Self-adjunctions and matrices

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Abstract

It is shown that the multiplicative monoids of Temperley–Lieb algebras are isomorphic to monoids of endomorphisms in categories where an endofunctor is adjoint to itself. Such a self-adjunction is found in a category whose arrows are matrices, and the functor adjoint to itself is based on the Kronecker product of matrices. This self-adjunction underlies the orthogonal group case of Brauer’s representation of the Brauer centralizer algebras.

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1. Introduction

As an offshoot of Jones’ polynomial approach to knot and link invariants, Temperley–Lieb algebras have played in the 1990s a prominent role in knot theory and low-dimensional topology (see [24,29,32]). In this paper we show that the multiplicative monoids of Temperley–Lieb algebras are closely related to the general notion of adjunction, one of the fundamental notions of category theory, and of mathematics in general (see [30]). More precisely, we show that these monoids are isomorphic to monoids of endomorphisms in categories involved in one kind of self-adjoint situation, where an endofunctor is adjoint to itself.

Early work on Temperley–Lieb algebras established the importance of a self-dual object in a monoidal category for understanding the categorial underpinnings of the matter (see [34,12,20] and papers cited therein). The result of the present paper is in the wake of this earlier work, and should not be surprising.

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We find a self-adjunction in categories whose arrows are matrices, where the functor adjoint to itself is based on the Kronecker product of matrices. This self-adjunction underlies the orthogonal group case of Brauer’s representation of the Brauer algebras, which can be restricted to the Temperley–Lieb subalgebras of the Brauer algebras (see [4,33, Section 3, 18, Section 3]). Thereby, building on ideas similar to, but not quite the same as, those that lead to the representation of braid groups in Temperley–Lieb algebras, which is due to Jones, we obtain a representation of braid groups in matrices. The question whether the representation of braid groups in Temperley–Lieb algebras is faithful is raised in [19]. (The review [1] provides a good survey of questions about the linear representation of braid groups.)

The representation of monoids of Temperley–Lieb algebras in matrices provides a faithful, i.e. isomorphic, representation in matrices of a certain brand of these algebras; this is the representation mentioned above, which originates in [4]. The faithfulness of this representation is established in [18, Section 3] (an elementary self-contained proof of the same fact may also be found in [8]).

Although this is a paper in the theory of Temperley–Lieb algebras, reading it does not presuppose acquaintance with works of that particular theory, except for the sake of motivation. In the latter sections of our paper, where we deal with matters of category theory, we presuppose some, rather general, acquaintance with a few notions from that field. They are all explained in Mac Lane’s book [30].

In the paper we proceed as follows. We first present by generators and relations monoids for which we will show later that they are engendered by categories involved in self-adjoint situations. These categories engender monoids, whose names will be indexed by $n$, when we consider a total binary operation on all arrows defined with the help of composition. Our categories engender monoids of a different kind, with names indexed by $\omega$, when we consider just composition, restricting ourselves to endomorphisms in the category. We deal first with the monoids related to the most general notion of self-adjunction, which we tie to the label $L$, and next with those related to the more particular notion of self-adjunction, tied to the label $K$, which we encounter in connection with Temperley–Lieb algebras.

Next we establish by relying on [9,3] that our monoids are isomorphic to monoids made of equivalence classes of diagrams which in knot theory would be called planar tangles, without crossings, and which we call friezes. In these representations, there are two different notions of equivalence of friezes: the $L$ notion is based purely on planar ambient isotopies, whereas the $K$ notion allows circles to cross lines, which is forbidden in the $L$ notion. So the mathematical content of the most general notion of self-adjunction is caught by the notion of planar ambient isotopy. The diagrammatic representation of the $K_n$ monoids is not new, but $L_n$ and its diagrammatic representation do not seem to have been treated in detail so far.

A theorem connecting self-adjunction to diagrams analogous to friezes is stated without proof in [12, Theorem 4.1.1, p. 172]. It is not clear, however, whether $L$ or $K$ notions are meant (it seems the self-adjunction is of the $L$ kind, while the diagrams are of the $K$ kind). Adjunction, as it occurs in symmetric monoidal closed categories, is connected to diagrams like ours in [11,25]. This connection between adjunction and diagrams is made also, more or less explicitly, in many of the papers mentioned above,
about categorial matters tied to Temperley–Lieb algebras; in particular, in [34,13] (see also [7, Section 4.10]). The completeness results of [34,13] are established by showing that each equivalence move of the Reidemeister kind in the diagrams corresponds to an equation. Our style is somewhat different, because we rely on normal forms that are not mentioned in the other approach.

We consider also a third notion of self-adjunction, \( \mathcal{J} \) self-adjunction, in whose diagrammatic representation we do not take account of circles at all. This notion is more strict than \( \mathcal{K} \) self-adjunction. In [9] it is shown for this third notion that it is maximal in the sense that we could not extend it with any further assumption without trivializing it. This maximality is an essential ingredient in our proof that we have in matrices an isomorphic representation of the monoids of Temperley–Lieb algebras. However, this maximality need not serve only for that particular goal, which can be reached by other means, by relying on [18, Section 3] and [8] or on [10]. Maximality can serve to establish the isomorphism of other nontrivial representations of the monoids of Temperley–Lieb algebras.

We deal with matters involving categories latter in the paper. There we introduce our categories of matrices, and exhibit the \( \mathcal{K} \) self-adjunction involved in them. We deal with the orthogonal group case of Brauer’s representation and with our representation of braid groups in matrices in the last two sections.

At the beginning our style of exposition will be rather formal, and it might help the reader while going through Sections 2–4 to take a look at Sections 5–7, and perhaps also at Sections 10–11, to get some motivation.

2. The monoids \( \mathcal{L}_\omega \) and \( \mathcal{K}_\omega \)

The monoid \( \mathcal{L}_\omega \) has for every \( k \in \mathbb{N}^+ = \mathbb{N} - \{0\} \) a generator \( |k| \), called a cup, and a generator \( \lceil k \rceil \), called a cap. The terms of \( \mathcal{L}_\omega \) are defined inductively by stipulating that the generators and \( 1 \) are terms, and that if \( t \) and \( u \) are terms, then \( (tu) \) is a term. As usual, we will omit the outermost parentheses of terms. In the presence of associativity we will omit all parentheses, since they can be restored as we please.

The monoid \( \mathcal{L}_\omega \) is freely generated from the generators above so that the following equations hold between terms of \( \mathcal{L}_\omega \) for \( l \leq k \):

\[
\begin{align*}
1t &= t, \quad t1 &= t, \\
(1) \quad t(uv) &= (tu)v, \\
(2) \quad |k|l &= |l||k + 2|, \\
(cup) \quad l|k| &= [k + 2][l], \\
(cap) \quad |l||k| &= [k + 2][l], \\
(cup-cap 1) \quad |l|[k + 2] &= |k||l|, \\
(cup-cap 2) \quad [k + 2][l] &= |l||k|, \\
(cup-cap 3) \quad |k||k + 1| &= 1.
\end{align*}
\]
The monoid \( \mathcal{K}_o \) is defined as the monoid \( \mathcal{L}_o \) save that we have the additional equation

\[(\cup \cap 4) \quad [k][k] = [k+1][k+1],\]

which, of course, implies

\[[k][k] = [l][l].\]

To understand the equations of \( \mathcal{L}_o \) and \( \mathcal{K}_o \) it helps to have in mind their diagrammatic interpretation of Sections 5–7 (see in particular the diagrams corresponding to \([k]\) and \([k]\) at the beginning of Section 6).

Let \([k]\) be an abbreviation for \([k][k]\), and let us call such terms circles. Then (cup-cap 4) says that we have only one circle, which we designate by \(c\). In \( \mathcal{K}_o \) we have the equations

\[[k]c = c[k],\]
\[[k]c = c[k],\]

which yield the equation \(tc = ct\) for any term \(t\).

3. Finite multisets, circular forms and ordinals

Let an \( o \)-monoid be a monoid with an arbitrary unary operation \(o\), and consider the free commutative \( o \)-monoid \( \mathcal{F} \) generated by the empty set of generators. In \( \mathcal{F} \) the operation \(o\) is a one-one function.

The elements of \( \mathcal{F} \) may be designated by parenthetical words, i.e. well-formed words in the alphabet \{\((,\)\}, which will be precisely defined in a moment, where the empty word stands for the unit of the monoid, concatenation is monoid multiplication, and \(o(a)\) is written simply \((a)\). Parenthetical words are defined inductively as follows:

(0) the empty word is a parenthetical word;
(1) if \(a\) is a parenthetical word, then \((a)\) is a parenthetical word;
(2) if \(a\) and \(b\) are parenthetical words, then \(ab\) is a parenthetical word.

We consider next several isomorphic representations of \( \mathcal{F} \), via finite multisets, circular forms in the plane and ordinals.

If we take that \((\) is the empty multiset, then the elements of \( \mathcal{F} \) of the form \((a)\) may be identified with finite multisets, i.e. the hierarchy of finite multisets obtained by starting from the empty multiset \(\emptyset\) as the only urelement. To obtain a more conventional notation for these multisets, just replace \((\) everywhere by \(\emptyset\), replace the remaining left parentheses (by left braces \{and the remaining right parentheses) by right braces\}, and put in commas where concatenation occurs.
The elements of $F$ may also be identified with nonintersecting finite collections of circles in the plane factored through homeomorphisms of the plane mapping one collection into another (cf. [22, Section II]). For this interpretation, just replace $(a)$ by $(\alpha)$. Since we will be interested in particular in this plane interpretation, we call the elements of $F$ circular forms. The empty circular form is the unit of $F$. When we need to refer to it we use $e$. We refer to other circular forms with parenthetical words.

The free commutative $o$-monoid $F$ has another isomorphic representation in the ordinals contained in the ordinal $\omega_0 = \min \{\xi \mid \omega^\xi = \xi\}$, i.e. in the ordinals lesser than $\omega_0$. By Cantor’s Normal Form Theorem (see, for example, [26, VII.7, Theorem 2, p. 248] or [28, IV.2, Theorem 2.14, p. 127]), for every ordinal $\beta \in \omega_0$ there is a unique finite ordinal $\gamma_0 \geq \cdots \geq \gamma_n$, such that $\beta = \omega^{\gamma_0} + \cdots + \omega^{\gamma_n}$. The natural sum $\beta$ of

$$\alpha = \omega^{\alpha_1} + \cdots + \omega^{\alpha_n}, \quad \alpha_1 \geq \cdots \geq \alpha_n,$$  

$$\beta = \omega^{\beta_1} + \cdots + \omega^{\beta_m}, \quad \beta_1 \geq \cdots \geq \beta_m,$$

is defined as $\omega^{\gamma_0} + \cdots + \omega^{\gamma_n}$ where $\gamma_0, \ldots, \gamma_n$ is obtained by permuting the sequence $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m$ so that $\gamma_1 \geq \cdots \geq \gamma_n, m$ (this operation was introduced by Hessenberg; see [26, p. 252] or [28, p. 130]). We also have $\beta \neq 0 \neq \alpha = \beta$. The natural sum $\#$ and the ordinal sum $+$ do not coincide in general: $\#$ is commutative, but $+$ is not (for example, $\omega = 1 + \omega \neq \omega + 1$, but $1 \# \omega = \omega \# 1 = \omega + 1 = \omega^0 + \omega^0$). However, if $\alpha_1 \geq \cdots \geq \alpha_n$, then $\omega^{\alpha_1} + \cdots + \omega^{\alpha_n} = \omega^{\alpha_1} \# \cdots \# \omega^{\alpha_n}$.

