A NEW PROOF OF THE FAITHFULNESS OF BRAUER’S REPRESENTATION OF TEMPERLEY–LIEB ALGEBRAS

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The faithfulness of the orthogonal group case of Brauer’s representation of the Brauer centralizer algebras restricted to their Temperley–Lieb subalgebras, which was established by Vaughan Jones, is here proved in a new, elementary and self-contained, manner.

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1. Introduction

Temperley–Lieb algebras, whose roots are in statistical mechanics (see [12, 4, Appendix II.b]), play a prominent role in knot theory and low-dimensional topology. They have entered this field through Jones’ representation of Artin’s braid groups in a version of these algebras (see [7, Sec. 11], and works of Jones cited therein), on which the famous Jones polynomial of knot theory is based (see also [9–11]). The version of Temperley–Lieb algebra we deal with here, which is pretty standard (see [9], and works by Kauffman and others cited therein), is defined as follows.

The Temperley–Lieb algebra $T_n$, where $n \geq 2$, is an associative algebra over a field of scalars whose nature is not important for us in this paper (in [9, p. 9], it is taken to be the field of rational functions $P/Q$ with $P, Q \in \mathbb{Z}[x, x^{-1}]$). The
basis of $T_n$, which we will call $J_n$, is freely generated from a set of generators $1^n, h_1^n, \ldots, h_{n-1}^n$ with multiplication subject to the monoid equations (i.e. multiplication is associative and $1^n$ is a unit for it) and the equations

\begin{align}
    h_i^n h_j^n &= h_j^n h_i^n, \quad i - j \geq 2, \\
    h_i^n h_{i \pm 1} h_i^n &= h_i^n, \\
    h_i^n h_i^n &= h_i^n. 
\end{align}

For the definition of multiplication in $T_n$, Eq. (3) is replaced by

$$h_i^n h_i^n = ph_i^n$$

with $p$ a specified nonzero scalar (in [9], it is taken to be $-x^2 - x^{-2}$, but other values are found in other treatments of Temperley–Lieb algebras). The basis $J_n$ of $T_n$ is finite: its cardinality is the $n$th Catalan number $(2^n)! / (n!(n+1)!)$). The algebra $T_n$ is the vector space whose basis is $J_n$.

For those $T_n$ where $p$ in (3p) is a natural number greater than or equal to 2, there is a representation in matrices due to Brauer (see [2] and [13, Sec. 3]. This is the orthogonal group case of Brauer’s representation restricted to the Temperley–Lieb subalgebra of the Brauer algebra. The faithfulness, i.e. isomorphism, of this representation was established by Jones in [8, Sec. 3, Theorem 3.4, p. 330] by referring to his technique of Markov trace. Our purpose in this paper is to give a different, elementary and self-contained, proof of that faithfulness. We believe our proof is worth publishing because of its aesthetic value.

The computational interest of Brauer’s representation is lessened by the fact that for $T_n$ we have to pass to $p^n \times p^n$ matrices, and so get an exponential growth. However, this growth is to be expected, since for sufficiently large $n$ the $n$th Catalan number is greater than $(2 - \varepsilon)^{2n}$ (this can be computed with the help of Stirling’s theorem). When they were first introduced in [12], Temperley–Lieb algebras were represented in matrices in a manner different from Brauer’s (see [4, Appendix II.b, p. 264]), again with an exponential growth.

Brauer’s representation is provided by assigning to the elements of the basis $J_n$ of $T_n$ particular 0–1 matrices that satisfy (1), (2) and (3p) for $p$ a natural number greater than or equal to 2. For the faithfulness of the representation it suffices to establish that the list of matrices assigned to the elements of $J_n$ is linearly independent. Our proof of linear independence proceeds as follows. We introduce a linear order on the matrices, and establish that every matrix has an entry with 1 where all the matrices preceding it in the order have 0.

It is shown in [3] that Brauer’s representation of Temperley–Lieb algebras is based on the fact that the Kronecker product of matrices gives rise to an endofunctor of the category of matrices that is adjoint to itself. The result of this paper is interesting because of applications in the area covered by [3].
2. Ordering $\mathcal{J}_n$

For $n \geq 2$ and $1 \leq i \leq n - 1$, let $h_{i,1}^n$ be $h_i^n$, and for $2 \leq k \leq i$ let $h_{i,k}^n$ be $h_{i,k-1}^n h_{i,k+1}^n$. We call the expressions $h_{i,k}^n$, for $1 \leq k \leq i$, blocks.

