules we extend GL (or GL') in order to obtain a sequent-system which corresponds to a given extension of $L$ as GL corresponds to $L$ in (I):

<table>
<thead>
<tr>
<th>$L +$</th>
<th>urlog +</th>
<th>$GL$ (GL') +</th>
</tr>
</thead>
<tbody>
<tr>
<td>(31) $(A_1 \circ A_2) \circ A_3 \rightarrow (A_1 \circ (A_2 \circ A_3))$</td>
<td>$(a_1 \cdot a_2) \cdot a_3 \leq a_1 \cdot (a_2 \cdot a_3)$</td>
<td>(Assoc) ↓</td>
</tr>
<tr>
<td>(32) $(A_1 \circ (A_2 \circ A_3)) \rightarrow ((A_1 \circ A_2) \circ A_3)$</td>
<td>$a_1 \cdot (a_2 \cdot a_3) \leq (a_1 \cdot a_2) \cdot a_3$</td>
<td>(Assoe) ↑</td>
</tr>
<tr>
<td>(33) $(A_2 \circ A_1) \rightarrow (A_1 \circ A_2)$</td>
<td>$a_2 \cdot a_1 \leq a_1 \cdot a_2$</td>
<td>(Perm)</td>
</tr>
<tr>
<td>(34) $A \rightarrow (A \circ A)$</td>
<td>$a \leq a \cdot a$</td>
<td>(Contr)</td>
</tr>
<tr>
<td>(35) $(A \circ A) \rightarrow A$</td>
<td>$a \cdot a \leq a$</td>
<td>(Expans)</td>
</tr>
<tr>
<td>(36) $A \rightarrow \top$</td>
<td>$a \leq 1$</td>
<td>(Thin)</td>
</tr>
<tr>
<td>(37) $\bot \rightarrow A$</td>
<td>$0 \leq a$</td>
<td>(Thin R)</td>
</tr>
<tr>
<td>(38) $(\top \circ A) \rightarrow A$</td>
<td>$1 \cdot a \leq a$</td>
<td>(Intr Ωl)</td>
</tr>
<tr>
<td>(39) $(A \circ \top) \rightarrow A$</td>
<td>$a \cdot 1 \leq a$</td>
<td>(Intr Ωr)</td>
</tr>
<tr>
<td>(40) $A \rightarrow (\top \circ A)$</td>
<td>$a \leq 1 \cdot a$</td>
<td>(Elim Ωl)</td>
</tr>
<tr>
<td>(41) $A \rightarrow (A \circ \top)$</td>
<td>$a \leq a \cdot 1$</td>
<td>(Elim Ωr)</td>
</tr>
<tr>
<td>(42) $\top \rightarrow (\top \circ \top)$</td>
<td>$1 \leq 1 \cdot 1$</td>
<td>(Contr Ω)</td>
</tr>
<tr>
<td>(43) $(\top \circ \top) \rightarrow \top$</td>
<td>$1 \cdot 1 \leq 1$</td>
<td>(Expans Ω)</td>
</tr>
<tr>
<td>(44) $(\top \circ A) \rightarrow (A \circ A)$</td>
<td>$1 \cdot a \leq a \cdot a$</td>
<td>(Contr l)</td>
</tr>
<tr>
<td>(45) $(A \circ \top) \rightarrow (A \circ A)$</td>
<td>$a \cdot 1 \leq a \cdot a$</td>
<td>(Contr r)</td>
</tr>
<tr>
<td>(46) $(A \circ A) \rightarrow (\top \circ A)$</td>
<td>$a \cdot a \leq 1 \cdot a$</td>
<td>(Expans l)</td>
</tr>
<tr>
<td>(47) $(A \circ A) \rightarrow (A \circ \top)$</td>
<td>$a \cdot a \leq a \cdot 1$</td>
<td>(Expans r)</td>
</tr>
<tr>
<td>(48) $(B \circ A) \rightarrow A$</td>
<td>$b \cdot a \leq a$</td>
<td>(Thin l)</td>
</tr>
<tr>
<td>(49) $(A \circ B) \rightarrow A$</td>
<td>$a \cdot b \leq a$</td>
<td>(Thin r)</td>
</tr>
</tbody>
</table>

It is easy to find the expressions in the first and second column which would correspond to (Canc l), (Canc r), (Dupl l) and (Dupl r). For example, to (Canc l) would correspond $(B_1[\top] \circ B_2[A]) \rightarrow (B_1[A] \circ B_2[A])$ and $b_1[1] \cdot b_2[a] \leq b_1[a] \cdot b_2[a]$, where a schema $B[A]$ is used as in Section 2.1, and $b[a]$ is defined analogously.

Note that in all extensions of $L$ we have envisaged above, every theorem is of the form $A \rightarrow B$.

The systems in $P$ without $\wedge$ and $\vee$, obtained by extending $L_0$ of Section 2.4, i.e. (1)-(6), with schemata from the first column will be sound and complete, in the sense of (0), with respect to right-residuated partially ordered groupoids which satisfy the corresponding conditions in the second column. Also, their Lindenbaum algebras will be free in the classes of all corresponding groupoids.
We can assert that these systems give the \((\rightarrow, \circ, \top, \bot)\) fragments of the corresponding extensions of \(L\) whenever cut-elimination is available for the corresponding sequent-systems of the third column.

In the remainder of this section, and in Sections 2.7 — 2.9, we shall concentrate on some extensions of \(L\) which are of interest either because similar systems have been considered in the literature, or because they correspond to natural \(urlogs\), or natural sequent-systems.