Let $\omega^-$ be the unary operation that assigns to every $\alpha \in \omega_0$ the ordinal $\omega^\alpha \in \omega_0$. Then it can be shown that the commutative $o$-monoid $(\omega_0, \#, 0, \omega^-)$ is isomorphic to $F$ by the isomorphism $i : \omega_0 \rightarrow F$ such that $i(0)$ is the empty word and

$$i(\omega^{\alpha_1} + \cdots + \omega^{\alpha_n}) = i(\omega^{\alpha_1} \# \cdots \# \omega^{\alpha_n}) = (i(\alpha_1)) \cdots (i(\alpha_n)).$$

That the function $i^{-1} : F \rightarrow \omega_0$ defined inductively by

$$i^{-1}(e) = 0,$$  

$$i^{-1}(ab) = i^{-1}(a) \# i^{-1}(b),$$  

$$i^{-1}((a)) = \omega^{-1}(a)$$

is the inverse of $i$ is established by easy inductions relying on the fact that

$$i(\alpha \# \beta) = i(\alpha)i(\beta).$$

It is well known in proof theory that the ordinal $\omega_0$ and natural sums play an important role in Gentzen’s proof of the consistency of formal Peano arithmetic PA (see [14, Paper 8, Section 4]). Induction up to any ordinal lesser than $\omega_0$ is derivable in PA; induction up to $\omega_0$, which is not derivable in PA, is not only sufficient, but also necessary, for proving the consistency of PA (see [14, Paper 9]).
From the isomorphism of $\mathcal{F}$ with $\langle \varepsilon_0, \varepsilon, 0, \omega^- \rangle$ we obtain immediately a normal form for the elements of $\mathcal{F}$. Circular forms inherit a well-ordering from the ordinals, and we have the following inductive definition. The empty word is in normal form, and if $a_1, \ldots, a_n$, $n \geq 1$, are parenthetical words in normal form such that $a_1 \geq \cdots \geq a_n$, then $(a_1) \cdots (a_n)$ is in normal form. We call this normal form of parenthetical words the Cantor normal form.

Let a commutative $o$-monoid be called solid iff it satisfies

$$(\text{solid}) \quad o(a) = o(1) a,$$

where 1 is the unit of the monoid. The free solid commutative $o$-monoid $\mathcal{F}'$ generated by the empty set of generators is isomorphic to the structure $\langle \mathbb{N}, +, 0, \cdots + 1 \rangle$ by the isomorphism that assigns to $n$ the sequence of $n$ pairs ( ). So (solid) makes $\langle \varepsilon_0, \varepsilon, 0, \omega^- \rangle$ collapse into $\langle \varepsilon, \varepsilon, 0, \cdots + 1 \rangle$. For $k \in \mathbb{N}$, let $k\mathbb{N} = \{kn \mid n \in \mathbb{N} \}$ and $k\mathbb{N}^+ = \{k^n \mid n \in \mathbb{N} \}$. If $k \geq 1$, then $\langle \mathbb{N}, +, 0, \cdots + 1 \rangle$ is isomorphic to $\langle k\mathbb{N}, +, 0, \cdots + k \rangle$, which for $k \geq 2$ is isomorphic to $\langle k\mathbb{N}, +, 1, \cdots \cdots, k \rangle$.

The equation (solid) is what a unary function $o: \mathcal{A} \rightarrow \mathcal{A}$, for a monoid $\mathcal{A}$, has to satisfy to be in the image of the Cayley monomorphic representation of $\mathcal{A} \in \mathcal{A}^{\mathcal{A}}$, which assigns to every $a \in \mathcal{A}$ the function $f_a \in \mathcal{A}^{\mathcal{A}}$ such that $f_a(b) = ab$. In the presence of (solid), the function $f_{o(a)}$ will be equal to $o \circ f_a$. The equation (solid) can be replaced by $o(ab) = o(a)b$, and in commutative $o$-monoids it could, of course, as well be written $o(ab) = ao(1)$.

4. Normal forms in $\mathcal{L}_\omega$ and $\mathcal{H}_\omega$

For $k \in \mathbb{N}^+$ let $c_k^0$ be the term $1$ of $\mathcal{L}_\omega$. For $\omega > 0$ an ordinal in $\varepsilon_0$ whose Cantor normal form is $\omega^{\omega_1} + \cdots + \omega^{i_1}$ let the term $c_k^m$ of $\mathcal{L}_\omega$ be defined inductively as

$$[k] c_k^{m+1} \cdot [k] \cdots [k] c_k^m \cdot [k].$$

Next, let $a_k^0$ be the term $[k]$, and let $a_k^m$ be the term $[k] c_k^{m+1}$. Similarly, let $b_k^0$ be the term $[k]$, and let $b_k^m$ be the term $c_k^{m+1} [k]$. Consider terms of $\mathcal{L}_\omega$ of the form

$$b_{j_1}^{n_{j_1}} \cdots b_{j_m}^{n_{j_m}} c_k^{i_1} \cdots c_k^{i_l} a_{i_1}^{n_{i_1}} \cdots a_{i_l}^{n_{i_l}},$$

where $n, m, l \geq 0$, $n + m + l \geq 1$, $j_1 < \cdots < j_m$, $k_1 < \cdots < k_l$, $i_1 < \cdots < i_l$, and for every $p \in \{1, \ldots, l\}$ we have $i_p \neq 0$. If $n$ is 0, the sequence $a_{i_1}^{n_{i_1}} \cdots a_{i_l}^{n_{i_l}}$ is empty, and analogously if $m$ or $l$ is 0. Terms of $\mathcal{L}_\omega$ of this form and the term $1$ will be said to be in normal form.

In the definition of normal form we could have required that $k_1 > \cdots > k_\alpha$, or, as a matter of fact, we could have imposed any other order on these particular indices, with the same effect. We have chosen the order above for the sake of definiteness. (Putting aside complications involving the terms $c_k^m$ and the ordinals, the idea of our normal form may be found in [2, p. 106].)

The following lemma is proved in [9].
Normal Form Lemma. Every term is equal to a term in normal form.

Let $c$ stand for $[k]$, where $k \in \mathbb{N}^+$. Let $c^0$ be the empty sequence, and let $c^{n+1}$ be $c^n \cdot c$. Consider terms of $\mathcal{L}_{\omega}$ of the form

$$\lceil j_1 \rceil \ldots \lceil j_m \rceil \cdot c^l \lceil i_1 \rceil \ldots \lceil i_n \rceil,$$

where $n, m, l \geq 0$, $n + m + l \geq 1$, $j_1 > \cdots > j_m$ and $i_1 < \cdots < i_n$. Terms of this form and the term $1$ will be said to be in $K$-normal form. (We could as well put $c^l$ on the extreme left, or on the extreme right, or, actually, anywhere, but for the sake of definiteness, and, by analogy with the normal form of $\mathcal{L}_{\omega}$, we put $c^l$ in the middle.)

We can easily derive from the Normal Form Lemma for $\mathcal{L}_{\omega}$ the Normal Form Lemma for $\mathcal{K}_{\omega}$, which says that every term is equal in $\mathcal{K}_{\omega}$ to a term in $\mathcal{K}$-normal form. For that it is enough to use the uniqueness of $c$ and $tc = ct$. However, the Normal Form Lemma for $\mathcal{K}_{\omega}$ has a much simpler direct proof. In $\mathcal{K}_{\omega}$ we have in fact assumed (solid), and the ordinals in $\omega$ have collapsed into natural numbers.

5. Friezes

A one-manifold with boundary is a topological space whose points have open neighbourhoods homeomorphic to the real intervals $(-1, 1)$ or $[0, 1)$, the boundary points having the latter kind of neighbourhoods. For $a > 0$ a real number, let $R_a = [0, \infty) \times [0, a]$. Let $\{(x, a) \mid x \geq 0\}$ be the top of $R_a$ and $\{(x, 0) \mid x \geq 0\}$ the bottom of $R_a$.

An $\omega$-diagram $D$ in $R_a$ is a compact one-manifold with boundary with denumerably many connected components embedded in $R_a$ such that the intersection of $D$ with the top of $R_a$ is $t(D) = \{(i, a) \mid i \in \mathbb{N}^+\}$ the intersection of $D$ with the bottom of $R_a$ is $b(D) = \{(i, 0) \mid i \in \mathbb{N}^+\}$ and $t(D) \cup b(D)$ is the set of boundary points of $D$.

It follows from this definition that every $\omega$-diagram has denumerably many components homeomorphic to $[0, 1]$, which are called threads, and at most a denumerable number of components homeomorphic to $S^1$, which are called circular components. The threads and the circular components make all the connected components of an $\omega$-diagram. All these components are mutually disjoint. Every thread has two end points that belong to the boundary $t(D) \cup b(D)$. When one of these end points is in $t(D)$ and the other in $b(D)$, the thread is transversal. A transversal thread is vertical when the first coordinates of its end points are equal. A thread that is not transversal is a cup when both of its end points are in $t(D)$, and it is a cap when they are both in $b(D)$.

A frieze is an $\omega$-diagram with a finite number of cups, caps and circular components. Although many, but not all, of the definitions that follow can be formulated for all $\omega$-diagrams, and not only for friezes, we will be interested here only in friezes, and we will formulate our definitions only with respect to them. The notion of frieze corresponds to a special kind of tangle of knot theory, in which there are no crossings (see [5, p. 99], [31, Chapter 9], [21, Chapter 12]).
For $D_1$ a frieze in $R_a$ and $D_2$ a frieze in $R_b$, we say that $D_1$ is $L$-equivalent to $D_2$, and write $D_1 \cong_L D_2$, iff there is a homeomorphism $h: R_a \to R_b$ such that $h[D_1] = D_2$ and for every $i \in \mathbb{N}^+$ we have $h(i,0) = (i,0)$ and $h(i,a) = (i,b)$. It is straightforward to check that $L$-equivalence between friezes is indeed an equivalence relation.

This definition is equivalent to a definition of $L$-equivalence in terms of ambient isotopies. The situation is analogous to what one finds in knot theory, where one can define equivalence of knots either in terms of ambient isotopies or in a simpler manner, analogous to what we have in the preceding paragraph. The equivalence of these two definitions is proved with the help of Alexander’s trick (see [5, Chapter 1B]), an adaptation of which also works in the case of $L$-equivalence.

For $D_1$ a frieze in $R_a$ and $D_2$ a frieze in $R_b$, we say that $D_1$ is $K$-equivalent to $D_2$, and write $D_1 \cong_K D_2$, iff there is a homeomorphism $h: D_1 \to D_2$ such that for every $i \in \mathbb{N}^+$ we have $h(i,0) = (i,0)$ and $h(i,a) = (i,b)$. It is clear that this defines an equivalence relation on friezes, which is wider than $L$-equivalence: namely, if $D_1 \cong_L D_2$, then $D_1 \cong_K D_2$, but the converse need not obtain. If $D_1$ and $D_2$ are without circular components, then $D_1 \cong_K D_2$ iff $D_1 \cong_K D_2$. The relation of $K$-equivalence takes account only of the number of circular components, whereas $L$-equivalence takes also account of whether circular components are one in another, and, in general, in which region of the diagram they are located.

If $i$ stands for $(i,a)$ and $-i$ stands for $(i,0)$, we may identify the end points of each thread in a frieze in $R_a$ by a pair of integers in $\mathbb{Z} - \{0\}$. For $M$ an ordered set and for $a,b \in M$ such that $a < b$, let a segment $[a,b]$ in $M$ be $\{z \in M | a \leq z \leq b\}$. The numbers $a$ and $b$ are the end points of $[a,b]$. We say that $[a,b]$ encloses $[c,d]$ iff $a < c$ and $d < b$. A set of segments is nonoverlapping iff every two distinct segments in it are either disjoint or one of these segments encloses the other.

We may then establish a one-to-one correspondence between the set $\Theta$ of threads of a frieze and a set $S_\Theta$ of nonoverlapping segments in $\mathbb{Z} - \{0\}$. Every element of $\mathbb{Z} - \{0\}$ is an end point of a segment in $S_\Theta$. Since enclosure is irreflexive and transitive, $S_\Theta$ is partially ordered by enclosure. This is a tree-like ordering without root, with a finite number of branching nodes. For example, in the frieze
the set $\Theta$ of threads corresponds to the following tree in $S_\Theta$:

```
[−4,−3]       [1,2]
   |           |
[−5,−2]       [−1,3]       [4,5]
   |           |
[−10,−9]      [−8,−7]       [−6,6]
   |           |
[−11,7]       |
   |           |
[−12,8]       |
   |           |
[−13,9]       |
   |           |
   ...         |
   |           |
   [−13 − n, 9 + n]
   |           |
   ...         |
```

The branching points of this tree are $[−6,6]$ and $[−11,7]$. This tree-like ordering of $S_\Theta$ induces an isomorphic ordering of $\Theta$.

If from a frieze $D$ in $R_\sigma$ we omit all the threads, we obtain a disjoint family of connected sets in $R_a$, which are called the regions of $D$. Every circular component of $D$ is included in a unique region of $D$. The closure of a region of $D$ has a border that includes a nonempty set of threads. In the tree-like ordering, this set must have a lowest thread, and all the other threads in the set, if any, are its immediate successors. Every thread is the lowest thread for some region. In our example, in the region in which one finds as circular components a circle and a square, the lowest thread is the one corresponding to $[−11,7]$, and its immediate successors correspond to $[−10,−9]$, $[−8,−7]$ and $[−6,6]$. Assigning to every region of a frieze the corresponding lowest thread in the border establishes a one-to-one correspondence between regions and threads.