Every element of $\mathcal{J}_n$ different from $1^n$ is denoted by a unique expression in Jones’ normal form

$$h_{i_1,k_1}^n \cdots h_{i_l,k_l}^n$$

where $l \geq 1$, $i_1 < \cdots < i_l$ and $i_1 - k_1 < \cdots < i_l - k_l$ (see [6, Sec. 4.1.4; 4, Sec. 2.8; 1 and 3, Sec. 10]). Let $h_{i,k}^0$ stand for $1^n$ when $k = 0$, and let us also call these expressions blocks. Then every element of $\mathcal{J}_n$ is denoted by a unique expression in the normal form

$$h_{i_1,k_1}^1 \cdots h_{i_{n-1},k_{n-1}}^1$$

where $0 \leq k_i \leq i$, and for every $k_i, k_j > 0$ such that $i < j$ we have $i - k_i < j - k_j$. So we may identify every element of $\mathcal{J}_n$ by a sequence $k_1 \cdots k_{n-1}$ subject to the conditions on $k_i$ and $k_j$ that we have just stated.

Let these sequences be ordered lexicographically from the right; in other words, let

$$k_1' \cdots k_{n-1}' < k_1 \cdots k_{n-1}$$

if and only if for some $i$ we have $k_i' < k_i$ and for every $j > i$ we have $k_j' = k_j$. This induces a linear order on $\mathcal{J}_n$, which we call the lexicographical order of $\mathcal{J}_n$.

3. Matrices, Relations and Graphs for $\mathcal{J}_n$

Let $p$ be a natural number greater than or equal to 2, and let $E_p$ be the $1 \times p^2$ matrix that for $1 \leq i, j \leq p$ has the entries

$$E_p(1, (i-1)p + j) = \delta(i,j),$$

where $\delta$ is the Kronecker delta. For example, $E_2$ is $[1 0 0 1]$ and $E_3$ is $[1 0 0 0 1 0 0 0 0 1]$. Let $E_p'$ be the transpose of $E_p$, and let us assign to $h_{i,k}^n$, $1 \leq k \leq n - 1$, the $p \times p$ matrix $I_{p^{n-k-1}} \otimes E_p' E_p \otimes I_{p^{k-1}}$, where $I_m$ is the $m \times m$ identity matrix with the entries $I_m(i,j) = \delta(i,j)$, and $\otimes$ is the Kronecker product of matrices (see [5, Chap. VII.5, pp. 211–213]). We assign to $1^n$ the matrix $I_p^n$.

Every $n \times m$ matrix $A$ whose entries are only 0 and 1 may be identified with a binary relation $R_A \subseteq n \times m$ such that $A(i,j) = 1$ if and only if $(i,j) \in R_A$. Every binary relation may of course be drawn as a bipartite graph. Here are a few examples of such graphs for matrices we have introduced up to now, with $p = 2$: 

![Graphs](image-url)
If \(1^n, h_1^n, \ldots, h_{n-1}^n\) denote the 0–1 matrices we have assigned to these expressions, then it can be verified that these matrices satisfy Eqs. (1), (2) and (3), with multiplication being matrix multiplication (see [3, Secs. 17–20]). If \(1^n, h_1^n, \ldots, h_{n-1}^n\) denote the corresponding binary relations, then for multiplication being composition of binary relations Eqs. (1)–(3) are satisfied. In both cases we also have the monoid equations.

Composition of binary relations is easy to read from bipartite graphs. Here is an example:

By so composing binary relations we can assign to every element of \(J_n\) a binary relation, and then from this binary relation we can recover the 0–1 matrix assigned to our element of \(J_n\). For this to provide an isomorphic representation of \(T_n\) in matrices it is sufficient (and also necessary) to establish that the list of 0–1 matrices assigned to the elements of \(J_n\) is linearly independent. The remainder of this paper is devoted to establishing this fact.

4. Diagonals

We may use \(1^n, h_1^n, \ldots, h_{n-1}^n\), and expressions obtained from these by multiplying, to denote either the 0–1 matrices assigned to these expressions in the previous section, or the corresponding binary relations, or the corresponding bipartite graphs. So we may speak of the 1 entries of \(h_v^n\), conceived as a matrix, which correspond to the ordered pairs of \(h_v^n\), conceived as a binary relation, which correspond to the edges of \(h_v^n\), conceived as a bipartite graph. When formulating our results we will stick, however, to the terminology of binary relations (but it helps intuition if examples of such relations are drawn as bipartite graphs).