Let \(LA\) be the system obtained by extending \(L\) with (31) and (32), and let \(LAP\) be \(LA\) extended with (33). The \((\rightarrow, \circ)\) fragment of \(LA\) (i.e., (1)-(6), (31), (32)) corresponds closely to the Lambek calculus of syntactic categories of [13]. In the Lambek calculus we have an ordering relation instead of the main \(\rightarrow\) of theorems of \(LA\), and we have two operations, corresponding to the left and right residual respectively, instead of the single \(\rightarrow\) of \(LA\), which corresponds to the right residual. In the same way \(L\) corresponds to the nonassociative Lambek calculus of [14]. We could extend \(L\) and \(LA\) with an additional binary connective \(\leftarrow\), corresponding to the left residual; for this connective we would have the double-line rule:

\[
\frac{(A \circ C) \rightarrow B}{C \rightarrow (B \leftarrow A)}.
\]

Analogously, in sequent-systems we would have:

\[
\frac{A, X \vdash Y[B]}{X \vdash Y[B \leftarrow A]}
\]

However, in \(LAP\) the connectives \(\rightarrow\) and \(\leftarrow\) would become equivalent, as the corresponding operations become equivalent in a commutative version of the Lambek calculus.

2.6. The extensions \(E^+\). Let us now consider an extension of \(L\) from Section 2.5 in which (38) and (40) are provable, and let us call this extension \(E\). It is easy to show that in the corresponding sequent-system the following double-line rule will be derivable:

\[
\frac{A \dashv B}{\varnothing \vdash A \rightarrow B}.
\]

(In sequent-systems with the single-conclusion property, in the presence of \((\rightarrow), (\circ), (\top)\) and \((\bot)\), the double-line rule (horiz) is interderivable with the following special form of (Intr \(\varnothing I\)) and (Elim \(\varnothing I\)):

\[
\frac{Z \vdash Y}{\varnothing, Z \vdash Y}.
\]

If moreover we have (Cut), this last double-line rule amounts to (Intr \(\varnothing I\)) and
(Elim $\emptyset I$), as we have shown at the end of Section 1.3. This double-line rule also amounts to (Intr $\emptyset I$) and (Elim $\emptyset I$) in the presence of (Assoc) and (Perm) only.

If we have (horiz), the statement corresponding to (I) of Section 2.3 is equivalent to the following statement:

(E) $\emptyset \vdash A \rightarrow B$ is provable in $GE$ iff $A \rightarrow B$ is provable in $E$.

We can then ask for what system $E^+$ we have that:

(I$^+$) $\emptyset \vdash A$ is provable in $GE$ iff $A$ is provable in $E^+$.

The system $E$ is contained in $E^+$, i.e., the theorems of $E^+$ of the form $A \rightarrow B$ must coincide with the theorems of $E$. But $E^+$ will also have theorems which are not of the form $A \rightarrow B$, and hence cannot be theorems of $E$. For example, $\top \vdash A$ is a theorem of $E^+$.

It is easy to show with the help of (T) and (horiz) that the following double-line rule is derivable in $GE$:

Concerning the analogous double-line rule for systems in $P$:

(50) $\vdash A$

we can prove the following lemma:

**Lemma 6.** $\emptyset \vdash A$ is provable in $GE$ iff $A$ is provable in $E$ plus (50) iff $A$ is provable in $E$ plus (50)$\dagger$.

**Proof.** We have:

$\emptyset \vdash A$ in $GE$ iff $\top \vdash \top \rightarrow A$ in $GE$, by (neces)

$\iff \top \rightarrow A$ in $E$, by (E).

So, if $\emptyset \vdash A$ is provable in $GE$, then $A$ is provable in $E$ plus (50)$\dagger$, and hence also in $E$ plus (50).

For the converse implications we proceed by induction on the length of proof of $A$ in $E$ plus (50). If $A$ is an axiom, we use (E). If $A$ is obtained by a rule of $E$ of the form:

\[
\frac{A \quad B}{C}
\]

(this rule is either (2), (3), (6), (9), (12) or (13)) with the help of (horiz), we can show that in $GE$ we can derive:

\[
\frac{\emptyset \vdash A \quad \emptyset \vdash B}{\emptyset \vdash C}.
\]

Finally, if $A$ is obtained by (50), we use (neces). q.e.d.
By this lemma, we can take either $E$ plus $(50)$, or $E$ plus $(50)^\uparrow$, as our system $E^+$. However, since $GE$ is closed under (neces), and not only (neces)$^\uparrow$, our system $E^+$ will be closed under $(50)^\downarrow$, even if we do not assume this rule as primitive. Because of that, we shall take in the sequel that $E^+$ is the system obtained by extending $E$ with $(50)$.

It is easy to see that in $E^+$ so defined we can derive *modus ponens*:

$$
\begin{array}{c}
A \\
\hline
A \rightarrow B \\
\end{array}
\quad \Rightarrow 
\begin{array}{c}
B
\end{array}
$$

and *adjunction*:

$$
\begin{array}{c}
A \\
\hline
B
\end{array}
\quad ,
\begin{array}{c}
A \wedge B
\end{array}
$$

In systems where we have $A \rightarrow A$ and *modus ponens*, the Rule of Replacement (17) amounts to the following rule:

$$
A \Leftrightarrow B \\
\hline
C^A
$$

In $L$, and similar systems without *modus ponens*, in general we do not have this rule. (However, $L$ is closed under this rule provided $C$ is not $A$, i.e. $A$ is a proper subformula of $C$.)

