A Brauerian representation of split preorders

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Split preorders are preordering relations on a domain whose composition is defined in a particular way by splitting the domain into two disjoint subsets. These relations and the associated composition arise in categorial proof theory in connection with coherence theorems. Here split preorders are represented isomorphically in the category whose arrows are binary relations and whose composition is defined in the usual way. This representation is related to a classical result of representation theory due to Richard Brauer.

1 Introduction

A split preorder is a preordering, i.e. a reflexive and transitive relation $R \subseteq W^2$ such that $W$ is equal to the disjoint union of two sets $X$ and $Y$. Every preorder may be conceived as a split preorder, but split preorders are not composed in the ordinary manner. The set $X$ is conceived as a domain, $Y$ as a codomain, and composition of split preorders is defined in a manner that takes this into account. The formal definition of this composition is not straightforward, and it will be given in the next section. It is, however, supported by geometric intuitions (see the end of the next section).

Split preorders and their compositions are of interest in logic because they arise as relations associated to graphs that have been used in categorial proof theory for coherence results, which settled questions of identity criteria for proofs (see [2] for a survey of this topic). We will consider this matter in Section 5.

A particular kind of split preorder is made of split equivalences, namely those split preorders that are equivalence relations, which in categorial proof theory represent generality of proofs. The paper [6] is devoted to this topic.

It was shown in [6] that the category whose arrows are split equivalences on finite ordinals can be represented isomorphically in the category whose arrows are binary relations between finite ordinals, where composition is defined in the usual simple way. This representation is related to Brauer’s representation of Brauer algebras [1], which it generalizes in a certain sense (see [6, Section 6]).

In this paper we will generalize the results of [6]. We will represent the category whose arrows are split preorders, and not only split equivalences, in the category Rel whose arrows are binary relations between certain sets. If our split preorders are relations on finite sets $W$, the category Rel may be taken to have as arrows binary relations between finite ordinals as in [6]. Our representation in Rel generalizes [6], which, in turn, generalizes Brauer’s representation, whence its name.

In the next section we will introduce the category SplPre whose arrows are split preorders. Sections 3 and 4 are devoted to representing isomorphically SplPre in Rel, and also provide a proof that SplPre is indeed a category. Section 3 deals with some general matters concerning the representation of arbitrary preordering relations, which we apply in Section 4. Section 5 explores connections with categorial proof theory. We give some examples of deductive systems covering fragments of propositional logic, and of the split preorders associated with proofs.

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2 The category SplPre

Let \( \mathcal{M} \) be a family of sets, whose members are \( X, Y, Z, \ldots \). For \( i \in \{s, t\} \) (where \( s \) stands for “source” and \( t \) for “target”), let \( \mathcal{M}' \) be a family of sets in one-to-one correspondence with \( \mathcal{M} \). We denote by \( \mathcal{M}^i \) the element of \( \mathcal{M}' \) corresponding to \( X \in \mathcal{M} \). We assume further that for every \( X \in \mathcal{M} \) there is a bijection \( i_X : X \rightarrow X' \). Finally, we assume that for every \( U \in \mathcal{M}, V \in \mathcal{M}' \) and \( W \in \mathcal{M}^i \), the sets \( U, V \) and \( W \) are mutually disjoint.

For \( X, Y \in \mathcal{M} \), let a split relation of \( \mathcal{M} \) be a triple \(<R, X, Y>\) such that \( R \subseteq (X^s \cup Y^t)^2 \). The set \( X^s \cup Y^t \) may be conceived as the disjoint union of \( X \) and \( Y \). A split relation \(<R, X, Y>\) is a split preorder iff \( R \) is a preorder, i.e. a reflexive and transitive relation. As usual, we write sometimes \( xRy \) for \((x, y) \in R\).

We construct now a category called SplPre. Its set of objects is \( \mathcal{M} \). Its arrows from \( X \in \mathcal{M} \) to \( Y \in \mathcal{M} \) are the split preorders \(<R, X, Y>\), which will also be denoted by \( R : X \rightarrow Y \).