For \(n, k \geq 0\), let \(A_k^n\) be the set \(\{k+1, \ldots, k+p^n\} \subseteq \mathbb{N}\), and for \(1 \leq q \leq p\) let

\[ S_q A_k^{n+1} = \{(q-1)p^n + j \mid j \in A_k^n\}. \]

For example, if \(p = 2\), then \(A_0^2 = \{1, 2, 3, 4\}, A_1^2 = \{1, 2\}, A_2^2 = \{3, 4\}, A_0^1 = \{3\}, S_1 A_0^2 = A_1^1, S_2 A_0^2 = A_1^1\) and \(S_1 S_2 A_0^2 = S_1(S_2 A_0^2) = A_0^2\). Words in the alphabet \(\{S_q \mid 1 \leq q \leq p\}\) will be called \(S\)-words.

For \(W\) a word, let \(W^0\) be the empty word, and let \(W^{n+1}\) be \(W^n W\). Let \(|W|\) be the length of the word \(W\). We will use the letters \(W, V, U, W_1, \ldots, W', \ldots\) for \(S\)-words.

For \(1 \leq q_1, q_2 \leq p\), and \(q_1 \neq q_2\), let \(L\) be short for \(S_{q_1}\) and \(R\) for \(S_{q_2}\). (When \(p = 2\), in bipartite graphs \(L = S_1\) is interpreted as “left”, and \(R = S_2\) as “right”.) Then the following is an easy consequence of definitions.
Remark 1. If \( n \geq 2 \) and \( |V| = n - 2 \), then the pairs \((VLLA^n_0, VRRA^n_0)\) and \((VRA^n_0, VLLA^n_0)\) are in the binary relation \( h^n_{n-1} \).

From now on, we will abbreviate \((W_1A^n_0, W_2A^n_0)\) to \((W_1, W_2)\), omitting \( A^n_0\), and, moreover, we will take for granted that such pairs belong to a binary relation, without specifying it explicitly. Then the following remark generalizes Remark 1.

Remark 2. If \( |V| \geq 0, |W| \geq 0, i = 1 + |V| \) and \( n = i + 1 + |W| \), then \((VLLW, VRRW)\) and \((VRRW, VLLW)\) are in \( h^n_{n} \).

Next we establish the following lemma (due to the second author), which is fundamental for our proof.

Lemma 3. If \( |V| \geq 0, |W| \geq 0, k \geq 0, i = k + |V| \) and \( n = i + 1 + |W| \), then in \( h^n_{i,k} \) we have pairs of the following forms:

\begin{align*}
\text{(even) for } k &= 2l, \\
\text{(I even)} & \quad (V)(RL)^l LW, VL(LR)^l W, \\
\text{(II even)} & \quad (V)(LR)^l RW, VR(RL)^l W;
\end{align*}

\begin{align*}
\text{(odd) for } k &= 2l + 1, \\
\text{(I odd)} & \quad (V)(LR)^l LLW, VRR(LR)^l W, \\
\text{(II odd)} & \quad (V)(RL)^l RWW, VLL(RL)^l W.
\end{align*}

(Note that (II even) is obtained from (I even) by interchanging \( L \) and \( R \), and analogously for (II odd) and (I odd). Note also that in (even) the \( S \)-words in between \( V \) and \( W \) on the right-hand sides are obtained from the corresponding words on the left-hand sides by reading them in reverse order, while in (odd) we have to read them in reverse order and interchange \( L \) and \( R \).)

Proof of Lemma 3. If \( k = 0 \), then in \( h^n_{i,0} \), which is \( 1^n \), we have \((VLLW, VLLW)\) and \((VRRW, VRRW)\). For \( 1 \leq k \leq i \) we proceed by induction on \( k \) in \( h^n_{i,k} \). In the basis, when \( k = 1 \), we apply Remark 2. For the induction step, when \( 2 \leq k \leq i \), we have

\[ h^n_{i,k} = h^n_{i,k-1} h^n_{1-k+1}, \]

and suppose the lemma holds for \( h^n_{i,k-1} \).