Finally, it is clear that $E$ and $E^+$ have the same Lindenbaum algebra.

2.7. Extensions of $S$. We shall now consider the extensions of $L$ given in the following chart, where arrows represent proper inclusion:
If $E$ is one of the systems in this chart, $E^+$ is $E$ plus (50), as in Section 2.6.

The urlogs corresponding to $S$ are right-residuated lattice-ordered monoids, and those corresponding to $M$ are residuated lattice-ordered commutative monoids. With $RA$ these commutative monoids are square-increasing (i.e., they satisfy $a \leq a \cdot a$), and with $RAM$ they are idempotent.

It is easy to see that $G$-terms on the left of the turnstile in theorems of $GS$ correspond to finite (possibly empty) sequences of formulae of $P$. Analogously, the $G$-terms on the left of the turnstile in theorems of $GM$ correspond to finite (possibly empty) multisets, i.e. sets of occurrences, of formulae of $P$. (The importance of multisets in the proof theory of relevant logic is stressed in [16]. Sequent-systems where left $G$-terms would correspond to ordinal sets in the sense of [16], i.e. to sequences where repetitions are omitted, could be obtained with structural rules like (Canc l) and (Canc r).)

The system $M^+$ corresponds to the negationless fragment of the relevant logic $R$ minus distribution $((A \land (B \lor C)) \rightarrow ((A \land B) \lor (A \land C)))$ and minus contraction $((A \rightarrow (A \rightarrow B)) \rightarrow (A \rightarrow B))$. In the same way $RA^+$ corresponds to the negationless fragment of $R$ minus distribution. The system $RAM^+$ is a mingle extension of $RA^+$; in $RAM^+$ we have the mingle principle $A \rightarrow (A \rightarrow A)$. (The logic $R$ and its neighbours are extensively treated in [1]; for $RA^+$ see [6].)

The urlogs corresponding to $BCK_f$ are residuated lattice-ordered commutative monoids where 1 is maximal, and the urlogs corresponding to $BCK_H$ are such monoids where moreover 0 is minimal. These urlogs are related to $BCK$ algebras (if $\langle \mathcal{A}, \land, \lor, \land, 0, 1, 0 \rangle$ is such an urlog, with or without 0 minimal, and if $a \ast b = b \land a$, $0 = 1$ and $a \leq b \iff b \leq a$, then $\langle \mathcal{A}, \ast, 0, \leq \rangle$ is a $BCK$ algebra in the sense of [11], and $\langle \mathcal{A}, \ast, \lor, 0, \leq \rangle$ is a $BCK$ algebra with condition (S), in the sense of [12] and [18]). The system $BCK_f^+$ corresponds to the positive fragment of a logic associated with $BCK$ algebras (see [17]), and $BCK_H^+$ corresponds to this logic with a Heyting-type negation. This logic is essentially intuitionistic logic without contraction.

In the urlogs corresponding to $J$ we have $a \cdot b = a \cap b$. (Since $b \leq 1$, we have $a \cdot b \leq a \cdot 1$, and hence, $a \cdot b \leq a$. Similarly we obtain $a \cdot b \leq b$, which gives $a \cdot b \leq a \cap b$. For the converse, from $a \cap b \leq a$ and $a \cap b \leq b$, we have $(a \cap b) \cdot (a \cap b) \leq a \cdot b$. Then using $a \cap b \leq (a \cap b) \cdot (a \cap b)$ we obtain $a \cap b \leq a \cdot b$.) So, according to (xi), the operation $\setminus$ in these urlogs is the relative pseudo-complement (see [20], p. 54). Since every relatively pseudo-complemented lattice has a unit element, and since every such lattice is distributive, we can identify the urlogs corresponding to $J$ with relatively pseudo-complemented lattices. The urlogs corresponding to $H$ are relatively pseudo-complemented lattices with a zero element, i.e. Heyting algebras.

The system $J^+$ is an axiomatization of Johansson’s propositional logic, and $H^+$ is an axiomatization of Heyting’s propositional logic. In $J$ and $H$, and hence also in $J^+$ and $H^+$, we can prove $(A \circ B) \vDash (A \land B)$ and $(A \land (B \lor C)) \rightarrow ((A \land B) \lor (A \land C))$. The system $C^+$ is the classical propositional calculus. The systems $C$ and $C^+$ are connected to the sequent-system $GC$ of Section 1.7 by statements corresponding respectively to (I) and (I').
2.8. The $(\to, \land, \lor, \bot)$ fragments of extensions of $S^+$. It is of some interest to consider the fragments of our extensions of $S^+$ which do not contain the less usual logical operations $\circ$ and $\top$.

Let $P^-$ be the language $P$ without $\circ$ and $\top$, and let $S^-$ be the system in $P^-$ given by:

\begin{align*}
(1) \quad & A \to A \\
(51) & \frac{A \to B}{B} \\
(53) \quad & (A \to B) \to ((C \to A) \to (C \to B)) \\
(54) & \frac{A}{(A \to B) \to B} \\
(7) \quad & (A \land B) \to A \\
(52) & \frac{A \land B}{A} \\
(8) \quad & (A \land B) \to B \\
(55) \quad & ((C \to A) \land (C \to B)) \to (C \to (A \land B)) \\
(10) \quad & A \to (A \lor B) \\
(11) \quad & B \to (A \lor B) \\
(56) \quad & ((A \to C) \land (B \to C)) \to ((A \lor B) \to C).
\end{align*}

Note that (55) and (56) are already provable in $L$, and that (51) and (52) are derivable in all $E^+$ systems. Hence, the only new postulates of $S^-$ are (53) and (54).