The identity arrow \( 1_X : X \rightarrow X \) of SplPre is the split preorder that, for all \( i, j \in \{s, t\}, u \in X^i \) and \( v \in X^j \), \((u, v) \in 1_X \) iff \( i_X^{-1}(u) = j_X^{-1}(v) \). To define composition of arrows in SplPre we need some auxiliary notions. For every \( X, Y \in \mathcal{M} \), let the functions \( \varphi^s \) and \( \varphi^t \) from \( X \cup Y^t \) to \( X^s \cup Y^t \) resp. from \( X^s \cup Y \) to \( X^s \cup Y^t \) be defined by

\[
\varphi^s(x) = \begin{cases} s_X(x) & \text{if } x \in X, \\ x & \text{if } x \in Y^t, \end{cases} \quad \text{and} \quad \varphi^t(x) = \begin{cases} x & \text{if } x \in X^s, \\ t_Y(x) & \text{if } x \in Y. \end{cases}
\]

For a split relation \( R : X \rightarrow Y \), let the relations \( R^s \subseteq (X \cup Y^t)^2 \) and \( R^t \subseteq (X^s \cup Y)^2 \) for \( i \in \{s, t\} \) be defined by \((x, y) \in R^i \) iff \((\varphi^i(x), \varphi^i(y)) \in R \). Finally, for an arbitrary binary relation \( R \), let \( \text{Tr}(R) \) be the transitive closure of \( R \). Then for split preorders \( R : X \rightarrow Y \) and \( P : Y \rightarrow Z \) of \( \mathcal{M} \) we define their composition \( P \circ R : X \rightarrow Z \) by

\[
P \circ R = \text{def} \text{Tr}(R^{-t} \cup P^{-s}) \cap (X^s \cup Z^t)^2.
\]

It is clear that \( P \circ R : X \rightarrow Z \) is a split preorder of \( \mathcal{M} \).

For SplPre to be a category we need that for \( R : X \rightarrow Y \) the equations \( R \circ 1_X = 1_Y \circ R = R \) hold, and that \( \circ \) is associative, which is rather complicated to check directly. We will not try to do that here. So, for the time being, we don’t know yet whether SplPre is a category. We know only that it is a graph (a set of objects and a set of arrows) with a family of arrows \( 1_X \) for every object \( X \), and with a binary partial operation on arrows \( \circ \). This kind of structure is called a deductive system, according to [8]. We prove that SplPre is a category in the sections that follow.

The strictification of a preorder \( R \subseteq W^2 \) is the relation \( R' \subseteq W^2 \) such that \( xR'y \) iff \( xRy \) and \( x \neq y \). The strictification of a preorder is an irrefflexive relation that satisfies strict transitivity: If \( xR'y \) and \( yR'z \) and \( x \neq z \), then \( xR'z \). If \( \text{Ref}(R) \) is the reflexive closure of \( R \), it is clear that for a preorder \( R \) we have \( \text{Ref}(R') = R \). Conversely, for every irrefflexive strictly transitive relation \( P \subseteq W^2 \) we have that \( \text{Ref}(P) \subseteq W^2 \) is a preorder, whose strictification \( \text{Ref}(P)' \) is equal to \( P \).

So we may represent preorders by their strictifications. This we do when we draw diagrams representing split preorders. The composition \( \circ \) of two split preorders is illustrated in the following diagrams:

![Diagram of composition of split preorders](image-url)
Every binary relation between $X^s$ and $Y^t$ may be viewed as the strictification of a split preorder. The composition of such split preorders, which correspond to binary relations between members of $\mathcal{M}$, is then a simple matter: it corresponds exactly to composition of binary relations. If in $\text{SplPre}$ we keep as arrows only these split preorders, we obtain a category isomorphic to the category whose arrows are binary relations between members of $\mathcal{M}$. Problems with composition of split preorders arise if they don’t correspond to binary relations between members of $\mathcal{M}$.

### 3 Representing preorders by sets of functions

This section contains some general results concerning the representation of arbitrary preordering relations. They will be used in the next section to establish that our representation of $\text{SplPre}$ in the category whose arrows are binary relations is an isomorphism.

Let $W$ be an arbitrary set, and let $R \subseteq W^2$. Let $p$ be a set such that the ordinal $2 = \{0, 1\}$ is a subset of $p$, and let $\leq$ be a binary relation on $p$ which restricted to 2 is the usual ordering of 2. For $x, y \in W$, let the function $f_x : W \to p$ be defined as follows:

$$f_x(y) = \begin{cases} 1 & \text{if } xRy, \\ 0 & \text{if not } xRy. \end{cases}$$

The function $f_x$ is the characteristic function of the $R$-cone over $x$. Consider also the following set of functions:

$$\mathcal{F}(R) = \text{def} \{ f : W \to p : (\forall x, y \in W) (xRy \Rightarrow f(x) \leq f(y)) \}.$$

We can then establish the following propositions.