If \( k - 1 = 2l + 1 \), then for \( |V| = i - k \) and \( |W| = n - i - 1 \) from (I odd) of the induction hypothesis we obtain that

\[ (V)(LR)^l LLW, VRRR(LR)^l W) \]

is in \( h^n_{i,k-1} \), and from Remark 2, we obtain that

\[ (VRRR(LR)^l W, VLLR(LR)^l W) \]

is in \( h^n_{i-k+1} \), which yields (I even) for \( h^n_{i,k} \). We obtain analogously (II even) for \( h^n_{i,k} \) from (II odd) for \( h^n_{i,k-1} \) and Remark 2 (we just interchange \( L \) and \( R \)).
If \( k - 1 = 2l + 2 \), then for \( |V| = i - k \) and \( |W| = n - i - 1 \) from (I \( \text{even} \)) of the induction hypothesis we obtain that
\[
(VL(RL)^{l+1}W, VLL(LR)^{l+1}W)
\]
is in \( h^n_{i,k-1} \), and from Remark 2, we obtain that
\[
(VLL(LR)^{l+1}W, RR(LR)^{l+1}W)
\]
is in \( h^n_{i-k+1} \), which yields (I \( \text{odd} \)) for \( h^n_{i,k} \). We obtain analogously (II \( \text{odd} \)) for \( h^n_{i,k} \) from (II \( \text{even} \)) for \( h^n_{i-k-1} \) and Remark 2.

The pairs mentioned in Lemma 3 will be called the diagonals of the block \( h^n_{i,k} \) for \( 1 \leq k \leq i \). As particular cases of these diagonals we have the pairs mentioned in Remarks 1 and 2. For \( k = 0 \), where \( h^n_{i,0} \) is \( 1^n \), all the pairs of \( 1^n \) are the diagonals of \( 1^n \).

Let us say that a nonintersecting \( n \)-diagram \( D \) of the kind introduced by Kauffman, and considered in [1, 3], can be put into the pair \((W_1, W_2)\) when the threads of \( D \) join the same \( S \)-symbols in the \( S \)-word \( W_1W_2 \). Then if \( i = k > 0 \) and \( n = i + 1 \), one can put into the diagonals of \( h^n_{i,k} \) just the \( n \)-diagram corresponding to this block. An analogous property holds for the “diagonals” of other matrices, which will be defined below.

We can establish the following.

**Remark 4.** If \( |V| \geq 0, |W| \geq 0, k \geq 2, i = k + |V|, n = i + 1 + |W|, 1 \leq k' < k \) and \( |U| = n \), then in \( h^n_{i,k'} \) we do not have any pair of the following forms:

- (odd) for \( k = 2l + 1 \), \((U,VRR(LR)^lW)\) and \((U,VLL(RL)^lW)\),
- (even) for \( k = 2l + 2 \), \((U,VL(LR)^{l+1}W)\) and \((U,VR(LR)^{l+1}W)\).

This is an easy consequence of the definition of \( h^n_{i,k} \). It suffices to look at the right members of the diagonals of \( h^n_{i,k} \), which cannot occur as right members of any pair of \( h^n_{i,k'} \) for \( 1 \leq k' < k \).

**Remark 5.** If \( |V| \geq 0, |W| \geq 0, k \geq 0, i = k + |V| \geq 1, n = i + 1 + |W|, |U| = n \) and \( |U'| = i \), then

- (even) for \( k = 2l \),
- (I \( \text{even} \)) every pair \((U,VL(LR)^lW)\) of \( h^n_{i,k} \) is either a diagonal of \( h^n_{i,k} \) or \( U \) is of the form \( U'S_qW \) for a symbol \( S_q \) different from \( L \);
- (II \( \text{even} \)) every pair \((U,VR(RL)^lW)\) of \( h^n_{i,k} \) is either a diagonal of \( h^n_{i,k} \) or \( U \) is of the form \( U'S_qW \) for a symbol \( S_q \) different from \( R \);
- (odd) for \( k = 2l + 1 \),
- (I \( \text{odd} \)) every pair \((U,VRR(LR)^lW)\) of \( h^n_{i,k} \) is either a diagonal of \( h^n_{i,k} \) or \( U \) is of the form \( U'S_qW \) for a symbol \( S_q \) different from \( L \);
- (II \( \text{odd} \)) every pair \((U,VLL(RL)^lW)\) of \( h^n_{i,k} \) is either a diagonal of \( h^n_{i,k} \) or \( U \) is of the form \( U'S_qW \) for a symbol \( S_q \) different from \( R \).
This too is an easy consequence of the definition of \( h_{i,k}^n \).