The rule:

\begin{equation}
\frac{A \land (A \to B)}{C \to B}
\end{equation}

can replace (54) in $S^-$, since we have:

\begin{align*}
& \frac{A}{(A \to B) \to B} \quad \text{(54)} \\
\frac{C \to (A \to B)}{(A \to B) \to B} \quad \text{(53), (51)} \\
\frac{A \to B}{(A \to B) \to B} \quad \text{(57)}
\end{align*}

Let us introduce the following abbreviation:

\[ A_n \to B = \begin{cases} A_1 \to (\ldots \to (A_n \to B) \ldots) & \text{if } n \geq 1 \\ B & \text{if } n = 0. \end{cases} \]

We shall now show that the following rule (analogous to (Cut)) is derivable in $S^-$:

\begin{equation}
\frac{\overline{B}_n \to A \quad \overline{C}_m \to (A \to D)}{\overline{C}_m \to (\overline{B}_n \to D)}, \quad n \geq 0, \quad m \geq 0.
\end{equation}

We have:
Sequent-systems...

\[ \bar{C}_m \rightarrow (A \rightarrow D) \]

\[ \rightarrow (\bar{B}_n \rightarrow A) \]

\[ \text{eventually (53) and (51)} \]

\[ \bar{C}_m \rightarrow (\bar{B}_n \rightarrow A) \rightarrow (\bar{B}_n \rightarrow D) \]

\[ \text{eventually (53) and (51)} \]

\[ (\bar{C}_m \rightarrow (\bar{B}_n \rightarrow A) \rightarrow (\bar{B}_n \rightarrow D)) \rightarrow (\bar{C}_m \rightarrow (\bar{B}_n \rightarrow D)) \]

\[ \bar{C}_m \rightarrow (\bar{B}_n \rightarrow D). \]

The rules (51) and (57) are instances of (58), and it is not difficult to prove (53) from (1) and (58). Hence, we can replace (53), (51) and (54) in \( S^- \) by (58).

It is easy to show that the rules (2), (3), (9), (12) and (13) of Section 2.1 are derivable in \( S^- \). From (1) and (54) we also obtain immediately:

\[ (A \rightarrow A) \rightarrow B \rightarrow B. \]

Let the language \( GP^- \) be \( GP \) without \( \odot \) and \( \top \), and let \( GS^- \) be the sequent-system in \( GP^- \) obtained from \( GS \) by rejecting \( \odot \) and \( \top \). We shall show that:

\[ (I^-) \quad \emptyset \vdash A \text{ is provable in } GS^- \text{ iff } A \text{ is provable in } S^- \text{.} \]

With the cut-elimination results of Section 1.6 this will establish that \( S^- \) is indeed the \( (\rightarrow, \land, \lor, \perp) \) fragment of \( S^+ \).

First, we define a translation of single-conclusion sequents of \( GP^- \) into formulae of \( P^- \). Let \( C_1, \ldots, C_n \), where \( n \geq 0 \), be obtained from the G-term \( X \) of \( GP^- \) by deleting all occurrences of \( \emptyset \) and all parentheses which do not belong to formulae of \( P^- \). So, \( C_1, \ldots, C_n \) are all the occurrences of formulae of \( P^- \) which are subterms of \( X \), arranged in the order in which they appear in \( X \) (a formula of \( P^- \) may occur more than once in \( C_1, \ldots, C_n \)). Then we define our translation as follows:

\[ s(X \vdash A) = \bar{C}_n \rightarrow A \]

\[ s(X \vdash \emptyset) = \bar{C}_n \rightarrow \perp. \]

We shall now prove the following lemma:

**Lemma 7.** If \( X \vdash Y \) is provable in \( GS^- \), then \( s(X \vdash Y) \) is provable in \( S^- \).

**Proof.** By induction on the length of proof of \( X \vdash Y \) in \( GS^- \). If \( X \vdash Y \) is \( A \vdash A \), then in \( S^- \) we have \( A \rightarrow A \). If \( X \vdash Y \) is obtained by (Cut), we use (58). The case with \( (\rightarrow) \) is trivial. For the cases with \( (\land) \) and \( (\lor) \) we use the following theorems of \( S^- \):

\[ ((D \rightarrow (C \rightarrow A)) \land (D \rightarrow (C \rightarrow B))) \rightarrow (D \rightarrow (C \rightarrow (A \land B))) \]

\[ ((D \rightarrow (A \rightarrow C)) \land (D \rightarrow (B \rightarrow C))) \rightarrow (D \rightarrow ((A \lor B) \rightarrow C)). \]