The collection (possibly empty) of circular components in a single region of a frieze corresponds to a circular form (see Section 3), which can then be coded by an ordinal in $\iota_0$. In every frieze we can assign to every thread the ordinal that corresponds to the collection of circular components in the region for which this is the lowest thread. This describes all the circular components of a frieze. (In an $\omega$-diagram that is not a
frieze it is possible that one collection of circular components, which is in a region without lowest thread, is not covered.) Then it is easy to establish the following.

**Remark 1.** The friezes $D_1$ and $D_2$ are $\mathcal{L}$-equivalent if

(i) the end points of the threads in $D_1$ are identified with the same pairs of integers as the end points of the threads in $D_2$,

(ii) the same ordinals in $\varepsilon_0$ are assigned to the threads of $D_1$ and $D_2$ that are identified with the same pairs of integers.

This means that the $\mathcal{L}$-equivalence class of a frieze may be identified with a function $f : S_\varepsilon \to \varepsilon_0$, where the domain $S_\varepsilon$ of $f$ is a set of nonoverlapping segments in $\mathbb{Z} - \{0\}$.

**Remark 1.** The friezes $D_1$ and $D_2$ are $\mathcal{K}$-equivalent if

(i) the end points of the threads in $D_1$ are identified with the same pairs of integers as the end points of the threads in $D_2$,

(ii) $D_1$ and $D_2$ have the same number of circular components.

This means that the $\mathcal{K}$-equivalence class of a frieze may be identified with a pair $(S_\varepsilon, l)$ where $S_\varepsilon$ is a set of nonoverlapping segments in $\mathbb{Z} - \{0\}$, and $l$ is a natural number, which is the number of circular components.

The set of $\mathcal{L}$-equivalence classes of friezes is endowed with the structure of a monoid in the following manner. Let the *unit frieze* $I$ be $\{(i, j) | i \in \mathbb{N}^+ \text{ and } y \in [0, 1]\}$ in $R_1$. So $I$ has no circular components and all of its threads are vertical threads. We draw $I$ as follows:

```
| 1 | 2 | 3 |
```

For two friezes $D_1$ in $R_a$ and $D_2$ in $R_b$ let the *composition* of $D_1$ and $D_2$ be defined as follows:

$$D_2 \circ D_1 = \{(x, y + b) | (x, y) \in D_1\} \cup D_2.$$ 

It is easy to see that $D_2 \circ D_1$ is a frieze in $R_{a+b}$.

For $1 \leq i \leq 4$, let $D_i$ be a frieze in $R_a$, and suppose $D_1 \equiv \mathcal{L} D_3$ with the homeomorphism $h_1 : R_{a_1} \to R_{a_3}$ and $D_2 \equiv \mathcal{L} D_4$ with the homeomorphism $h_2 : R_{a_2} \to R_{a_4}$. Then $D_2 \circ D_1 \equiv \mathcal{L} D_4 \circ D_3$ with the homeomorphism $h : R_{a_1 + a_2} \to R_{a_2 + a_4}$ defined as follows. For $p^1$ the first and $p^2$ the second projection, let

$$h(x, y) = \begin{cases} 
( p^1(h_1(x, y - a_2)), p^2(h_1(x, y - a_2)) + a_4 ) & \text{if } y > a_2, \\
h_2(x, y) & \text{if } y \leq a_2.
\end{cases}$$

So the composition $\circ$ defines an operation on $\mathcal{L}$-equivalence classes of friezes.
We can then establish that

\[(1) \quad I \circ D \cong \mathcal{F} D, \quad D \circ I \cong \mathcal{F} D,\]

\[(2) \quad D_3 \circ (D_2 \circ D_1) \cong \mathcal{F} (D_3 \circ D_2) \circ D_1.\]

The equivalences of (1) follow from the fact that the threads of \(I \circ D\), \(D \circ I\) and \(D\) are identified with the same pairs of integers, because all the threads of \(I\) are vertical transversal threads, and from the fact that \(I\) has no circular component. Then we apply Remark 1 \(\mathcal{F}\). For the equivalence (2), it is clear that \(D_3 \circ (D_2 \circ D_1)\) is actually identical to \((D_3 \circ D_2) \circ D_1\). So the set of \(\mathcal{F}\)-equivalence classes of friezes has the structure of a monoid, and the monoid structure of the set of \(\mathcal{H}\)-equivalence classes of friezes is defined quite analogously. We will show for these monoids that they are isomorphic to \(\mathcal{L}_o\) and \(\mathcal{H}_o\) respectively.

### 6. Generating Friezes

For \(k \in \mathbb{N}^+\) let the **cup frieze** \(V_k\) be the frieze in \(R_1\) without circular components, with a single semicircular cup with the end points \((k,1)\) and \((k+1,1)\); all the other threads are straight line segments connecting \((i,0)\) and \((i,1)\) for \(i < k\) and \((i,0)\) and \((i,2,1)\) for \(i \geq k\). This frieze looks as follows:

![Cup Frieze](image1)

For \(k \in \mathbb{N}^+\) let the **cap frieze** \(A_k\) be the frieze in \(R_1\) that is defined analogously to \(V_k\) and looks as follows:

![Cap Frieze](image2)

Let a frieze without cups and caps be called a **circular frieze**. Note that according to this definition the unit frieze \(I\) is a circular frieze. For circular friezes the following lemma is proved in [9].

**Generating Circles Lemma.** Every circular frieze is \(\mathcal{F}\)-equivalent to a frieze generated from the unit frieze \(I\) and the cup and cap friezes with the operation \(\circ\).
Note that the unit frieze \( I \) is \( L \)-equivalent to \( V_k \circ A_{k+1} \) (or to \( V_{k+1} \circ A_k \)), for any \( k \in \mathbb{N}^+ \), so that, strictly speaking, the mentioning of \( I \) is superfluous in the preceding and in the following lemma, proved in [9] too.

**Generating Lemma.** Every frieze is \( L \)-equivalent to a frieze generated from the unit frieze \( I \) and the cap and cup friezes with the operation \( \circ \).

Since \( L \)-equivalence implies \( K \)-equivalence, we have the Generating Circles Lemma and the Generating Lemma also for \( L \)-equivalence replaced by \( K \)-equivalence.

It follows from the Generating Lemma that there are only denumerably many \( L \)-equivalence classes of friezes; the same holds a fortiori for \( K \)-equivalence classes. If we had allowed infinitely many cups or caps in friezes, then we would have a continuum of different \( L \) or \( K \)-equivalence classes of friezes (which is clear from the fact that we can code 0-1 sequences with such friezes). The corresponding monoids could not then be finitely generated, as \( L \circ \) and \( K \circ \) are. With infinitely many circular components we would have a continuum of different \( L \)-equivalence classes, but not so for \( K \)-equivalence classes (see Section 9).

### 7. \( L \circ \) and \( K \circ \) are monoids of friezes

Let \( \mathcal{F} \) be the set of friezes. We define as follows a map \( \delta \) from the terms of \( L \circ \) into \( \mathcal{F} \):

\[
\begin{align*}
\delta(\lfloor k \rfloor) &= V_k, \\
\delta(\lceil k \rceil) &= A_k, \\
\delta(1) &= I, \\
\delta(tu) &= \delta(t) \circ \delta(u).
\end{align*}
\]

We can then prove the following.

**Soundness Lemma.** If \( t = u \) in \( L \circ \), then \( \delta(t) \equiv_\mathcal{F} \delta(u) \).

**Proof.** We already verified in Section 5 that we have replacement of equivalents, and that Eqs. (1) and (2) of the axiomatization of \( L \circ \) are satisfied for \( I \) and \( \circ \). It just remains to verify the remaining equations, which is quite straightforward.

We have an analogous Soundness Lemma for \( K \circ \) and \( \equiv_\mathcal{F} \), involving the additional checking of (cup-cap 4).

Let \( [\mathcal{F}]_\mathcal{F} \) be the set of \( L \)-equivalence classes \( [D]_\mathcal{F} = \{D' : D \equiv_\mathcal{F} D'\} \) for all friezes \( D \) (and analogously with \( L \) replaced by \( K \)). This set is a monoid whose unit is \([I]_\mathcal{F}\) and whose multiplication is defined by taking that \([D_1]_\mathcal{F}[D_2]_\mathcal{F}\) is \([D_1 \circ D_2]_\mathcal{F}\). The Soundness Lemma guarantees that there is a homomorphism, defined via \( \delta \), from \( L \circ \) to the monoid \([\mathcal{F}]_\mathcal{F}\), and the Generating Lemma guarantees that this
homomorphism is onto. We have the same with \( L \) replaced by \( K \). It remains to establish that these homomorphisms from \( L \) onto \([F]_L\) and from \( K \) onto \([F]_K\) are also one-one.

The following lemma is proved in [9].

**Auxiliary Lemma.** If \( t \) and \( u \) are terms of \( L \) in normal form and \( \delta(t) \equiv_L \delta(u) \), then \( t \) and \( u \) are the same term.

With that we obtain the following.

**Completeness Lemma.** If \( \delta(t) \equiv_L \delta(u) \), then \( t = u \) in \( L \).

**Proof.** By the Normal Form Lemma of Section 4, for every term \( t \) and every term \( u \) of \( L \) there are terms \( t' \) and \( u' \) in normal form such that \( t = t' \) and \( u = u' \) in \( L \). By the Soundness Lemma, we obtain \( \delta(t) \equiv_L \delta(t') \) and \( \delta(u) \equiv_L \delta(u') \), and if \( \delta(t) \equiv_L \delta(u) \), it follows that \( \delta(t') \equiv_L \delta(u') \). Then, by the Auxiliary Lemma, the terms \( t' \) and \( u' \) are the same term, and hence \( t = u \) in \( L \).

The Auxiliary Lemma and the Completeness Lemma are easily obtained when \( L \) is replaced by \( K \). So we may conclude that our homomorphisms from \( L \) onto \([F]_L\) and from \( K \) onto \([F]_K\) are one-one, and hence \( L \) is isomorphic to \([F]_L\) and \( K \) is isomorphic to \([F]_K\).

We may also conclude that for every term \( t \) of \( L \) there is a unique term \( t' \) in normal form such that \( t = t' \) in \( L \). If \( t = t' \) and \( t = t'' \) in \( L \), then \( t' = t'' \) in \( L \), and hence, by the Soundness Lemma, \( \delta(t') \equiv_L \delta(t'') \). If \( t' \) and \( t'' \) are in normal form, by the Auxiliary Lemma we obtain that \( t' \) and \( t'' \) are the same term. We conclude analogously that the \( K \)-normal form is unique in the same sense with respect to \( K \).

8. The monoids \( L_n \) and \( K_n \)

The monoid \( L_n \) has for every \( i \in \{1, \ldots, n-1\} \) a generator \( h_i \), called a diapsis (plural diapsides), and also for every ordinal \( \alpha \in \omega_0 \) and every \( k \in \{1, \ldots, n+1\} \) a generator \( c_{\alpha}^k \), called a \( c \)-term. The number \( n \) here could in principle be any natural number, but the interesting monoids \( L_n \) have \( n \geq 2 \). When \( n \) is 0 or 1, we have no diapsides. The diapsis \( h_i \) corresponds to the term \([i]_0\) of \( L_0 \). The terms of \( L_n \) are obtained from these generators and 1 by closing under multiplication.

We assume the following equations for \( L_n \):

\[
\begin{align*}
(1) & \quad 1t = t, \quad t1 = t, \\
(2) & \quad t(uv) = (tu)v, \\
(c1) & \quad 1 = c_0^k, \\
(c2) & \quad c_k^z c_k^\beta = c_k^{z+\beta}, \\
(cc) & \quad c_k^z c_l^\beta = c_l^\beta c_k^z, \quad \text{for } k \neq l,
\end{align*}
\]
\[(h1) \quad h_i h_{j+2} = h_{j+2} h_i, \quad \text{for } i \leq j,
\]
\[(h2) \quad h_i h_{i+1} = h_i,
\]
\[(hc1') \quad c_k h_i = c_k^2 h_i, \quad \text{for } k \neq i + 1,
\]
\[(hc2') \quad c_{i+1} c_k h_i = c_i c_k h_i,
\]
\[(hc3) \quad c_{i+1} c_k h_i = c_{i+2} h_i.
\]

With the help of \((c2)\) we can derive \((cc)\) for \(k = l\) too.