Let \( t \) be in the normal form

\[
h_{1,k_1}^n \cdot \ldots \cdot h_{n-1,k_{n-1}}^n
\]

of Sec. 2, and let \( \max(t) = \max \{ i \mid k_i > 0 \} \). If for every \( i \) we have \( k_i = 0 \), then \( \max(t) \) is undefined, and \( t \) is equal to \( 1^n \).

**Remark 6.** If \( \max(t) = i \) and \( n = i + 1 + w \), then for every \( W \) with \( |W| = w \) there are \( V \) and \( U \) with \( |V| = |U| = i + 1 \) such that in \( t \) we have \((VW, UW)\), and there are no other pairs in \( t \) except those of such a form.

This is an easy consequence of the definition of \( h_{i,k}^n \).

If \( t \) is in normal form as above, then we define the *diagonals of \( t \)* inductively in terms of the diagonals of blocks. Formally, we define the diagonals of \( h_{1,k_1}^n \cdot \ldots \cdot h_{i,k_i}^n \) for every \( i \) such that \( 1 \leq i \leq n - 1 \). The diagonals of \( h_{1,k_1}^n \) are the diagonals of this block. If \((W_1, W_2)\) is a diagonal of \( h_{1,k_1}^n \cdot \ldots \cdot h_{j,k_j}^n \) with \( j < n - 1 \) and \((W_2, W_3)\) is a diagonal of \( h_{j,k_j+1}^n \), then \((W_1, W_3)\) is a diagonal of \( h_{1,k_1}^n \cdot \ldots \cdot h_{j,k_j+1}^n \).

**Remark 7.** If \( \max(t) = i \) and \( n = i + 1 + |W| \) for \( |W| \geq 0 \), then every diagonal of \( t \) is of the form \((VW, UW)\) or of the form \((VR,WU)\) for some \( V \) and \( U \) such that \(|V| = |U| = i \).

From the definition above it is not clear that every \( t \) has diagonals. If \( t \) is equal to \( 1^n \), then it certainly has diagonals. For the rest we have the following lemma.

**Lemma 8.** Consider \( h_{1,k_1}^n \cdot \ldots \cdot h_{n-1,k_{n-1}}^n \) in normal form. If \( t \) is \( h_{1,k_1}^n \cdot \ldots \cdot h_{j,k_j}^n \) for \( j \leq n - 1 \), \(|V| \geq 0 \), \(|W| \geq 0 \), \( \max(t) = i = |V| + k_i \geq 1 \), \( n = i + 1 + |W| \) and \(|U| = n \), then, for some \( V \) and every \( W \), in \( t \) we have diagonals of the following forms:

- (even) for \( k_i = 2l \),
  - (I even) \((U, V L(R) L) W\),
  - (II even) \((U, V R(R) R) W\);

- (odd) for \( k_i = 2l + 1 \),
  - (I odd) \((U, V R(R) R) W\),
  - (II odd) \((U, V L(L) L) W\).

**Proof.** We proceed by induction on \( i \). For \( i = 1 \), we apply Lemma 3. For \( i \geq 2 \), let \( t \) be \( t'h_{i,k_i}^n \cdot \ldots \cdot h_{j,k_j}^n \), and suppose the lemma holds for \( t' \). If \( t' \) is equal to \( 1^n \), it suffices to apply Lemma 3 to \( h_{i,k_i}^n \). Otherwise, let \( \max(t') = i' \). We have to consider four cases.