The cases with \( (\perp) \), (Assoc), (Intr \( \emptyset \) l), (Intr \( \emptyset \) r), (Elim \( \emptyset \) l) and (Elim \( \emptyset \) r) are all trivial. q.e.d.
As an immediate corollary of Lemma 7 we have \((\neg\neg)\) from left to right. The converse is proved by a straightforward induction on the length of proof of \(A\) in \(S^-\). For example, for (53) we have the following proof in \(GL\) plus \((\text{Assoc})\downarrow\):

\[
\frac{C \rightarrow A \vdash C \rightarrow A}{C \rightarrow A, C \vdash A} \quad \frac{A \rightarrow B \vdash A \rightarrow B}{A \rightarrow B, A \vdash B} \quad \frac{(\rightarrow)\uparrow}{A \rightarrow B, (C \rightarrow A, C) \vdash B} \quad (\text{Assoc})\downarrow
\]

\[
\frac{(A \rightarrow B, C \rightarrow A), C \vdash B}{\vdots} \quad (\rightarrow)\downarrow
\]

\[
A \rightarrow B \vdash (C \rightarrow A) \rightarrow (C \rightarrow B)
\]

which shows that (53) is tied with \((\text{Assoc})\downarrow\). For (54), in \(GL\) plus \((\text{Elim Or})\) we can derive:

\[
\frac{A \rightarrow B \vdash A \rightarrow B}{\emptyset \vdash A} \quad \frac{A \rightarrow B, \emptyset \vdash B}{A \rightarrow B, A \vdash B} \quad (\rightarrow)\uparrow
\]

\[
\frac{(\rightarrow)\uparrow}{\emptyset \vdash B} \quad (\text{Cut})
\]

\[
\frac{A \rightarrow B \vdash B}{A \rightarrow B, \emptyset \vdash B} \quad (\text{Elim Or})
\]

which shows that (54) is tied with \((\text{Elim Or})\). A careful examination of our induction would show that we don’t need \((\text{Assoc})\uparrow\), \((\text{Intr Or})\) and \((\text{Elim Or})\) to prove \((\neg\neg)\) from right to left.

Hence, we have proved that \(S^-\) gives the \((\rightarrow, \wedge, \vee, \perp)\) fragment of \(S^+\). Similarly we can prove that the \((\rightarrow, \wedge, \vee, \perp)\) fragment

of \(M^+\) is given by \(M^- = S^- + (60) (A \rightarrow (B \rightarrow C)) \rightarrow (B \rightarrow (A \rightarrow C))\) (alternatively, \(M^- = S^- + (61) A \rightarrow ((A \rightarrow B) \rightarrow B)\), cf. [1], p. 80; (54) is superfluous in \(M^-\));

of \(RA^+\) is given by \(RA^- = M^- + (62) (A \rightarrow (A \rightarrow B)) \rightarrow (A \rightarrow B)\);

of \(BCK^+_\widetilde{J}\) is given by \(BCK^-_\widetilde{J} = M^- + (63) A \rightarrow (B \rightarrow A)\);

of \(J^+\) is given by \(J^- = BCK^-_\widetilde{J} + (62)\);

of \(BCK^+_H\) is given by \(BCK^-_H = BCK^-_\widetilde{J} + (37) \perp \rightarrow A\);

of \(H^+\) is given by \(H^- = J^- + (37)\) ((1) is superfluous in \(BCK^-_\widetilde{J}\) and its extensions). Note that since in Section 1.6 we did not prove cut-elimination for sequent systems with \((\text{Expans})\), we cannot assert anything here about \(RAM^-\). For systems with \((\text{Thin R})\) we did have cut-elimination in the presence of \((\text{Intr Or})\), \((\text{Intr Or})\) and \((\text{Thin})\).

### 2.9. Implicational fragments of extensions of \(L\)

We have seen in Section 2.4 that the implicational fragment of \(L\) is made of \(A \rightarrow A\) only. On the other hand, in the implicational fragment of the system \(LA\) of Section 2.5 we already have (53), as can be seen from the proof on \(GL\) plus \((\text{Assoc})\downarrow\) after the proof of Lemma 7 in the last section.
The implicational fragment of LA is given by the system LA\_ with the axiom-schema (1) \( A \rightarrow A \) and the rule (58'), obtained from (58) by replacing \( n \geq 0 \) by \( n \geq 1 \). The rules (2) and (3) can be obtained from (58'): the rule (2) is just the instance of (58') where \( n = 1 \) and \( m = 0 \), and for (3) we have:

\[
\begin{align*}
(A_1 \rightarrow B) \rightarrow (A_1 \rightarrow B) & \quad B \rightarrow B, \\
A \rightarrow A_1 & \quad (A_1 \rightarrow B_1) \rightarrow (A_1 \rightarrow B_1) \\
(A_1 \rightarrow B_1) \rightarrow (A \rightarrow B_1) & \quad (A_1 \rightarrow B) \rightarrow (A \rightarrow B_1).
\end{align*}
\]

The rule (58') can be replaced in LA\_ by the following two rules:

\[
\begin{align*}
B \_ n \rightarrow A & \quad (A \rightarrow D) \\
A \rightarrow D & \quad (B \_ n \rightarrow D).
\end{align*}
\]

The rule (2) is the instance of (51') where \( n = 1 \). The rules (51') and (54') would become (51) and (54) respectively by putting \( n = 0 \).

Let GLA\_ be the sequent-system given by (Id), (Cut), (Assoc) and (\( \rightarrow \)). To prove that LA\_ is indeed the implicational fragment of LA, we first establish that \( A \vdash B \) is provable in GLA\_ iff \( A \rightarrow B \) is provable in LA\_. From left to right, we use a translation like the translation \( s \) of Section 2.8 to establish a lemma analogous to Lemma 7 (note that (58') is exactly what we need for the step with (Cut) in the proof of this lemma). From right to left, we proceed by induction on the length of proof of \( A \rightarrow B \) in LA\_. Once we have established our equivalence, we apply the cut-elimination results of Section 1.6.