With \(h_i\) defined as \(\lceil i \rceil \lfloor i \rfloor\) and \(c_k^2\) defined as in Section 4, we can check easily that all the equations above hold in \(L\). We can make this checking also with friezes. So \(L_n\) is a submonoid of \(L\).

An \(n\)-frieze is a frieze such that for every \(k \geq n + 1\) we have a vertical thread identified with \([-k, k]\) and for every \(k \geq n + 2\) the ordinal of circular components assigned to the thread \([-k, k]\) is 0. Each \(n\)-frieze without circular components may be conceived up to \(L\)-equivalence or \(N\)-equivalence, which here coincide, as an element of the free (noncommutative) o-monoid generated by the empty set of generators (cf. Section 3). This is because the threads of each \(n\)-frieze without circular components are identified with a rooted subtree of \(S_{\Theta}\) (see Section 5), whose root is \([- (n + 1), n + 1]\), and this rooted tree may be coded by a parenthetical word.

If \(F_n\) is the set of \(n\)-friezes, let \([F_n]_L\) be the set of \(L\)-equivalence classes of these friezes, and analogously for \([F_n]_K\). The set \([F_n]_L\) has the structure of a monoid defined as for \([F]_L\).

Then it can be shown that the monoid \([F_n]_L\) is isomorphic to \(L_n\) with the help of a map \(\delta : L_n \rightarrow F_n\) that maps a diapsis \(h_k\) into the diapsidal \(n\)-frieze \(H_k\), which is the \(n\)-frieze in \(R_b\), for some \(b > 1\), without circular components, with a single semicircular cap with the end points \((k, b)\) and \((k + 1, b)\), and a single semicircular cap with the end points \((k, 0)\) and \((k + 1, 0)\); all the other threads are vertical threads orthogonal to the \(x\) axis. A diapsidal \(n\)-frieze \(H_k\) looks as follows:

```
\[
\begin{array}{c}
\begin{array}{cccc}
\vdots & \cdots & \zeta & \cdots \\
1 & k-1 & k & k+1 & k+2 \\
\end{array}
\end{array}
\]
```

The \(c\)-term \(c_k^2\) is mapped by \(\delta\) into the frieze

```
\[
\begin{array}{c}
\begin{array}{ccc}
\cdots & \zeta & \cdots \\
1 & k-1 & k \\
\end{array}
\end{array}
\]
```

where \(\zeta\) stands for an arbitrary circular form corresponding to \(\zeta\). We also have \(\delta(1) = I\) and \(\delta(tu) = \delta(t) \circ \delta(u)\), as before. It is clear that the unit frieze \(I\) is an \(n\)-frieze for every \(n \in N\), and that the composition of two \(n\)-friezes is an \(n\)-frieze.
We will not go into the details of the proof that we have an isomorphism here, because we do not have much use for $L_n$ in this work. A great part of this proof is analogous to what we had for $K_n$, or to what we have for $\mathcal{N}$ below. The essential part of the proof is the definition of unique normal form for elements of $L_n$. Here is how such a normal form would look like.

For $1 \leq j \leq i \leq n - 1$ and $\alpha, \beta \in \varepsilon_0$, let the block $h_{\alpha, \beta}^{i,j}$ be defined as

$$c_{i+1}^\alpha h_i h_{i-1} \cdots h_{j+1} h_j c_{j+1}^\beta.$$  

A term of $L_n$ in normal form will be $1$, or it looks as follows:

$$c_{k_1}^{\gamma_1} \cdots c_{k_l}^{\gamma_l} h_{i_1,c_1}^{b_1} \cdots h_{i_n,c_n}^{b_n},$$

where $n, l \geq 0$, $k_1 < \cdots < k_l$, $b_1 < \cdots < b_n$, and $c_1 < \cdots < c_n$. All the $c$-terms on the left-hand side are such that they could be permuted with all the blocks, and pass to the right-hand side; i.e. they would not be “captured” by a block. We must also make a choice for the indices $k_p$ of these $c$-terms to ensure uniqueness, and $\gamma_p$ should not be 0.

One way to define the monoid $\mathcal{N}_n$ is to have the same generators as for $L_n$, and the following equations, which we add to those of $L_n$:

$$c_k^{\alpha} = c_{k+1}^{\alpha},$$

$$c_k^\beta = c_{k+1}^\beta.$$

The first equation has the effect of collapsing the ordinals in $\varepsilon_0$ into natural numbers (as the equation (solid) of Section 3), while the second equation has the effect of making superfluous the lower index of $c$-terms.

An alternative, and simpler, axiomatization of $\mathcal{N}_n$ is obtained as follows. The monoid $\mathcal{N}_n$ has for every $i \in \{1, \ldots, n - 1\}$ a generator $h_i$, called again a diapsis, and also the generator $c$, called the circle. The terms of $\mathcal{N}_n$ are obtained from these generators and $1$ by closing under multiplication. We assume the following equations for $\mathcal{N}_n$:

(1) $1t = t, \quad t1 = t,$

(2) $t(\alpha \beta) = (\alpha \beta)t,$

(h1) $h_i h_{j+2} = h_{j+1} h_i, \quad$ for $i \leq j$,

(h2) $h_i h_{i+1} h_i = h_i,$

(hc1) $h_i c = ch_i,$

(hc2) $h_i h_i = ch_i.$

Eqs. (h1), (h2) and (hc2), which may be derived from Jones’ paper [17, p. 13], and which appear in the form above in many works of Kauffman (see [24,23, Section 6] and references therein), are usually tied to the presentation of Temperley–Lieb algebras. They may, however, be found in Brauer algebras too (see [33, p. 180–181]).
With \( h_i \) defined as \( \lceil i \rceil \lfloor i \rfloor \) and \( c \) defined as \( \lfloor i \rfloor \lceil i \rceil \) we can check easily that \( \mathcal{K}_n \) is a submonoid of \( \mathcal{K}_\infty \).

For \( 1 \leq j \leq i \leq n - 1 \), let the block \( h_{[i,j]} \) be defined as \( h_i h_{i+1} \cdots h_{j+1} h_j \). The block \( h_{[i,i]} \), which is defined as \( h_i \), will be called singular. (One could conceive \( \lfloor i \rfloor \) as the infinite block \( \cdots h_{i+2} h_{i+1} h_i \), whereas \( \lceil i \rceil \) would be \( h_i h_{i+1} h_{i+2} \cdots \).) Let \( c^1 \) be \( c \), and let \( c^{l+1} \) be \( c^l c \).

A term is in Jones normal form if it is either of the form \( c^l h_{[b_1,a_1]} \cdots h_{[b_k,a_k]} \) for \( l, k \geq 0 \), \( l + k \geq 1 \), \( a_1 < \cdots < a_k \) and \( b_1 < \cdots < b_k \), or it is the term \( 1 \) (see [17, Section 4.1.4, p. 14]). As before, if \( l = 0 \), then \( c^l \) is the empty sequence, and if \( k = 0 \), then \( h_{[b_1,a_1]} \cdots h_{[b_k,a_k]} \) is empty.

Then we can prove the following lemma as in [3]. (A lemma with the same content is established in a different manner in [17, pp. 13–14], [15, pp. 87–89].)

**Normal Form Lemma.** Every term of \( \mathcal{K}_n \) is equal in \( \mathcal{K}_n \) to a term in Jones normal form.

We ascertainment above that the unit frieze \( I \) of Section 5 is an \( n \)-frieze for every \( n \in \mathbb{N} \). We have also defined there what is the diapsidal \( n \)-frieze \( H_i \) for \( i \in \{1, \ldots, n-1\} \). The circular \( n \)-frieze \( C \) is the \( n \)-frieze that differs from the unit frieze \( I \) by having a single circular component, which, for the sake of definiteness, we choose to be a circle of radius \( 1/4 \), with centre \( (1/2, 1/2) \). We have also mentioned that the composition of two \( n \)-friezes is an \( n \)-frieze. Then we can prove the following lemma as in [3]. (Different, and more sketchy, proofs of this lemma may be found in [32, Chapter VIII, Section 26, 23, Section 6]; in [2, Proposition 4.1.3] one may find a proof of something more general, and somewhat more complicated.)

**Generating Lemma.** Every \( n \)-frieze is \( \mathcal{K} \)-equivalent to an \( n \)-frieze generated from \( I \), \( C \) and the diapsidal \( n \)-friezes \( H_i \), for \( i \in \{1, \ldots, n-1\} \), with the operation \( \circ \).

Let \( \mathcal{D}_n \) be the set of \( n \)-friezes. We define as follows a map \( \delta \) from the terms of \( \mathcal{K}_n \) into \( \mathcal{D}_n \):

\[
\delta(h_i) = H_i, \\
\delta(c) = C, \\
\delta(1) = I, \\
\delta(tu) = \delta(t) \circ \delta(u).
\]

We can then prove easily the following.

**Soundness Lemma.** If \( t = u \) in \( \mathcal{K}_n \), then \( \delta(t) \cong_{\mathcal{K}} \delta(u) \).

We want to show that the homomorphism from \( \mathcal{K}_n \) to \( [\mathcal{F}_n]_{\mathcal{K}} \) defined via \( \delta \), whose existence is guaranteed by the Soundness Lemma, is an isomorphism. The Generating
Lemma guarantees that this homomorphism is onto, and it remains to establish that it is one-one. The proof of that is based on the following lemmata, proved in [3].

**Key Lemma.** If $t$ is the term $h_{[b_1,a_1]} \cdots h_{[b_k,a_k]}$ with $a_1 < \cdots < a_k$ and $b_1 < \cdots < b_k$, then $T_{b(t)}$ is $a_1, \ldots, a_k$ and $B_{b(t)}$ is $b_1, \ldots, b_k$.

**Auxiliary Lemma.** If $t$ and $u$ are terms of $\mathcal{K}_n$ in Jones normal form and $\delta(t) \equiv_x \delta(u)$, then $t$ and $u$ are the same term.

**Completeness Lemma.** If $\delta(t) \equiv_x \delta(u)$, then $t = u$ in $\mathcal{K}_n$.

This last lemma is proved analogously to the Completeness Lemma of the preceding section by using the Normal Form Lemma, the Soundness Lemma and the Auxiliary Lemma of the present section. With this lemma we have established that $\mathcal{K}_n$ is isomorphic to $[\mathcal{F}_n]_x$.

9. The monoid $\mathcal{J}_\omega$

Let $\mathcal{J}_\omega$ be the monoid defined as $\mathcal{L}_\omega$ save that for every $k \in \mathbb{N}^+$ we require also $[k][k] = 1$,

i.e. $[k] = 1$. It is clear that all the equations of $\mathcal{H}_\omega$ are satisfied in $\mathcal{J}_\omega$, but not conversely. In $\mathcal{J}_\omega$ circles are irrelevant.

The monoid $\mathcal{J}_n$ is obtained by extending $\mathcal{H}_n$ with $c^1 = 1$, or $c = 1$. Alternatively, we may omit $c$-terms, or the generator $c$, and assume only Eqs. (1), (2), (h1) and (h2) of the preceding section, together with the idempotency of $h_i$, namely, $h_i h_i = h_i$. (These axioms may be found in [17, p. 13].) The monoids $\mathcal{J}_n$ are submonoids of $\mathcal{J}_\omega$.

Let a $\mathcal{J}$-frieze be an $\omega$-diagram with a finite number of cups and caps and denumerably many circular components. (Instead of “denumerably many circular components” we could put “$\kappa$ circular components for a fixed infinite cardinal $\kappa$”; for the sake of definiteness, we chose $\kappa$ to be the least infinite cardinal $\omega$.) We define $\mathcal{H}$-equivalence of $\mathcal{J}$-friezes as for friezes, and we transpose other definitions of Section 5 to $\mathcal{J}$-friezes in the same manner. It is clear that the following holds.

**Remark** 1. $\mathcal{J}$. The $\mathcal{J}$-friezes $D_1$ and $D_2$ are $\mathcal{H}$-equivalent iff the end points of the threads in $D_1$ are identified with the same pairs of integers as the end points of the threads in $D_2$.

So we need not pay attention any more to circular components.

The unit $\mathcal{J}$-frieze is defined as the unit frieze $I$ save that we assume that it has denumerably many circular components, which are located in some arbitrary regions. With composition of $\mathcal{J}$-friezes defined as before, the set of $\mathcal{H}$-equivalence classes of $\mathcal{J}$-friezes makes a monoid.

By adapting the argument in Sections 6 and 7, we can show that this monoid is isomorphic to $\mathcal{J}_\omega$. We do not need any more the Generating Circles Lemma, since
circular \( J \)-friezes are \( \mathcal{H} \)-equivalent to the unit \( J \)-frieze. The cup and cap friezes \( V_k \) and \( A_k \) have now denumerably many circular components, which are located in some arbitrary regions.