**Proposition 1** The relation $R$ is reflexive iff for all $x \in W$, $f_x(x) = 1$.

**Proposition 2** The relation $R$ is transitive iff for all $x \in W$, $f_x \in \mathcal{F}(R)$.

**Proof.**

$(\Rightarrow)$ Suppose $yRz$. If $f_x(y) = 0$, then $f_x(y) \leq f_x(z)$, and if $f_x(y) = 1$, then $f_x(z) = 1$ by the transitivity of $R$.

$(\Leftarrow)$ If $yRz$ implies $f_x(y) \leq f_x(z)$, then $yRz$ implies $f_x(y) = 0$ or $f_x(z) = 1$, which means that $yRz$ and $xRy$ implies $xRz$. \qed

**Proposition 3** If $R$ is a preorder, then

$(\forall x, y \in W) (xRy \Leftrightarrow (\forall f \in \mathcal{F}(R)) f(x) \leq f(y)).$

**Proof.** Note first that in $(\forall f \in \mathcal{F}(R)) f(x) \leq f(y)$, then $1 \leq f_x(y)$ by the left-to-right directions of Propositions 1 and 2, and so $xRy$. \qed

**Proposition 4** If $\leq$ is a preorder, then $R$ is a preorder iff $(\forall f \in \mathcal{F}(R)) f(x) \leq f(y)$.

**Proof.** Suppose $\leq$ is a preorder. Then we obtain the reflexivity of $R$ by taking $x = y$ in $(\forall f \in \mathcal{F}(R)) f(x) \leq f(y)$ and by using the reflexivity of $\leq$. For the transitivity of $R$, suppose $xRy$ and $yRz$. Then for every $f \in \mathcal{F}(R)$ we have $f(x) \leq f(y) \leq f(z)$, and by the transitivity of $\leq$ and by $(\forall f \in \mathcal{F}(R)) f(x) \leq f(y)$ we obtain $xRz$. \qed

As an immediate consequence of Proposition 3 we have the following

**Proposition 5** If $R, P \subseteq W^2$ are preorders, then $R = P$ iff $\mathcal{F}(R) = \mathcal{F}(P)$.
4 Representing SplPre in Rel

Let $\text{Rel}$ be the category whose objects are sets in a certain universe, and whose arrows are binary relations between these sets. Let $I_a \subseteq a \times a$ be the identity relation on the set $a$, and the composition $R_2 \circ R_1 \subseteq a \times c$ of $R_1 \subseteq a \times b$ and $R_2 \subseteq b \times c$ is $\{(x, y) : (\exists z \in c) \{xR_1 z \text{ and } zR_2 y\}\}$.

Then let $p$ be a set in which the ordinal 2 is included, as in the preceding section, and let the relation $\leq$ on $p$ be a linear order such that every subset of $p$ has a greatest element and 0 is the least element of $p$. A finite ordinal $p \geq 2$ with the usual ordering satisfies these conditions, but in general $p$ need not be finite. Then we define a map $F_p$ from the objects of SplPre to the objects of $\text{Rel}$ by setting $F_p(X) = p^X$, namely, the set of all functions from $X$ to $p$. So $p^X$ has to be an object of $\text{Rel}$.

Given functions $f_1 : X \to p$ and $f_2 : Y \to p$, let $[f_1, f_2] : X^s \cup Y^t \to p$ be defined by

$$[f_1, f_2](u) = \begin{cases} f_1(s_{X}^{-1}(u)) & \text{if } u \in X^s, \\ f_2(t_{Y}^{-1}(u)) & \text{if } u \in Y^t. \end{cases}$$

For $R : X \to Y$ an arrow of SplPre, and for $f_1 : X \to p$ and $f_2 : Y \to p$ we define the arrow $F_p(R)$ of $\text{Rel}$ by

$$(f_1, f_2) \in F_p(R) \iff [f_1, f_2] \in \mathcal{F}(R),$$

where $\mathcal{F}(R)$ is the set of functions defined as in the preceding section. Here $W$ is $X^s \cup Y^t$. Then we can prove the following propositions.