For the implicational fragments of our extensions of \( S^+ \) (these fragments coincide with the implicational fragments of the corresponding extensions of \( S \), as explained in Section 2.6), we proceed in principle as for the (\( \rightarrow \), \( \land \), \( \lor \), \( \bot \)) fragments in Section 2.8. It is possible to show that the implicational fragment of \( S^+ \) is given by \( S_{\rightarrow} = (1), (53), (51), (54) \) (see [24], p. 168);

of \( M^+ \) is given by \( M_{\rightarrow} = S_{\rightarrow} + (60) \)

(alternatively, \( M_{\rightarrow} = S_{\rightarrow} + (61); (54) \) is superfluous in \( M_{\rightarrow} \); cf. with the axioms of \( BCI \) in [19], p. 316);

of \( RA^+ \) is given by \( R_{\rightarrow} = M_{\rightarrow} + (62) \) (cf. [1], p. 88);

of \( BCK^+_f \) and \( BCK^+_h \) is given by \( BCK_{\rightarrow} = M_{\rightarrow} + (63) \);

of \( J^+ \) and \( H^+ \) is given by \( H_{\rightarrow} = BCK_{\rightarrow} + (62) \)

((1) is superfluous in \( BCK_{\rightarrow} \) and \( H_{\rightarrow} \); cf. [19], p. 316).

We shall conclude this section with a brief remark on strict implication. The system \( S4_{\rightarrow} \), which gives the strict-implicational fragment of the modal logic \( S4 \), is obtained by extending \( S_{\rightarrow} \) with:

\[
\begin{align*}
(62) & \quad (A \rightarrow (A \rightarrow B)) \rightarrow (A \rightarrow B) \\
(60') \quad (A \rightarrow ((B_1 \rightarrow B_2) \rightarrow C)) \rightarrow ((B_1 \rightarrow B_2) \rightarrow (A \rightarrow C)).
\end{align*}
\]
(63') \((A_1 \rightarrow A_2) \rightarrow (B \rightarrow (A_1 \rightarrow A_2))\).

(It is a matter of routine to prove that (53), (62) and (60') can be replaced by 
\((C \rightarrow (A \rightarrow B)) \rightarrow ((C \rightarrow A) \rightarrow (C \rightarrow B))\), and that (54) is superfluous in \(S4_{\sim}\))

The sequent-system \(GS4_{\sim}\) corresponding to \(S4_{\sim}\), in the sense that \(\emptyset \vdash A\) is provable in \(GS4_{\sim}\) iff \(A\) is provable in \(S4_{\sim}\), would be given by (Id), (Cut), (\(\rightarrow\)), (Intr\(\emptyset\)), (Intr\(\emptyset\)) and (Elim\(\emptyset\)), plus (Assoc) and (Contr) with \(Z, Z_1, Z_2, Z_3\) replaced by \(A, A_1, A_2\) and \(A_3\), and plus the following two quasi-structural rules corresponding to (60') and (63') respectively:

\[
\frac{X[A, B_1 \rightarrow B_2] \vdash Y}{X[B_1 \rightarrow B_2, A] \vdash Y}, \quad \text{(Perm S4)}
\]
\[
\frac{X[A_1 \rightarrow A_2] \vdash Y}{X[A_1, A_2, B] \vdash Y}, \quad \text{(Thin S4)}
\]

These quasi-structural rules indicate that a logic built on \(S4_{\sim}\) would not belong exactly to the family of logics we are investigating here.

2.10. Negation in \(L\) and its extensions. As we have already remarked in Section 2.1, we may assume that \(\sim A\) is defined in the language \(P\) as \(A \sim \bot\). We can then enquire what are the properties of this negation in \(L\) and its extensions.

Let us first consider those systems where \(\bot\) is idle, i.e., where nothing specific being assumed about it, \(\bot\) behaves as an arbitrary propositional variable. Such is the system \(L\), and such are all the systems in the chart of Section 2.7, as their \(+\) counterparts, except those extending \(BCK_{\mathcal{H}}\). We shall say that these systems have a purely implicational negation. The best known system with a purely implicational negation is the Johansson propositional calculus.

Let \(P\) be the language which differs from \(P\) by having instead of \(\bot\) the unary connective \(\sim\) as primitive. Let also \(\sim\) be defined in \(P\) as above. Then \(P\) is a proper sublanguage of \(P\): to the formulae of \(P\) correspond only those formulae of \(P\) in which \(\bot\) does not occur except as a consequent of implications, i.e. \(\bot\) occurs only in subformulae of the form \(A \rightarrow \bot\). We shall now show what systems in \(P\) capture the \(P\) fragments of our systems in \(P\) with purely implicational negation:

**Lemma 8.1.** Let \(E\) be \(L\), or an extension of \(L\) in \(P\), with purely implicational negation, where all theorems are of the form \(B \rightarrow C\). Next, let \(E\) be the system in \(P\) obtained by extending the axioms and rules of \(E\) with:

\[
\begin{align*}
\text{(64)} \quad & \sim(B \circ A) \\
\text{(65)} \quad & \frac{\sim B}{(A \rightarrow B) \rightarrow \sim A} \\
\text{(66)} \quad & \frac{\sim A \quad \sim B}{\sim(A \vee B)}
\end{align*}
\]
Then for every formula \( A \) of \( P_\bot \) we have that \( A \) is provable in \( E_\bot \) iff \( A \) is provable in \( E \).