A \( J \)-\( n \)-frieze is defined as an \( n \)-frieze save that it has denumerably many circular components. Then we can show by adapting the argument in the preceding section that \( J \_n \) is isomorphic to the monoid \( [FJ_n]_X \) of \( \mathcal{H} \)-equivalence classes of \( J \)-\( n \)-friezes.

An alternative proof that the map from \( J \_n \) to \( [FJ_n]_X \), defined analogously to what we had in the preceding section, is one-one may be obtained as follows. One can establish that the cardinality of \( [FJ_n]_X \) is the \( n \)th Catalan number \((2n)!/(n!(n+1)!))\) (see the comment after the definition of \( n \)-friezes in the preceding section; see also [23, Section 6.1], and references therein). Independently, one establishes as in [17, p. 14] that the number of terms of \( J \_n \) in Jones normal form is also the \( n \)th Catalan number. So, by the Normal Form Lemma of the preceding section, the cardinality of \( J \_n \) is at most the \( n \)th Catalan number. Since, by the Generating Lemma of that section, it is known that the map above is onto, it follows that it is one-one. This argument is on the lines of the argument in [6, Note C, pp. 464–465], which establishes that the standard presentation of symmetric groups is complete with respect to permutations. It can also be adapted to give an alternative proof of the Completeness Lemma of the preceding section, which is not based on the Key Lemma and the Auxiliary Lemma of that section.

It is shown in [9] that \( J \_\omega \) is maximal in the following sense. Let \( t \) and \( u \) be terms of \( L \_\omega \) such that \( t = u \) does not hold in \( J \_\omega \). If \( \mathcal{X} \) is defined as \( J \_\omega \) save that we require also \( t = u \), then for every \( k \in \mathbb{N}^+ \) we have \([k][k] = 1 \) in \( \mathcal{X} \). With the same assumptions, for some \( n \in \mathbb{N} \) we have that \( \mathcal{X} \) is isomorphic to the monoid \( \mathbb{Z}/n \), i.e. the additive commutative monoid \( \mathbb{Z} \) with equality modulo \( n \).

10. Self-adjunctions

A self-adjunction, which we will also call \( L \)-adjunction, is an adjunction in which an endofunctor is adjoint to itself, which means that it is both left and right adjoint to itself (for the general notion of adjunction see [30, Chapter IV]). More precisely, an \( L \)-adjunction is \( \langle \mathcal{A}, \circ, 1, F, \varphi, \gamma \rangle \) where \( \langle \mathcal{A}, \circ, 1 \rangle \) is a category, which means that for \( f : a \to b \), \( g : b \to c \) and \( h : c \to d \) arrows of \( \mathcal{A} \) we have the equations

\[
\text{cat 1)} \quad f \circ 1_a = f, \quad 1_b \circ f = f,
\]
\[
\text{cat 2)} \quad h \circ (g \circ f) = (h \circ g) \circ f;
\]

\( F \) is a functor from \( \mathcal{A} \) to \( \mathcal{A} \), which means that we have the equations

\[
\text{fun 1)} \quad F1_a = 1_{Fa},
\]
\[
\text{fun 2)} \quad F(g \circ f) = Fg \circ Ff;
\]

\( \varphi \) is a natural transformation (the counit of the adjunction) with components \( \varphi_a : FFa \to a \), and \( \gamma \) is a natural transformation (the unit of the adjunction) with components
\( \gamma_a : a \to \text{FF} a \), which means that we have the equations

\[
\begin{align*}
(nat \ \varphi) \quad & f \circ \varphi_a = \varphi_b \circ \text{FF} f, \\
(nat \ \gamma) \quad & \text{FF} f \circ \gamma_a = \gamma_b \circ f;
\end{align*}
\]

and, finally, we have the triangular equations

\[
\begin{align*}
(\varphi \gamma) \quad & F \varphi_a \circ \gamma_F a = \varphi_F a \circ F \gamma_a = 1_F a.
\end{align*}
\]

We will call the equations from (cat 1) to (\varphi \gamma) we have displayed above the \( L \)-equations.

A \( K \)-adjunction is an \( L \)-adjunction that satisfies the additional equation

\[
(\varphi \gamma K) \quad \text{F} (\varphi_a \circ \gamma_a) = \varphi_F a \circ \gamma_F a.
\]

The \( L \)-equations plus this equation will be called \( K \)-equations.

Let \( \kappa_a \) be an abbreviation for \( \varphi_a \circ \gamma_a : a \to a \). Then in every \( L \)-adjunction we have that \( f \circ \kappa_a = \kappa_b \circ f \), and (\varphi \gamma K) is expressed by \( F \kappa_a = \kappa_F a \). We will see that the arrows \( \kappa_a \) are in general not equal to the identity arrows \( 1_a \) in arbitrary \( K \)-adjunctions, but they have some properties of identity arrows: they commute with other arrows, and they are preserved by the functor \( F \).

A \( J \)-adjunction is an \( L \)-adjunction that satisfies the additional equation

\[
(\varphi \gamma J) \quad \varphi_a \circ \gamma_a = 1_a,
\]

i.e. \( \kappa_a = 1_a \). The \( L \)-equations plus this equation will be called \( J \)-equations. Every \( J \)-adjunction is a \( K \)-adjunction, but not vice versa, as we will see later.

11. Free self-adjunctions

The free \( L \)-adjunction generated by an arbitrary object, which we will denote by 0, is defined as follows. The category of this self-adjunction, which we will call \( \mathcal{L}_c \), has the objects 0, \( F0 \), \( \text{FF} 0 \), etc., which may be identified with the natural numbers 0, 1, 2, etc.

An arrow-term of \( \mathcal{L}_c \) will be a word \( f \) that has a type \((n,m)\), where \( n,m \in \mathbb{N} \). That \( f \) is of type \((n,m)\) is expressed by \( f : n \to m \). Now we define the arrow-terms of \( \mathcal{L}_c \) inductively. We stipulate first for every \( n \in \mathbb{N} \) that \( 1_n : n \to n \), \( \varphi_n : n + 2 \to n \) and \( \gamma_n : n \to n + 2 \) are arrow-terms of \( \mathcal{L}_c \). Next, if \( f : m \to n \) is an arrow-term of \( \mathcal{L}_c \), then \( Ff : m + 1 \to n + 1 \) is an arrow-term of \( \mathcal{L}_c \), and if \( f : m \to n \) and \( g : n \to k \) are arrow-terms of \( \mathcal{L}_c \), then \( (g \circ f) : m \to k \) is an arrow-term of \( \mathcal{L}_c \). As usual, we do not write parentheses in \((g \circ f)\) when they are not essential.

On these arrow-terms we impose the \( L \)-equations, where \( a \) and \( b \) are replaced by \( m \) and \( n \), and \( f, g \) and \( h \) stand for arrow-terms of \( \mathcal{L}_c \). Formally, we take the smallest equivalence relation \( \equiv \) on the arrow-terms of \( \mathcal{L}_c \) satisfying, first, congruence conditions
with respect to $F$ and $\circ$, namely,

- if $f \equiv g$, then $Ff \equiv Fg$,
- if $f_1 \equiv f_2$ and $g_1 \equiv g_2$, then $g_1 \circ f_1 \equiv g_2 \circ f_2$,

provided $g_1 \circ f_1$ and $g_2 \circ f_2$ are defined, and, second, the conditions obtained from the $L$-equations by replacing the equality sign by $\equiv$. Then we take the equivalence classes of arrow-terms as arrows, with the obvious source and target, all arrow-terms in the same class having the same type. On these equivalence classes we define $1$, $\varphi$, $\gamma$, $F$ and $\circ$ in the obvious way. This defines the category $L_c$, in which we have clearly an $L$-adjunction.

The category $L_c$ satisfies the following universal property. If $\lambda$ maps the object 0 into an arbitrary object of the category $A$ of an arbitrary $L$-adjunction, then there is a unique functor $\Lambda$ of $L$-adjunctions (defined in the obvious way, so that the $L$-adjunction structure is preserved) such that $\lambda$ maps 0 into $\lambda(0)$. This property characterizes $L_c$ up to isomorphism with a functor of $L$-adjunctions. This justifies calling free the $L$-adjunction of $L_c$.

The category $K_c$ of the free $K$-adjunction and the category $J_c$ of the free $J$-adjunction, both generated by 0, are defined as $L_c$ save that we replace everywhere $L$ by $K$ and $J$, respectively. So Eqs. (12) come into play. The categories $K_c$ and $J_c$ satisfy universal properties analogous to the one above.

Let $BDC^m$ be $\gamma_m \circ BCR_m$ as in the preceding section, let $BDC^0_m$ be $1_m$, and let $BDC^l_m = BDC^m \circ BDC^l_m$.

It is easy to show by induction on the length of derivation that if $f \equiv g$ in $J_c$, then for some $k,l \in \mathbb{N}$ we have $f \circ \kappa^k = g \circ \kappa^l$ in $K_c$. (The converse implication holds trivially.) We will establish in the next section that if $f \equiv g$ in $J_c$ but not in $K_c$, then $k$ must be different from $l$.

12. $L_c$ and $L_\circ$

Let $F^0$ be the empty sequence, and let $F^{k+1} = F^k F$. On the arrows of $L_c$, we define a total binary operation $\ast$ based on composition of arrows in the following manner. For $f : m \to n$ and $g : k \to l$,

$$
\begin{align*}
g \ast f & = \begin{cases} 
g \circ F^{k-n} f & \text{if } n \leq k, \\
F^{n-k} g \circ f & \text{if } k \leq n.
\end{cases}
\end{align*}
$$

Next, let $f \equiv g$ iff there are $k,l \in \mathbb{N}$ such that $F^k f = F^l g$ in $L_c$. It is easy to check that $\equiv$ is an equivalence relation on the arrows of $L_c$, which satisfies moreover

$$(congr*) \text{ if } f_1 \equiv g_1 \text{ and } g_1 \equiv g_2, \text{ then } g_1 \ast f_1 \equiv g_2 \ast f_2.$$

For every arrow $f$ of $L_c$, let $[f]$ be $\{g \mid f \equiv g\}$, and let $L_c^\ast$ be $\{[f] \mid f \text{ is an arrow of } L_c\}$. With

$$
1 = \text{def } [1_0],
$$

$$
[g][f] = \text{def } [g \ast f].
$$
we can check that $L_c^*$ is a monoid. We will show that this monoid is isomorphic to
the monoid $L_o^*$.

Consider the map $\psi$ from the arrow-terms of $L_c$ to the terms of $L_o$ defined inductively by

$$
\psi(1_n) \text{ is } 1,
\psi(\varphi_n) \text{ is } [n+1],
\psi(\gamma_n) \text{ is } [n+1],
\psi(Ff) \text{ is } \psi(f),
\psi(g \circ f) \text{ is } \psi(g)\psi(f).
$$

Let $m,n \in \mathbb{N}$. If in a frieze we have for every $k \in \mathbb{N}^+$ that $(m+k,0)$ and $(n+k,a)$
are the end points of a transversal thread, i.e., $[-(m+k),n+k]$ identifies this thread, and there are no circular components in the regions that correspond to $[-(m+k+1),n+k+1]$, i.e., the ordinal 0 of circular components is assigned to this thread, then we say that this frieze is of type $(n,m)$. Note that a frieze of type $(n,m)$ is also of
type $(n+k,m+k)$. The $n$-friezes of Section 8 are the friezes of type $(n,n)$.

We can easily establish the following by induction on the length of $f$.

**Remark I.** For every arrow-term $f: n \to m$ of $L_c$, the frieze $\delta(\psi(f))$ is of type $(n,m)$.

We also have the following.

**Remark II.** If the frieze $\delta(t)$ is of type $(n,m)$, then $\delta(t[n+1]) \equiv_{\mathcal{S}} \delta([m+1]t)$ and
$\delta(t[n+1]) \equiv_{\mathcal{S}} \delta([m+1]t)$.

Then we can prove the following lemma.