**Proposition 6** $F_p(1_X) = I_{F_p(X)}$.

**Proof.** For $f_1, f_2 : X \to p$ and $i, j \in \{s, t\}$ we have $[f_1, f_2] \in \mathcal{F}(1_X)$ iff

$$(\forall u \in X^i)(\forall v \in X^j) (i_{X}^{-1}(u) = j_{X}^{-1}(v) \Rightarrow [f_1, f_2](u) \leq [f_1, f_2](v)).$$

So if $[f_1, f_2] \in \mathcal{F}(1_X)$, then for $u = s(X)(x)$ and $v = t(X)(x)$ we have $[f_1, f_2](u) \leq [f_1, f_2](v)$, which means $f_1(x) \leq f_2(x)$. By setting $u = t(X)(x)$ and $v = s(X)(x)$ we obtain $f_2(x) \leq f_1(x)$, and by the antisymmetry of $\leq$ we obtain $f_1(x) = f_2(x)$. So from $[f_1, f_2] \in \mathcal{F}(1_X)$ we have inferred $f_1 = f_2$.

For the converse, suppose $f_1 = f_2$, and for $u \in X^i$ and $v \in X^j$ let $i_{X}^{-1}(u) = j_{X}^{-1}(v)$. If $i = j$, then $[f_1, f_2](u) \leq [f_1, f_2](v)$ by the reflexivity of $\leq$. If $i \neq j$, then $f_1(i_{X}^{-1}(u)) = f_2(j_{X}^{-1}(v))$, and hence $[f_1, f_2](u) \leq [f_1, f_2](v)$ by the reflexivity of $\leq$.

Note that antisymmetry of the linear ordering $\leq$ on $p$ was used in the proof of Proposition 6.

**Proposition 7** $F_p(P \ast R) = F_p(P) \circ F_p(R)$.

**Proof.** Suppose $R : X \to Y$ and $P : Y \to Z$. We have to show that for $f_1 : X \to p$ and $f_2 : Z \to p$ such that

$$(\ast\ast) \quad (\forall u, v \in X^s \cup Y^t) (u(P \ast R)v \Rightarrow [f_1, f_2](u) \leq [f_1, f_2](v))$$

there is an $f_3 : Y \to p$ such that the following two statements are satisfied:

$$(\ast R) \quad (\forall u, v \in X^s \cup Y^t) (uRv \Rightarrow [f_1, f_3](u) \leq [f_1, f_3](v)),$$

$$(\ast P) \quad (\forall u, v \in X^s \cup Y^t) (uPv \Rightarrow [f_3, f_2](u) \leq [f_3, f_2](v)).$$

For $y \in Y$ let

$$X_y = \text{def} \{x \in X : (s(X)(x), y) \in \text{Tr}(R^{-t} \cup P^{-s})\},$$

$$Z_y = \text{def} \{z \in Z : (t(Z)(z), y) \in \text{Tr}(R^{-t} \cup P^{-s})\}.$$ 

Then we define $f_3$ as follows:

$$f_3(y) = \begin{cases} \max \{[f_1(x) : x \in X_y] \cup [f_2(z) : z \in Z_y]\} & \text{if } X_y \cup Z_y \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

This definition is correct because we assumed that every nonempty subset of $p$ has a greatest element.
We verify that $f_3$ satisfies $(\ast R)$. Suppose $uRv$.

(1) If $u, v \in X^*$, then $f_1(s_X^{-1}(u)) \leq f_1(s_X^{-1}(v))$ by $(\ast \ast)$, and hence $[f_1, f_3](u) \leq [f_1, f_3](v)$.

(2) Suppose $u, v \in Y^*$. Since $uRv$, we must have $X_{\tau_Y^{-1}}(u) \subseteq X_{\tau_Y^{-1}}(v)$ and $Z_{\tau_Y^{-1}}(u) \subseteq Z_{\tau_Y^{-1}}(v)$, and hence $f_3(t_{\tau_Y^{-1}}(u)) \leq f_3(t_{\tau_Y^{-1}}(v))$. So $[f_1, f_3](u) \leq [f_1, f_3](v)$.

(3) Suppose $u \in X^*$ and $v \in Y^*$. Since $uRv$, we must have $s_X^{-1}(u) \in X_{\tau_Y^{-1}}(v)$, and hence

$$f_1(s_X^{-1}(u)) \leq f_3(t_{\tau_Y^{-1}}(v)).$$

So $[f_1, f_3](u) \leq [f_1, f_3](v)$.