**Proof.** Suppose \( A \) is provable in \( E_\bot \). To show that \( A \) is also provable in \( E \) it is enough to establish that in \( L \), and hence also in \( E \), we have the following theorem and rules:

\[
\begin{align*}
(A \rightarrow C) & \rightarrow C; \\
(B \rightarrow C) & \rightarrow (A \rightarrow (A \rightarrow C)) \\
A \rightarrow C & \rightarrow B \rightarrow C \\
(A \lor B) & \rightarrow C \\
A \rightarrow B & \rightarrow B \rightarrow C \\
& \rightarrow A \rightarrow C
\end{align*}
\]

Next, let \( A_1, \ldots, A_n \) be all the subformulae of a formula \( A \) of \( P_\bot \), and let \( f_A = \delta_f (\neg A_1 \circ A_1) \lor \cdots \lor (\neg A_n \circ A_n) \). For every \( i, 1 \leq i \leq n, \) in \( E_\bot \) we can prove \( \neg A_i \leftrightarrow (A_i \rightarrow f_A) \), since we have:

\[
\begin{align*}
(1), (2), (10), (11) & \rightarrow (\neg A_1 \circ A_1) \ldots (\neg A_n \circ A_n) \\
(\neg A_1 \circ A_1) & \rightarrow (A_1 \rightarrow f_A) \\
A_i \rightarrow (A_i \rightarrow f_A) & \rightarrow (A_i \rightarrow f_A) \\
(14) & \rightarrow (A_i \rightarrow f_A) \\
(A_i \rightarrow f_A) & \rightarrow \neg A_i
\end{align*}
\]

Let \( A_j \) be a subformula of \( A \), and let \( A_j^f \) be obtained from \( A_j \) by replacing every subformula of \( A_j \) of the form \( \neg A_i \) by \( A_i \rightarrow f_A \). In \( E_\bot \) we can prove \( \neg A_i \leftrightarrow A_j^f \). To show that, we use \( \neg A_i \leftrightarrow (A_i \rightarrow f_A) \) and the Rule of Replacement (17), which holds in \( E_\bot \) in virtue of the rules (3), (6), (15), (16) and

\[
\begin{align*}
A \rightarrow B & \rightarrow B \rightarrow A
\end{align*}
\]

This last rule is derivable in \( E_\bot \) with the help of (3'), (2), \( \neg B \rightarrow (B \rightarrow f_A \rightarrow B) \) and \( (A \rightarrow f_A \rightarrow B) \rightarrow \neg A \).

Suppose now that \( A \) is provable in \( E \). Since \( \bot \) behaves in \( E \) as an arbitrary propositional variable, in \( E \) we can prove the schema \( A^B \) obtained from \( A \) by substituting uniformly the schematic letter \( B \) for \( \bot \). But then \( A^B \) is provable in \( E_\bot \), and hence \( A^f \) is provable in \( E_\bot \). We shall now show that it follows that \( A \) is provable in \( E_\bot \).

Since \( A \) is a formula of \( P_\bot \) provable in \( E \), and since all theorems of \( E \) in \( P \) are of the form \( B \rightarrow C \), we have that \( A \) is either of the form \( A_1 \rightarrow A_2 \) or of the
form $\neg A_1$. Suppose $(A_1 \rightarrow A_2)'$ is provable in $E_\neg$. So, $A_1' \rightarrow A_2'$ is provable in $E_\neg$. Since in $E_\neg$ we have $A_1 \Leftrightarrow A_1'$ and $A_2 \Leftrightarrow A_2'$, by using (2) we have that $A_1 \rightarrow A_2$ is provable in $E_\neg$. Suppose $(\neg A_1)'$ is provable in $E_\neg$. So, $A_1' \rightarrow f_A$ is provable in $E_\neg$. Since in $E_\neg$ we have $A_1 \Leftrightarrow A_1'$, by using (2) we can prove $A_1 \rightarrow f_A$ in $E_\neg$, and since in $E_\neg$ we also have $\neg f_A$ (see the proof above, using (64) and (66)), by using (67) we can prove $\neg A_1$ in $E_\neg$. So, if $A'$ is provable in $E_\neg$, then $A$ is provable in $E_\neg$. q.e.d.

Since in all the extensions of $L$ we have envisaged in Section 2.5 all theorems are of the form $B \rightarrow C$, Lemma 8.1 covers all of these extensions which have a purely implicational negation. In the extensions $E^+$ of Section 2.6 not all theorems are of the form $B \rightarrow C$. However, for these extensions we can prove the following:

**Lemma 8.2.** Let $E$ be an extension of $L$ in $P$, with a purely implicational negation, in which we can derive modus ponens (51). Next, let $E_\neg$ be the system in $P_\neg$, obtained by extending the axioms and rules of $E$ with (64), (65) and (66). Then for every formula $A$ of $P_\neg$ we have that $A$ is provable in $E_\neg$ iff $A$ is provable in $E$.

**Proof.** The proof is analogous to the proof of Lemma 8.1, save that now we do not need (67) to show that if $A'$ is provable in $E_\neg$, then $A$ is provable in $E_\neg$. This implication follows easily by using $A \Leftrightarrow A'$ and modus ponens, which is derivable in $E_\neg$ too. q.e.d.

Note that the rule (67) is derivable in the $E$ of Lemma 8.2, with the help of (2), $\neg B \rightarrow (B \rightarrow f_A \rightarrow B)$, $(A \rightarrow f_A \rightarrow B) \rightarrow \neg A$ and modus ponens.