**\psi Lemma.** If $f = g$ in $L_c$, then $\psi(f) = \psi(g)$ in $L_o$.

**Proof.** We proceed by induction on the length of the derivation of $f = g$ in $L_c$. All
the cases are quite straightforward except when $f = g$ is an instance of $(\text{nat } \varphi)$ or
$(\text{nat } \gamma)$, where we use Remarks I and II. In case $f = g$ is an instance of $(\varphi \gamma)$ we use
$(\text{cup-cap } 3)$. □

As an immediate corollary we have that if $f \equiv_{\mathcal{S}} g$, then $\psi(f) = \psi(g)$ in $L_o$. Hence
we have a map from $L_c^*$ to $L_o^*$, which we also call $\psi$, defined by $\psi([f]) = \psi(f)$.
Since

$$
\psi([1_0]) = \psi(1_0) = 1,
\psi([g \ast f]) = \psi(g)\psi(f),
$$

this map is a monoid homomorphism.
Consider next the map $\chi$ from the terms of $\mathcal{L}_o$ to the arrow-terms of $\mathcal{L}_c$ defined inductively by
\[
\begin{align*}
\chi(1) & \quad \text{is} \quad 1_0, \\
\chi([k]) & \quad \text{is} \quad \varphi_{k-1}, \\
\chi([k]) & \quad \text{is} \quad \gamma_{k-1}, \\
\chi(tu) & \quad \text{is} \quad \chi(t) * \chi(u).
\end{align*}
\]

Then we establish the following lemmata.

$\chi$ Lemma. If $t = u$ in $\mathcal{L}_o$, then $\chi(t) \equiv_{\mathcal{F}} \chi(u)$.

Proof. We proceed by induction on the length of the derivation of $t = u$ in $\mathcal{L}_o$. The cases where $t = u$ is an instance of (1) and (2) are quite straightforward. If $t = u$ is an instance of (cup), we have
\[
\varphi_{k-1} \circ F^{k-1} \varphi_{l-1} = F^{k-1} \varphi_{l-1} \circ \varphi_{k+1}, \quad \text{by (nat } \varphi).\]

If $t = u$ is an instance of (cup-cap 1), we have
\[
F^{k-1} \varphi_{l-1} \circ \gamma_{k+1} = \gamma_{k-1} \circ F^{k-1} \varphi_{l-1}, \quad \text{by (nat } \gamma).\]

We proceed analogously for (cap), using (nat $\gamma$), and for (cup-cap 2), using (nat $\varphi$). Finally, if $t = u$ is an instance of (cup-cap 3), we have by ($$\varphi$$)
\[
F \varphi_{k-1} \circ \gamma_k = 1_k,
\]
\[
\varphi_{k-1} \circ F \gamma_{k-2} = 1_{k-1}.
\]

We have already established that $\equiv_{\mathcal{F}}$ is an equivalence relation that satisfies (congr $\ast$). So the lemma follows. $\square$

$\chi \psi$ Lemma. For every arrow-term $f$ of $\mathcal{L}_c$ we have $\chi(\psi(f)) \equiv_{\mathcal{F}} f$.

Proof. We proceed by induction on the length of $f$. We have
\[
\begin{align*}
\chi(\psi(1_n)) & \quad \text{is} \quad 1_0 \equiv_{\mathcal{F}} 1_n, \\
\chi(\psi(\varphi_n)) & \quad \text{is} \quad \varphi_n, \\
\chi(\psi(\gamma_n)) & \quad \text{is} \quad \gamma_n, \\
\chi(\psi(Ff)) & \quad \text{is} \quad \chi(\psi(f)) \\
& \equiv_{\mathcal{F}} f, \quad \text{by the induction hypothesis} \\
\chi(\psi(g \circ f)) & \quad \text{is} \quad \chi(\psi(g)) \ast \chi(\psi(f)) \\
& \equiv_{\mathcal{F}} g \circ f, \quad \text{by the induction hypothesis, (congr } \ast\text{) and the definition of } \ast. \quad \square
\end{align*}
\]
By a straightforward induction we can prove also the following lemma.

ψt-Lemma. For every term t of Lω we have that ψ(χ(t)) is the term t.

This establishes that L* and Lω are isomorphic monoids.

Let S* and J* be monoids defined analogously to L*, by replacing everywhere L by S and J respectively. Then we can easily extend the foregoing results to establish that S* is isomorphic to the monoid Sω and that J* is isomorphic to the monoid Jω.

We also have the following lemma.

L Cancellation Lemma. In every L-adjunction, for f, g : a → Fb or f, g : Fa → b, if Ff = Fg, then f = g.

Proof. Suppose Ff = Fg for f, g : a → Fb. Then

\[ F\varphi_b \circ FFf \circ \gamma_a = F\varphi_b \circ FFg \circ \gamma_a \]
\[ F\varphi_b \circ \gamma_{Fb} \circ f = F\varphi_b \circ \gamma_{Fb} \circ g, \text{ by (nat \( \gamma \))} \]
\[ f = g, \text{ by (\( \varphi \gamma \)) and (cat 1).} \]

Suppose Ff = Fg for f, g : Fa → b. Then

\[ \varphi_b \circ FFf \circ F\gamma_a = \varphi_b \circ FFg \circ F\gamma_a \]
\[ f \circ \varphi_{Fa} \circ F\gamma_a = g \circ \varphi_{Fa} \circ F\gamma_a, \text{ by (nat \( \varphi \))} \]
\[ f = g, \text{ by (\( \varphi \gamma \)) and (cat 1).} \]

As an instance of this lemma we obtain that Ff = Fg implies f = g in L* provided that for f, g : m → n we have m + n > 0. As a matter of fact, this implication holds for f, g : 0 → 0 too, but the proofs we know of that fact are rather involved, and are pretty lengthy. We know two proofs, which are both based on reducing arrow-terms of L* to a unique normal form that corresponds to c1 of Section 4 when m + n = 0. One of these normal forms is based on the normal form for terms of Lω in Section 4, but with a number of complications brought in by the types of arrow terms. The other normal form is a composition-free normal form in a particular language, and reduction to it (achieved in the style of Gentzen’s famous proof-theoretical cut elimination theorem; see [14, Paper 3]) is at least as complicated as reduction to the other normal form. We will omit these proofs, since the importance of the fact in question, on which we will not rely in the sequel, does not warrant spending too much on establishing it.

As a corollary of the L Cancellation Lemma, and of previously established results, we have that for f, g : m → n with m + n > 0

\[ f = g \text{ in } L^* \text{ iff } f \equiv_{L^*} g \]
\[ \text{iff } \psi(f) = \psi(g) \text{ in } L_ω \]
\[ \text{iff } \delta(\psi(f)) \equiv_{L} \delta(\psi(g)). \]
Since the $\mathcal{L}$ Cancellation Lemma applies also to $\mathcal{K}_c$, we have exactly analogous equivalences when $\mathcal{L}$ is replaced by $\mathcal{K}$. However, for this replacement we can rather easily lift the restriction $m+n>0$.

$\mathcal{K}_c$ Cancellation Lemma. In $\mathcal{K}_c$, if $Ff = Fg$, then $f = g$.

Proof. If in $f,g : m \to n$ we have $m+n>0$, we apply the $\mathcal{L}$ Cancellation Lemma. If $m=n=0$, then $f$ is equal either to $I_0$ or to $\varphi_0 \circ f'$, and $g$ is equal either to $I_0$ or to $\varphi_0 \circ g'$. Here $f'$ must be $f'' \circ \gamma_0$ and $g'$ must be $g'' \circ \gamma_0$. (So we could alternatively consider $f$ and $g$ being equal to $f'''' \circ \gamma_0$ or $g'''' \circ \gamma_0$, and reason analogously below.)

If both $f=g=I_0$, we are done. It is excluded that $f=I_0$ while $g=\varphi_0 \circ g'$. We have that $\psi(f)$ is $\psi(Ff)$ and $\psi(g)$ is $\psi(Fg)$. Since from $Ff = Fg$ in $\mathcal{K}_c$ it follows that $\delta(\psi(Ff)) \cong_X \delta(\psi(Fg))$, we have $\delta(\psi(f)) \cong_X \delta(\psi(g)))$. But $\delta(\psi(I_0))$ is not $\mathcal{K}$-equivalent to $\delta(\psi(\varphi_0 \circ g'' \circ \gamma_0))$, because, by Remark I, the frieze $\delta(\psi(g'))$ must be a 2-frieze, from which we obtain that there is at least one circular component in $\delta(\psi(\varphi_0 \circ g'' \circ \gamma_0))$. It is excluded in the same manner that $g=I_0$ while $f=\varphi_0 \circ f'$.

If $f=\varphi_0 \circ f'$ and $g=\varphi_0 \circ g'$, then from $\delta(\psi(\varphi_0 \circ f')) \cong_X \delta(\psi(\varphi_0 \circ g'))$, we conclude $\delta(\psi(f')) \cong_X \delta(\psi(g'))$. This is because $\delta(\psi(\varphi_0 \circ f'))$ and $\delta(\psi(\varphi_0 \circ g'))$ are both $\mathcal{K}$-equivalent to $\delta(c^k)$, for $k \geq 1$, while $\delta(\psi(f'))$ and $\delta(\psi(g'))$ must both be $\mathcal{K}$-equivalent to $\delta(c^{k-1}[1])$. But $f'$ and $g'$ are of type 0 $\to$ 2, and hence, by the $\mathcal{L}$ Cancellation Lemma, $f' = g'$ in $\mathcal{K}_c$, from which we obtain that $\varphi_0 \circ f' = \varphi_0 \circ g'$ in $\mathcal{K}_c$.  

This proof would not go through for $\mathcal{L}_c$, because $\delta(\psi(f'))$ need not be $\mathcal{L}$-equivalent to $\delta(\psi(g'))$. For example, with $h$ being $F(\varphi_3 \circ \gamma_2) \circ \gamma_2 \circ \gamma_0$, we have in $\mathcal{L}_c$

$$\varphi_0 \circ \varphi_2 \circ h = \varphi_0 \circ F^2 \varphi_0 \circ h,$$

but $\delta(\psi(\varphi_2 \circ h))$ is not $\mathcal{L}$-equivalent to $\delta(\psi(F^2 \varphi_0 \circ \gamma_3))$, and $\varphi_2 \circ h = F^2 \varphi_0 \circ h$ does not hold in $\mathcal{L}_c$. It holds in $\mathcal{K}_c$.

The $\mathcal{K}_c$ Cancellation Lemma implies that $f \cong_X g$ for $f : m \to n$ and $g : k \to l$ could be defined by $F^{k-n} f = g$ in $\mathcal{K}_c$ when $k \leq n$, and by $f = F^{n-k} g$ in $\mathcal{K}_c$ when $k \geq n$. So for arbitrary $f,g : m \to n$ we have established that $f = g$ in $\mathcal{K}_c$ iff $f \cong_X g$.

Since, by Remark I, for $f,g : n \to n$, where $n \in \mathbb{N}$, the friezes $\delta(\psi(f))$ and $\delta(\psi(g))$ are $n$-friezes, we can conclude that $\mathcal{K}_n$ is isomorphic to the monoid of endomorphisms $f : n \to n$ of $\mathcal{K}_c$. We have this isomorphism for every $n \in \mathbb{N}$, but the monoids $\mathcal{K}_n$ are interesting only when $n \geq 2$.

We could conclude analogously that $\mathcal{L}_n$ is isomorphic to the monoid of endomorphisms $f : n \to n$ of $\mathcal{L}_c$, relying on the proof of the isomorphism of $\mathcal{L}_n$ with $[\mathcal{K}_n]_{\mathcal{L}}$, which we have only indicated, and not given in Section 8. The $\mathcal{L}$ Cancellation Lemma guarantees this isomorphism for $\mathcal{L}_n$ if $n > 0$, though, as we mentioned above, at the cost of additional arguments this restriction can be lifted.

We have the following lemma for $\mathcal{J}$-adjunctions.

$\mathcal{J}$ Cancellation Lemma. In every $\mathcal{J}$-adjunction, for $f$ and $g$ arrows of the same type, if $Ff = Fg$, then $f = g$. 
Proof. Take \( f, g : a \to b \), for \( a \) and \( b \) arbitrary objects. From \( Ff = Fg \) we infer
\[
\varphi_b \circ FFf \circ \gamma_a = \varphi_b \circ FFg \circ \gamma_a,
\]
from which by \((\text{nat } \varphi)\) or \((\text{nat } \gamma)\), followed by \((\varphi \text{? } \mathcal{J})\) and \((\text{cat } 1)\), we obtain \( f = g \). □

By proceeding as in this proof in an arbitrary \( \mathcal{L} \)-adjunction we can conclude only that if \( Ff = Fg \), then \( f \circ \kappa_a = g \circ \kappa_a \).