(4) Suppose $u \in Y^*$ and $v \in X^*$.

(4.1) If $X_{\tau_Y^{-1}}(u) \cup Z_{\tau_Y^{-1}}(u) \neq \emptyset$, then either $f_3(t_{\tau_Y^{-1}}(u)) = f_1(x)$ for some $x \in X_{\tau_Y^{-1}}(u)$, or $f_3(t_{\tau_Y^{-1}}(u)) = f_2(z)$ for some $z \in Z_{\tau_Y^{-1}}(u)$, by the definition of $f_3$. So suppose, first, that $f_3(t_{\tau_Y^{-1}}(u)) = f_1(x)$ for some $x \in X_{\tau_Y^{-1}}(u)$. Since $uRv$, we must have $(s_X(x), v) \in \text{Tr}(R^{-1} \cup P^{-s})$. Hence $(s_X(x), v) \in P \ast R$, and we obtain $f_3(t_{\tau_Y^{-1}}(u)) = f_1(x) \leq f_1(s_X^{-1}(v))$ by $(\ast \ast)$. So $[f_1, f_3](u) \leq [f_1, f_3](v)$. We proceed analogously when $f_3(t_{\tau_Y^{-1}}(u)) = f_2(z)$ for some $z \in Z_{\tau_Y^{-1}}(u)$.

(4.2) If $X_{\tau_Y^{-1}}(u) \cup Z_{\tau_Y^{-1}}(u) = \emptyset$, then $f_3(t_{\tau_Y^{-1}}(u)) = 0$, and, since 0 is the least element of $p$, we have $[f_1, f_3](u) \leq [f_1, f_3](v)$.

We verify analogous that $f_3$ satisfies $(\ast P)$.

It remains to show that if for some $f_3 : Y \rightarrow P$ we have $(\ast R)$ and $(\ast P)$, then we have $(\ast \ast)$. Suppose $u(P \ast R)v$ and $u, v \in X^*$. Then for $l \geq 0$ there is a (possibly empty) sequence $y_1, \ldots, y_{2l}$ of elements of $Y$ such that

$$(u, t_Y(y_1)) \in R, (s_Y(y_1), s_Y(y_2)) \in P, (t_Y(y_2), t_Y(y_3)) \in R, \ldots, (t_Y(y_{2l}), v) \in R$$

(if $l = 0$, then $uRv$). Then by applying $(\ast R)$, $(\ast P)$ and the transitivity of $\leq$ we obtain $f_1(s_X^{-1}(u)) \leq f_1(s_X^{-1}(v))$, and hence $[f_1, f_2](u) \leq [f_1, f_2](v)$. We proceed analogously when $u, v \in Z^*$, or when $u \in X^*$ and $v \in Z^*$, or when $u \in Z^*$ and $v \in X^*$.

As an immediate consequence of Proposition 5 we have the following

**Proposition 8** If $F_p(R) = F_p(P)$, then $R = P$.

Since $F_p$ defined on the objects of $\text{SplPre}$ is clearly a one-one map, Proposition 8 means that $F_p$ defined on the arrows of $\text{SplPre}$ is a one-one map. Thus $F_p$ is an embedding of $\text{SplPre}$ into $\text{Rel}$, i.e., we have in $\text{Rel}$ as a subcategory an isomorphic copy of $\text{SplPre}$. Hence $\text{SplPre}$ is a category, as was promised.

If the family $\mathcal{M}$ satisfies the condition that for every $X, Y \in \mathcal{M}$ there is a $Z \in \mathcal{M}$ such that $Z$ is isomorphic to the disjoint union of $X$ and $Y$, the category $\text{SplPre}$ has the structure of a symmetric monoidal closed category (see [9, VII.7]).

If we take the subcategory of $\text{SplPre}$ whose arrows correspond to binary relations between members of $\mathcal{M}$, as explained at the end of Section 2, and if $\mathcal{M}$ is the set of objects of $\text{Rel}$, then this subcategory of $\text{SplPre}$ is isomorphic to $\text{Rel}$. When our representation of $\text{SplPre}$ in $\text{Rel}$ via the functor $F_p$ is restricted to this subcategory, it amounts to a nontrivial embedding of $\text{Rel}$ in $\text{Rel}$.