Let $P_{\neg\neg}$ be obtained from the language $P_\neg$ of Section 2.8 (i.e. the language $P$ without $\circ$ and $\top$) by replacing $\bot$ by $\neg$, as $P_\neg$ was obtained from $P$. For systems in $P_{\neg\neg}$ we cannot have (64), and hence Lemma 8.2 is not forthcoming. However, for the system $M_{\neg\neg}$ and its extensions in $P_{\neg\neg}$, we can prove the following lemma, analogous to Lemma 8.2:

**Lemma 9.** Let $E^\neg$ be $M_{\neg\neg}$, or an extension of $M_{\neg\neg}$ in $P_{\neg\neg}$, with purely implicational negation, and let $E_\neg_{\neg\neg}$ be the system in $P_{\neg\neg\neg}$ obtained by extending the axioms and rules of $E^\neg$ with:

\[(68) \quad (A \rightarrow \neg B) \rightarrow (B \rightarrow \neg A).\]

Then for every formula $A$ of $P_{\neg\neg\neg}$ we have that $A$ is provable in $E_{\neg\neg\neg}$ iff $A$ is provable in $E_{\neg\neg}\neg$.

Before proving this lemma let us first note that in every extension of $S_{\neg\neg}$, and hence also in every extension of $M_{\neg\neg}$, (68) can be replaced by the following two schemata:

\[(69) \quad A \rightarrow \neg \neg A\]

\[(70) \quad (A \rightarrow B) \rightarrow (\neg B \rightarrow \neg A)\]
since we have:

\[(1), (51) \quad \frac{(\neg A \to \neg A) \to (A \to \neg \neg A)}{A \to \neg \neg A}; \]

\[(3) \quad \frac{B \to \neg \neg B}{(A \to B) \to (A \to \neg \neg B)} \quad (69)\]

\[(2) \quad \frac{(A \to B) \to (A \to \neg \neg B)}{(A \to B) \to (\neg B \to \neg A)}; \quad (68)\]

\[(53), (2) \quad \frac{(A \to \neg B) \to (\neg \neg B \to \neg A)}{(A \to \neg B) \to ((B \to \neg \neg B) \to (B \to \neg A))} \quad (70)\]

\[(69), (57) \quad \frac{(A \to \neg B) \to (\neg \neg B \to \neg A)}{(A \to \neg B) \to (B \to \neg A)}. \]

It is a matter of routine to show that in every extension of \(M^-\), (69) and (70) can be replaced by (70) and

\[(71) \quad \neg \neg (A \to A). \]

Now we are going to prove Lemma 9:

**Proof of Lemma 9.** Suppose \(A\) is provable in \(E^-\). To show that \(A\) is provable in \(E\), it is enough to establish that in \(M^-\), and hence also in \(E^-\), we have (60): \(A \to \neg \neg (A \to \neg \neg A)\).

Next, let \(A_1, \ldots, A_n\) be all the subformulae of a formula \(A\) of \(P^-\), and let \(v_A = a_1(A_1 \to A_1) \land \cdots \land (A_n \to A_n)\). For every \(i, 1 \leq i \leq n\), in \(E^-\), we can prove \(\neg A_i \to (A_i \to \neg v_A)\), since we have:

\[(1), (2), (7), (8) \quad (1), (52) \quad (70), (2) \quad (68)\]

\[(60), (51) \quad (68), (51) \quad \frac{\neg A_i \to (v_A \to \neg A_i)}{\neg A_i \to (A_i \to \neg v_A)}; \quad \frac{v_A \to (A_i \to A_i)}{v_A \to (\neg A_i \to \neg A_i)} \quad \frac{v_A \to (v_A \to \neg A_i)}{v_A \to (A_i \to \neg v_A) \to \neg A_i} \quad \frac{(A \to \neg v_A) \to (v_A \to \neg A_i)}{(A \to \neg v_A) \to \neg A_i}; \]

Next we let \(A'\) be obtained from \(A\) by replacing every subformula of \(A\) of the form \(\neg A_i\) by \(A_i \to \neg v_A\), and we proceed as for the proof of Lemma 8.2 (the closure of \(E^-\) under the Rule of Replacement (17) is guaranteed by (70)). q.e.d.

If we replace \(L\) in Lemma 8.1 and 8.2 by \(M\), we can replace (64)-(67) in these lemmata by (68) and:

\[(50') \quad \frac{\neg A}{\top \to \neg \neg A} \]

This double-line rule is an instance of (50) of Section 2.6; so, in the extensions of \(M^+\) it is superfluous.

The systems of Section 2.7 which extend \(BCK_H\) do not have a purely implicational negation: in these systems (37) \(\bot \to A\) is assumed. To axiomatize
the theorems in $P_\bot$ of these systems, we add to their postulates:

\[(72) \quad \neg A \vDash (A \rightarrow \neg (B \rightarrow B)) \quad (74) \quad \frac{\neg A}{(B \rightarrow B) \rightarrow \neg A} \]

\[(73) \quad \neg (B \rightarrow B) \rightarrow A \]

The schema (72) can replace (68) in extensions of $M$ where $(A \rightarrow A) \rightarrow (B \rightarrow B)$ is provable (already $BCK_\bot$ is such an extension). Since in these extensions we have $\top \vDash (B \rightarrow B)$, the double-line rule (74) can replace (50'). Of course, (74) is superfluous for the extensions of $BCK^+_\bot$. The schema (73) can be replaced above by $\neg B \rightarrow (B \rightarrow A)$. We proceed analogously for the corresponding systems in $P_\bot^+$.

References


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