The \( \mathcal{J} \) Cancellation Lemma enables us to show that for arbitrary \( f, g : m \to n \) we have \( f = g \) in \( \mathcal{J}_c \) iff \( \psi(f) = \psi(g) \) in \( \mathcal{J}_c \), which means that we can check equations of \( \mathcal{J}_c \) through \( \mathcal{K} \)-equivalence of \( \mathcal{J} \)-frieses. The monoids \( \mathcal{J}_n \) are isomorphic to the monoids of endomorphisms of \( \mathcal{J}_c \).

We may now confirm what we stated at the end of the preceding section, namely, that if \( f = g \) in \( \mathcal{J}_c \) but not in \( \mathcal{K}_c \), then for some \( k, l \geq 0 \) such that \( k \neq l \) we have \( f \circ \kappa^k = g \circ \kappa^l \) in \( \mathcal{K}_c \). If \( f = g \) does not hold in \( \mathcal{K}_c \), but it holds in \( \mathcal{J}_c \), then the friezes \( \delta(\psi(f)) \) and \( \delta(\psi(g)) \) differ only with respect to the number of circular components.

13. Self-adjunction in \( \text{Mat}_\mathcal{F} \)

Let \( \text{Mat}_\mathcal{F} \) be the skeleton of the category of finite-dimensional vector spaces over a number field \( \mathcal{F} \) with linear transformations as arrows. A number field is any subfield of the field of complex numbers \( \mathbb{C} \), and hence it is an extension of the field of rational numbers \( \mathbb{Q} \). A skeleton of a category \( \mathcal{C} \) is any full subcategory \( \mathcal{C}' \) of \( \mathcal{C} \) such that each object of \( \mathcal{C} \) is isomorphic in \( \mathcal{C} \) to exactly one object of \( \mathcal{C}' \). Any two skeletons of \( \mathcal{C} \) are isomorphic categories, so that, up to isomorphism, we may speak of the skeleton of \( \mathcal{C} \).

More precisely, the objects of the category \( \text{Mat}_\mathcal{F} \) are natural numbers (the dimensions of our vector spaces), an arrow \( A : m \to n \) is an \( n \times m \) matrix, composition of arrows \( \circ \) is matrix multiplication, and the identity arrow \( 1_n : n \to n \) is the \( n \times n \) matrix with 1 on the diagonal and 0 elsewhere. (The number 0 is a null object in the category \( \text{Mat}_\mathcal{F} \), which, as far as we are here interested in this category, we could as well exclude.)

For much of what we say at the beginning concerning self-adjunction in \( \text{Mat}_\mathcal{F} \) it would be enough to assume that the scalars in \( \mathcal{F} \) are just elements of the commutative monoid \( \langle \mathbb{N}, +, 0 \rangle \). However, then we would not have vector spaces, but something more general, which has no standard name. Later (see Section 17) we will indeed need that the scalars make \( \mathbb{Q} \) or an extension of it.

Let \( p \in \mathbb{N}^+ \), and consider the functor \( p \otimes \) from \( \text{Mat}_\mathcal{F} \) to \( \text{Mat}_\mathcal{F} \) defined as follows: for the object \( m \) of \( \text{Mat}_\mathcal{F} \) we have that \( p \otimes m \) is \( pm \), and for the arrow \( B : m \to n \) of \( \text{Mat}_\mathcal{F} \), i.e. an \( n \times m \) matrix \( B \), let \( p \otimes B : pm \to pn \) be the Kronecker product \( 1_p \otimes B \) of the matrices \( 1_p \) and \( B \) (see [16, Chapter VII.5, pp. 211–213]). It is not difficult to check that \( p \otimes \) is indeed a functor. The essential properties of the Kronecker product \( \otimes \) we will need below are that \( \otimes \) is associative and that
\[
\chi(A \otimes B) = \chi A \otimes B = A \otimes \chi B.
\]
The functor $1 \otimes$ is just the identity functor on $\text{Mat}_F$. The interesting functors $p \otimes$ on $\text{Mat}_F$ will have $p \geq 2$.

Let $E_p$ be the $1 \times p^2$ matrix that for $1 \leq i, j \leq p$ has the entries

$$E_p(1, (i-1)p + j) = \delta(i,j),$$

where $\delta$ is the Kronecker delta. For example, $E_2$ is $[1 \ 0 \ 0 \ 1]$ and $E_3$ is $[1 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 1]$. Let $E'_p$ be the transpose of $E_p$. Then $\varphi_m$ is $E_p \otimes 1_m$, and $\gamma'_m$ is its transpose, i.e. $E'_p \otimes 1_m$. We can check that $\varphi$ and $\gamma$ are natural transformations, which satisfy moreover $(\varphi_\gamma)$ and $(\varphi_\gamma' \mathcal{N})$. Namely, we can check that $\langle \text{Mat}_F, \circ, 1, p \otimes, \varphi, \gamma \rangle$ is a $\mathcal{N}$-adjunction.

This self-adjunction is based on the fact that $2 \otimes A = A \oplus A$, where $A \oplus A$ is the sum of matrices

$$
\begin{bmatrix}
A & 0 \\
0 & A
\end{bmatrix}
$$

and behind this sum we have a bifunctor that is both a product and a coproduct. The category $\text{Mat}_F$ is a linear category in the sense of [27]; namely, in it finite products and coproducts are isomorphic—actually, they coincide. Finite products and coproducts coincide in the category of commutative monoids with monoid homomorphisms, of which the category of vector spaces over $F$ is a subcategory. Since we always have that the product bifunctor is right adjoint to the diagonal functor into the product category, and the coproduct bifunctor is left adjoint to this diagonal functor, by composing the product bifunctor, which in $\text{Mat}_F$ coincides with the coproduct bifunctor, with the diagonal functor we obtain in $\text{Mat}_F$ a self-adjoint functor.

The category $\text{Mat}_F$ is a strict monoidal category with the bifunctor $\otimes$, whose unit object is $1$. This category is also symmetric monoidal (see [30, Chapter VII.1,7]).

The self-adjunction of $p \otimes$ in $\text{Mat}_F$ is not a $\mathcal{J}$-adjunction for $p \geq 2$, because $\varphi_m \circ \gamma_m = p1_m$ (here the natural number $p$ is a scalar, and in $p1_m$ the matrix $1_m$ is multiplied by this scalar). However, there is still a possibility to interpret $\mathcal{J}_\omega$, which is derived from $\mathcal{J}_c$, in $\text{Mat}_F$, as we will see in the next section.

14. Representing $\mathcal{J}_\omega$ in $\text{Mat}_F$

For $p \in \mathbb{N}^+$, consider the operation $*$ on the arrows $A : p^n \to p^q$ and $B : p^k \to p^l$ of $\text{Mat}_F$, which is analogous to the operation $*$ of $\mathcal{L}_c$ in Section 12:

$$B * A = \begin{cases} B \circ (1_{p^{k-n}} \otimes A) & \text{if } n \leq k, \\
(1_{p^{n-k}} \otimes B) \circ A & \text{if } k \leq n. \end{cases}$$
Consider next the map $\eta_p$ from the terms of $\mathcal{L}_\omega$ to the arrows of $\text{Mat}_\mathcal{F}$ defined inductively as follows:

$\eta_p(1)$ is $1_1 : p^0 \to p^0$,
$\eta_p([k])$ is $\varphi_{p^{k-1}} : p^{k+1} \to p^{k-1}$,
$\eta_p([k])$ is $\gamma_{p^{k-1}} : p^{k-1} \to p^{k}$,
$\eta_p(tu)$ is $\eta_p(t) \ast \eta_p(u)$.

Next, let $A \equiv \mathcal{F}B$ in $\text{Mat}_\mathcal{F}$ iff there are numbers $k, l, m \in \mathbb{N}$ such that $p^m(1_{p^l} \otimes A) = 1_{p^l} \otimes B$ or $1_{p^l} \otimes A = p^m(1_{p^l} \otimes B)$ in $\text{Mat}_\mathcal{F}$. The relation $\equiv$ is an equivalence relation on the arrows of $\text{Mat}_\mathcal{F}$, congruent with respect to the operations $\ast$ and $1_{p^l} \otimes$. We can then prove the following lemma.

**$\eta_p$ Lemma.** For $p \geq 2$ we have $t = u$ in $\mathcal{F}_\omega$ iff $\eta_p(t) \equiv \mathcal{F} \eta_p(u)$ in $\text{Mat}_\mathcal{F}$.

**Proof.** From left to right we proceed, in principle, by induction on the length of derivation of $t = u$ in $\mathcal{F}_\omega$. However, most cases in this induction are already covered by the Lemma of Section 12, and by our having established in the preceding section that $\text{Mat}_\mathcal{F}$ is a $\mathcal{K}$-adjunction. The only case specific for $\mathcal{F}_\omega$, namely, when $t = u$ is an instance of $[k][k] = 1$, is covered by the fact that $\varphi_{p^{k-1}} \circ \gamma_{p^{k-1}} = p1_{p^{k-1}}$.

To prove the lemma from right to left, suppose that we do not have $t = u$ in $\mathcal{F}_\omega$, but $\eta_p(t) \equiv \mathcal{F} \eta_p(u)$ in $\text{Mat}_\mathcal{F}$. Then, by the left-to-right direction of the lemma, we should have in $\text{Mat}_\mathcal{F}$ an $\eta_p$ image of $\mathcal{F}_\omega$ extended with $t = u$. By the maximality result mentioned at the end of Section 9, we should have that $\eta_p([i][i]) \equiv \mathcal{F} \eta_p(1)$ in $\text{Mat}_\mathcal{F}$. We have, however, that $\eta_p([i][i])$ is $\gamma_{p^{i-1}} \circ \varphi_{p^{i-1}}$, and for no $k, l, m \in \mathbb{N}$ we can have $p^m(1_{p^l} \otimes (\gamma_{p^{i-1}} \circ \varphi_{p^{i-1}})) = 1_{p^l}$ or $1_{p^l} \otimes (\gamma_{p^{i-1}} \circ \varphi_{p^{i-1}}) = p^m1_{p^l}$, provided $p \geq 2$. From this the lemma follows.

An alternative proof of this lemma could be obtained by relying on the results of [10].

**15. Representing $\mathcal{K}_c$ in $\text{Mat}_\mathcal{F}$**

Let $p \in \mathbb{N}^+$, and consider the category $\mathcal{K}_c$ of the free $\mathcal{K}$-adjunction generated by the object 0 (see Section 11). We define inductively a functor $H_p$ from $\mathcal{K}_c$ to $\text{Mat}_\mathcal{F}$ in the following manner:

$H_p(0)$ is $1 = p^0$,
$H_p(m+1)$ is $pH_p(m) = p^{m+1}$,
$H_p(1_m)$ is $1_{p^m} : p^m \to p^m$,
$H_p(\varphi_m)$ is $\varphi_{p^m} : p^{m+2} \to p^m$,.
\[ H_p(\gamma_m) = \gamma_p^m : p^m \rightarrow p^{m+2}, \]
\[ H_p(Ff) = 1_p \otimes H_p(f) : p^{m+1} \rightarrow p^{n+1}, \quad \text{for } f : m \rightarrow n, \]
\[ H_p(g \circ f) = H_p(g) \circ H_p(f). \]

That this defines a functor indeed follows from the fact that \( \langle \text{Mat}_\mathcal{F}, \circ, 1, p^\otimes, \varphi, \gamma \rangle \) is a \( \mathcal{K} \)-adjunction, as established in Section 13.

The function \( H_p : \mathcal{L}_c \rightarrow \text{Mat}_\mathcal{F} \) on arrow-terms of \( \mathcal{L}_c \), which we have above, is not obtained by composing \( \psi : \mathcal{L}_c \rightarrow \mathcal{L}_\omega \) of Section 12 and \( \eta_p : \mathcal{L}_\omega \rightarrow \text{Mat}_\mathcal{F} \) of the preceding section, but we can check by induction on the length of \( f \) that \( H_p(f) \equiv^f \eta_p(\psi(f)) \) in \( \text{Mat}_\mathcal{F} \).

The functor \( H_1 \) is not faithful, since for every arrow \( f \) of \( \mathcal{K}_c \) we have \( H_1(f) = 1_1 \), but for \( p \geq 2 \) the functors \( H_p \) are faithful. As a matter of fact, these functors, which are one-one on objects, are one-one on arrows. This is shown by the following lemma.

**Faithfulness of \( H_p \).** For \( f \) and \( g \) arrow-terms of \( \mathcal{L}_c \) of the same type and \( p \geq 2 \), if \( H_p(f) = H_p(g) \) in \( \text{Mat}_\mathcal{F} \), then \( f = g \) in \( \mathcal{K}_c \).