5 Split preorders associated to proofs in fragments of logic

The language $\mathcal{L}$ of conjunctive logic is built from a nonempty set of propositional variables $P$ with the binary connective $\land$ and the propositional constant, i.e. nullary connective, $\top$ (the exact cardinality of $P$ is not important here). We use the schematic letters $A, B, C, \ldots$ for formulæ of $\mathcal{L}$.

We have the following axiomatic derivations for every $A$ and $B$ in $\mathcal{L}$:

$$1_A : A \rightarrow A, \quad \text{k}_{A,B}^1 : A \land B \rightarrow A, \quad \text{k}_{A,B}^2 : A \land B \rightarrow B, \quad \text{k}_A : A \rightarrow \top,$$
and the following inference rules for generating derivations:

\[
\frac{f : A \to B \quad g : B \to C}{g \circ f : A \to C}, \quad \frac{\text{frac } f : C \to A \quad g : C \to B(f, g) : C \to A \land B}{}
\]

This defines the deductive system \( \mathcal{D} \) of conjunctive logic (both intuitionistic and classical). In this system \( \top \) is included as an “empty conjunction”.

Let \( \text{SplPre} \) now be the category of split preorders of \( \omega = \{0, 1, 2, \ldots \} \). We define a map \( G \) from \( \mathcal{L} \) to the objects of \( \text{SplPre} \) by letting \( G(A) \) be the number of occurrences of propositional variables in \( A \). Next we define inductively a map, also denoted by \( G \), from the derivations of \( \mathcal{D} \) to the arrows of \( \text{SplPre} \):

\[
G(1_A) = 1_{G(A)},
\]

\[
(u, v) \in G(\hat{k}_{A,B}^1) \iff \ u = v \text{ or } (u \in G(A \land B)^f \text{ and } v \in G(A)^f \text{ and } s^{-1}_{G(A \land B)}(u) = t^{-1}_{G(A)}(v)),
\]

\[
(u, v) \in G(\hat{k}_{A,B}^2) \iff \ u = v \text{ or } (u \in G(A \land B)^f \text{ and } v \in G(B)^f \text{ and } s^{-1}_{G(A \land B)}(u) = t^{-1}_{G(B)}(v) + G(A)),
\]

\[
(u, v) \in G(\hat{k}_A) \iff \ u = v,
\]

\[
G(g \circ f) = G(g) \ast G(f),
\]

\[
(u, v) \in G((f, g)) \iff \ u = v \text{ or } (u \in G(C)^f \text{ and } v \in G(A \land B)^f \text{ and } t_{G(A \land B)}(u) \in G(f)) \quad \text{or} \quad (u \in G(C)^f \text{ and } v \in G(A \land B)^f \text{ and } t_{G(A \land B)}(v) - G(A)) \in G(g)).
\]

It is easy to check that for every arrow \( f \) of \( \mathcal{D} \), the relation \( G(f) \) is a preorder. When we draw this preorder, we may draw just the corresponding strictification, as we remarked in Section 2. For example, for \( p, q, r \in \mathcal{P} \), the graph of \( G((1_{(p \land q) \land \top}, 1_{(p \land q) \land \top}) \circ \hat{k}_{1_{(p \land q) \land \top},r}) \) is

\[
\begin{array}{c}
((p \land q) \land \top) \land r \\
0 \quad 1 \quad 2 \\
0 \quad 1 \quad 2 \quad 3 \\
((p \land q) \land \top) \land ((p \land q) \land \top)
\end{array}
\]

where the formulae of \( \mathcal{L} \) are written down to show where the ordinal 3 of the source and the ordinal 4 of the target come from. Actually, we may as well omit the ordinals from such drawings.

We may also replace uniformly the relations defined above as values of \( G \) by the converse relations. This shows that the information about the orientation of the edges is not essential. This information may be omitted from the drawings, provided that, in composing graphs, care is taken not to compose edges of a single graph with each other.

If we stipulate that for \( f, g : A \to B \) derivations of \( \mathcal{D} \) we have that \( f \) and \( g \) are equivalent iff \( G(f) = G(g) \), and define a proof of \( \mathcal{D} \) to be the equivalence class of a derivation, then proofs of \( \mathcal{D} \) would constitute arrows of a category \( \mathcal{C} \), whose objects are formulae of \( \mathcal{D} \), with the obvious sources and targets. In this particular case, where \( \mathcal{D} \) is our deductive system for conjunctive logic, the category \( \mathcal{C} \) will be the free cartesian category generated by the set of propositional variables \( \mathcal{P} \) as the generating set of objects (this set may be conceived as a discrete category). This fact about \( \mathcal{C} \) follows from the coherence result for cartesian categories treated in [3] and [10]. (Cartesian categories are categories with all finite products, including the empty product, i.e. terminal object. The category \( \mathcal{C} \) can be equationally presented; see [8, Chapter I.3], or [3].)

When we replace the split preorders \( G(f) \) defined above by their transitive and symmetric closures, we obtain the equivalence relations of [6], which are also split preorders. The category \( \mathcal{C} \) induced by these split preorders is, however, again the free cartesian category generated by \( \mathcal{P} \).
The language of disjunctive logic is dual to the language we had above: instead of $\land$ and $\top$ we have $\lor$ and $\bot$ in $\mathcal{L}$, and instead of $k^i$, $\hat{k}$ and $(\cdot, \cdot)$ we have in the corresponding deductive system $\mathcal{D}$

$$\hat{k}_{A,B}^1 : A \to A \lor B, \quad \hat{k}_{A,B}^2 : B \to A \lor B, \quad \hat{k}_A : \bot \to A$$

and

$$f : A \to C \quad g : B \to C \quad [f, g] : A \lor B \to C.$$ Equivalence of derivations induced by maps $G$ dual to those above makes of the proofs of $\mathcal{D}$ the free category with all finite coproducts generated by $\mathcal{P}$.

Let us now assume we have in $\mathcal{D}$ both $\land$ and $\lor$, but without $\top$ and $\bot$, and let us introduce the category $\mathcal{C}$ of proofs of $\mathcal{D}$ as we did above. To define the new map $G$, we combine the two kinds of map $G$ defined previously, paying attention to order (for example, both in $G(\hat{k}_{A,B}^i)$ and $G(\hat{k}_{A,B}^i)$ edges connecting the domain and the codomain must be oriented in the same direction). The category $\mathcal{C}$ will be the free category with nonempty finite products and coproducts generated by $\mathcal{P}$. This follows from the coherence result of [5]. In the presence of $\top$ and $\bot$, we would obtain a particular brand of bicartesian category, according to the coherence result of [4]. If we replace the split preorders $G(f)$ considered here for conjunctive-disjunctive logic by their transitive and symmetric closures, as we did above for conjunctive logic, the resulting category $\mathcal{C}$ will not be any more the free category with nonempty finite products and coproducts (see [6]).

Up to now, in this section, the split preorders that were values of $G$ always corresponded to relations between finite ordinals (see the end of Sections 2 and 4). We will next consider split preorders that are not such.

The coherence result of [7] for symmetric monoidal closed categories without the unit object $I$ can be used to show that split preorders for a deductive system corresponding to the tensor-implication fragment of intuitionistic linear logic give rise to the free symmetric monoidal closed category without $I$. These split preorders may be taken as equivalence relations, but this is not necessary.

Suppose we add intuitionistic implication $\to$ to conjunctive or conjunctive-disjunctive logic, and take the equivalence of derivations induced by split preorders that for the derivations from $p \land (p \to q)$ to $q$ and from $q$ to $p \to (p \land q)$ correspond to the following graphs

$$\begin{align*}
&\begin{array}{c}
p \land (p \to q) \\
q
\end{array} \quad \begin{array}{c}
q \\
p \to (p \land q)
\end{array}
\end{align*}$$

In this case, however, the category $\mathcal{C}$ would not be the free cartesian closed or free bicartesian closed category generated by $\mathcal{P}$ (for counterexamples see [5, Section 1] and [111]). Taking the transitive and symmetric closures of our split preorders would again not give this free cartesian closed or free bicartesian closed category.

In [6] we considered the category $Gen$ of split preorders on $\omega$ that are equivalence relations, and represented $Gen$ in a subcategory of $Rel$ via a functor that amounts to a special case of the functor $F_p$ considered in this paper. In [6], the set $p$ is a finite ordinal, and $F_p(n) = p^n$ is taken to be a finite ordinal too. This representation of $Gen$ is connected to Brauer’s representation of Brauer algebras of [1], as explained in [6, Section 6]). This is the reason why we call our representation of split preorders Brauerian. It is a generalization of Brauer’s representation, and of the Brauerian representation of $Gen$ of [6].

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