**Proof.** Suppose \( H_p(f) = H_p(g) \) in \( \text{Mat}_\mathcal{F} \), but not \( f = g \) in \( \mathcal{K}_c \). If \( f = g \) in \( \mathcal{F}_c \), then, as we have seen at the very end of Section 12, for some \( k, l \geq 0 \) such that \( k \neq l \) we have \( f \circ \kappa_0^k = g \circ \kappa_0^l \) in \( \mathcal{K}_c \). But then in \( \text{Mat}_\mathcal{F} \) we have
\[ p^k H_p(f) = p^l H_p(f), \]
which is impossible, since \( H_p(f) \) is never a zero matrix.

So we do not have \( f = g \) in \( \mathcal{F}_c \). Hence, according to what we have established before the end of Section 12, we do not have \( \psi(f) = \psi(g) \) in \( \mathcal{F}_\omega \). Then, by the \( \eta_p \) Lemma of the preceding section, we do not have \( \eta_p(\psi(f)) \equiv^f \eta_p(\psi(g)) \) in \( \text{Mat}_\mathcal{F} \). However, from \( H_p(f) = H_p(g) \) in \( \text{Mat}_\mathcal{F} \) it follows that \( \eta_p(\psi(f)) \equiv^f \eta_p(\psi(g)) \) in \( \text{Mat}_\mathcal{F} \), which yields a contradiction. \( \square \)

So in \( \text{Mat}_\mathcal{F} \) we have an isomorphic representation of \( \mathcal{K}_c \). We also have for every \( n \in \mathbb{N} \) isomorphic representations of the monoids \( \mathcal{K}_n \) as monoids of endomorphisms of \( p^n \), provided \( p \geq 2 \).

Our proof of the faithfulness of these representations of \( \mathcal{K}_n \) relies on the maximality result mentioned at the end of Section 9. An alternative proof is obtained either as in [9], or by relying on the faithfulness result of [18, Section 3] and [8], as mentioned in the Introduction.

16. The algebras End\((p^n)\)

Let \( \text{End}(p^n) \) be the set of all endomorphisms \( A : p^n \rightarrow p^n \) in \( \text{Mat}_\mathcal{F} \), i.e. of all \( p^n \times p^n \) matrices in \( \text{Mat}_\mathcal{F} \). Let us first consider \( \text{End}(p^n) \) when \( p = 2 \). We have remarked at the end of the preceding section that we have in \( \text{End}(2^n) \) an isomorphic representation of the monoid \( \mathcal{K}_n \). Let us denote by \( h_k^n \) the representation of the diapsis \( h_k = [k] \times [k] \) of \( \mathcal{K}_n \) in \( \text{End}(2^n) \). The matrix \( h_k^n \) is \( 1_{2^n \times 2^n} \otimes (\gamma_2^{k-1} \circ \varphi_2^{k-1}) \). To define
\(\gamma_{2k-1}\) and \(\varphi_{2k-1}\) we need the matrix \(E_2\), namely \([1 \ 0 \ 0 \ 1]\), and its transpose \(E'_2\). The matrix \(E'_2 \circ E_2\) is
\[
\begin{bmatrix}
1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1
\end{bmatrix}.
\]
So \(h^n_k = 1_{2k-1} \otimes (E'_2 \circ E_2) \otimes 1_{2k-1}\).

For example, in \(\text{End}(2^2)\) the diapsis \(h_1\) is represented by the matrix \(h_1^1\), which is \(\gamma_1 \circ \varphi_1 = E'_2 \circ E_2\). In \(\text{End}(2^3)\) the diapsides \(h_1\) and \(h_2\) are represented by \(h_1^1 = 1_2 \otimes h_2^1\) and \(h_2^2 = \gamma_2 \circ \varphi_2 = (E'_2 \circ E_2) \otimes 1_2\).

Every \(n \times m\) matrix \(A\) whose entries are only 0 and 1 may be identified with a binary relation \(R_A \subseteq n \times m\) such that \(A(i, j) = 1\) iff \((i, j) \in R_A\). Every binary relation may of course be drawn as a bipartite graph. Here are a few examples of such graphs for matrices we have introduced up to now, with \(p = 2\):

\[
\begin{array}{cccccc}
E_2 & E'_2 & h_1^1 & h_1^2 & h_2^2
\end{array}
\]

In \(\text{End}(2^4)\) we have \(h_1^1 = 1_2 \otimes h_1^2\), \(h_1^3 = 1_2 \otimes h_1^3\) and \(h_2^2 = \gamma_4 \circ \varphi_4 = (E'_2 \circ E_2) \otimes 1_2\), etc.

For \(p > 2\) in \(\text{End}(p^n)\) the unit \(1\) of \(\mathcal{K}_n\) is represented by the \(2^n \times 2^n\) identity matrix \(1_{2^n}\), whose entries are \(1_{2^n}(i, j) = \delta(i, j)\), where \(\delta\) is Kronecker’s delta. As usual, we denote this matrix also by \(I\). The circle \(c\) is represented by \(1_{2^n} \otimes (E'_2 \circ E_2) = 1_{2^n} \otimes [2] = 2I_{2^n} = 2I\).

We proceed analogously when \(p > 2\) in \(\text{End}(p^n)\). Then \(h^n_k = 1_{p^{n-k-1}} \otimes (\gamma_{p^{k-1}} \circ \varphi_{p^{k-1}}) = 1_{p^{n-k-1}} \otimes (E'_p \circ E_p) \otimes 1_{p^{k-1}}\), the unit matrix \(I\) is the identity matrix \(1_{p^n}\), and the circle is represented by \(pI\).

If \(1^n, h_1^n, \ldots, h_{n-1}^n, c\) denote the 0-1 matrices we have assigned to these expressions, then these matrices satisfy the equations of \(\mathcal{K}_n\), with multiplication being matrix multiplication. If \(1^n, h_1^n, \ldots, h_{n-1}^n\) denote the corresponding binary relations, then for multiplication being composition of binary relations the equations of \(\mathcal{J}_n\) are satisfied.

Composition of binary relations is easy to read from bipartite graphs. Here is an example:
By so composing binary relations we can assign to every element of $J_n$ a binary relation, and then from this binary relation we can recover the 0-1 matrix assigned to our element of $K_n$.

However, $\text{End}(p^n)$ is a richer structure than $K_n$. It is an associative $F$ algebra under matrix addition $+$, the multiplication of a matrix $A$ by a scalar $\alpha$ (which is written $\alpha A$) and matrix multiplication, which we continue to write as composition $\circ$. We will consider in the next section representations of braid groups in the algebras $\text{End}(p^n)$.

The representation of $K_n$ in $\text{End}(p^n)$ we dealt with above is obtained by restricting to $K_n$ the orthogonal group case of Brauer’s representation of Brauer algebras from [4] (see also [33, Section 3, 18, Section 3]).

17. Representing braid groups in $\text{End}(p^n)$

The braid group $B_n$ has for every $k \in \{1, \ldots, n - 1\}$ a generator $\sigma_k$. The number $n$ here could in principle be any natural number, but, as for $L_n$, $K_n$ and $J_n$, the interesting groups $B_n$ have $n \geq 2$. When $n$ is 0 or 1, we have no generators $\sigma_k$. The terms of $B_n$ are obtained from these generators and $I$ by closing under inverse $^{-1}$ and multiplication. The following equations are assumed for $B_n$:

1. $1t = t\mathbf{1} = t$,
2. $t(uv) = (tu)v$,
3. $tt^{-1} = t^{-1}t = \mathbf{1}$,

$(\sigma0)$ $\sigma_i \sigma_j = \sigma_j \sigma_i$, for $|i - j| \geq 2$,

$(\sigma3)$ $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$.

We can replace (3) by

$(3.1)$ $I^{-1} = \mathbf{1}$,

$(3.2)$ $(tu)^{-1} = u^{-1}t^{-1}$,

$(\sigma2)$ $\sigma_i \sigma^{-1}_i = \sigma^{-1}_i \sigma_i = \mathbf{1}$.

(In naming Eqs. $(\sigma3)$ and $(\sigma2)$ we paid attention to the fact that $(\sigma3)$ corresponds to the third Reidemeister move, and $(\sigma2)$ to the second.)

Inspired by the bracket equations (see [24, pp. 11, 15] and references therein), we define inductively as follows a map $\rho$ from the terms of $B_n$ to $\text{End}(p^n)$:

$$\rho(\mathbf{1}) \text{ and } \rho(\mathbf{1}^{-1}) \text{ are } I,$$

$$\rho(\sigma_i) \text{ is } \alpha_i h^n_i + \beta_i I,$$

$$\rho(\sigma_i^{-1}) \text{ is } \alpha'_i h^n_i + \beta'_i I,$$

$$\rho(tu) \text{ is } \rho(t) \circ \rho(u),$$

$$\rho((tu)^{-1}) \text{ is } \rho(u^{-1}) \circ \rho(t^{-1}).$$
We will now find conditions for $\alpha_i$, $\alpha'_i$, $\beta_i$ and $\beta'_i$ sufficient to make $\rho$ a group homomorphism from $B_n$ to $\text{End}(p^n)$.

The equations (1), (2), (3.1) and (3.2) are always satisfied. The equation (σ0) will also be satisfied always, because $h_i^n \circ h_j^n = h_j^n \circ h_i^n$ when $|i - j| \geq 2$. This is the equation (h1) of $K_n$.

Consider now the equation (σ3). For $\rho(\sigma, \sigma_{i+1}) = \rho(\sigma_{i+1}, \sigma)$ to hold in $\text{End}(p^n)$, we compute that it is sufficient if we have $\beta_i = \beta_{i+1}$, $\alpha_i = \alpha_{i+1}$, $\beta'_i = \beta'_{i+1}$ and $\alpha'_i = \alpha'_{i+1}$, so that

\[ \rho(\sigma_i) \quad \text{is} \quad \alpha h_i^n + \beta I, \]
\[ \rho(\sigma^{-1}_i) \quad \text{is} \quad \alpha' h_i^n + \beta' I, \]

together with

\[ p = -x^{-1} \beta - x^{-1} \beta. \]

Since for (σ3) we required $p = -x^{-1} \beta - x^{-1} \beta$, we obtain $\alpha' = \alpha^{-1}$. So with the clauses

\[ \rho(\sigma_i) = \alpha h_i^n + \beta I, \]
\[ \rho(\sigma^{-1}_i) = \alpha^{-1} h_i^n + \beta^{-1} I, \]
\[ p = -x^{-1} \beta - x^{-1} \beta, \quad \text{i.e.} \quad x^{-1} = (-p \pm \sqrt{p^2 - 4})/2, \]

which amount to the clauses:

\[ \rho(\sigma_i) = \alpha(h_i^n + (2/(-p \pm \sqrt{p^2 - 4}))I), \]
\[ \rho(\sigma^{-1}_i) = \alpha^{-1}(h_i^n + ((-p \pm \sqrt{p^2 - 4})/2)I), \]

we obtain that $\rho$ is a homomorphism from $B_n$ to $\text{End}(p^n)$, i.e. a representation of $B_n$ in $\text{End}(p^n)$.

The conditions we have found sufficient to make $\rho$ a representation of $B_n$ in $\text{End}(p^n)$ are also necessary, since the 0-1 matrices in our representation of $K_n$ in $\text{End}(p^n)$ are linearly independent (see [18, Section 3] and also [8] for an elementary self-contained proof). Actually, for the necessity of our conditions we have to prove linear independence just for the 0-1 matrices in the representation of $K_3$. Then we have only five of these matrices, whose linear independence one can easily check in case $p$ is equal to 2 or 3 by listing them all, and by finding for each an entry with 1 where all the others have 0. In [18,8] linear independence is established for every $n \geq 2$ in $K_n$ and every $p \geq 2$. 
The clauses of the bracket equations
\[
\rho(\sigma_i) = xh_i^n + x^{-1}I,
\]
\[
\rho(\sigma_i^{-1}) = x^{-1}h_i^n + xI,
\]
\[
p = -x^2 - x^{-2},
\]
are obtained from ours by requiring that \(\beta = x^{-1}\), and by not requiring as we do that \(p\) be a natural number. So our representation is in a certain sense more general, but it requires that \(p\) be a natural number.

If \(p\) is \(2\) in our representation, then we obtain that \(\beta = -x\). In this case, however, \(\rho(\sigma_i\sigma_i) = \rho(\sigma_i+1\sigma_i+1) = x^2I\), and the representation is not faithful. Is this representation faithful for \(p > 2\)? (The question whether the representation of braid groups in Temperley–Lieb algebras based on the bracket equations is faithful is raised in [19]).

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References