- (even–even) If \( k_{i'} = 2l' \) and \( k_i = 2l_i \), then, by the induction hypothesis, for \(|V'| = i' - k_{i'} \) and \(|W'| = n - i' - 1 \) in \( t' \) we have
  - (I even) \((U, V' L(L) L) W\)',
  - (II even) \((U, V' R(R) R) W\)'.

where
and by Lemma 3 for $|V| = i - k_i$ and $|W| = n - i - 1$ in $h_{1,k_i}^n$, we have

$$\begin{align*}
(\text{I even}) & \quad (V(RL)^iLW, VL(LR)^iW), \\
(\text{II even}) & \quad (V(LR)^iRW, VR(RL)^iW).
\end{align*}$$

Since $t$ is in normal form, we have $i' < i$ and $i' - k_{i'} < i - k_i$, and so we have $|W| < |W'|$ and $|V'| < |V|$. Then we can replace $W'$ in (I even) and (II even) by $W''LW$ and $W''RW$ for $|W''| \geq 0$, and we can replace $V$ in (I even) and (II even) by $V'LV''$ and $V'RV''$ for $|V''| \geq 0$. We have $|V''| = |W''| + 2(i' - l_i)$, and so $|V''|$ is even if and only if $|W''|$ is even. If both $|V''|$ and $|W''|$ are even, then we make $(LR)^iW''$ equal to $V''(LR)^i$, and $(RL)^iW''$ equal to $V''(RL)^i$. If both $|V''|$ and $|W''|$ are odd, then we make $(LR)^iW''$ equal to $V''(RL)^i$, and $(RL)^iW''$ equal to $V''(LR)^i$.

We proceed analogously in the remaining three cases, where $k_{i'}$ is even and $k_i$ odd, where $k_{i'}$ is odd and $k_i$ even, and, finally, where they are both odd.

\section{5. Linear Independence for $\mathcal{J}_n$}

We are now ready to prove the following result, which guarantees linear independence for the representation of $\mathcal{J}_n$.

\textbf{Theorem 9.} If $t'$ precedes $t$ in the lexicographical order, then the diagonals of $t$ are not pairs of $t'$.

\textbf{Proof.} Let $t$ be $h_{1,k_1}^n \cdots h_{n-1,k_{n-1}}^n$, let $t'$ be $h_{1,k'_1}^n \cdots h_{n-1,k'_{n-1}}^n$, and let $t'$ precede $t$ in the lexicographical order. Then there is an $i$ such that $k'_i < k_i$ and for every $j > i$ we have $k'_j = k_j$. We make an induction on the number $\nu$ of $j$'s such that $j > i$ and $k'_j = k_j > 0$.

If $\nu = 0$, then max($t$) = $i$. If $t'$ is 1$^n$, then Theorem 9 follows by Remark 7. If $t'$ is not 1$^n$, then let max($t'$) = $i'$. Hence, $k_i$ and $k'_i$ are both greater than 0, and we must have $i' \leq i$. Then two cases are possible.

(1) Suppose $i' < i$; that is, 0 = $k'_i < k_i$. Then, by Remark 6, every pair of $t'$ must be of the form $(VW', U'W')$ for $|V'| = |U'| = i' + 1$ and $|W'| = n - i' - 1$. On the other hand, by Remark 7, every diagonal of $t$ is of the form $(VLW, URW)$ or $(VRW, ULW)$ for $|W| = |U| = i$ and $|W| = n - i - 1$. Since $|W| < |W'|$, every diagonal of $t$ is not a pair of $t'$.

(2) Suppose $i' = i$; that is, 0 < $k'_i < k_i$. By Lemma 3, the second member of every diagonal of $t$ is of one of the following forms: $VL(LR)^iW$ and $VR(RL)^iW$ for $k_i = 2l$, $VRR(LR)^iW$ and $VLL(RL)^iW$ for $k_i = 2l + 1$, where $|V| = i - k_i$ and $|W| = n - i - 1$. On the other hand, by Remark 4, no pair of $t'$ can have such second members.

Suppose now for the induction step that $\nu > 0$. If max($t$) = $j$ and max($t'$) = $j'$, then $j = j' \geq 2$ and $k_j = k'_j$. Every diagonal $(W_1,W_3)$ of $t$ can be obtained
by composing a diagonal \((W_1, W_2)\) of \(h_{1,k_1}^n \cdots h_{j-1,k_{j-1}}^n\) and a diagonal \((W_2, W_3)\) of \(h_{j,k_j}^n\). By the induction hypothesis, in \(h_{1,k_1}^n \cdots h_{j-1,k_{j-1}}^n h_{j,0}^n \cdots h_{n-1,0}^n\) we do not have the pair \((W_1, W_2)\), which is a diagonal of \(h_{1,k_1}^n \cdots h_{j-1,k_{j-1}}^n h_{j,0}^n \cdots h_{n-1,0}^n\), since the former expression precedes the latter in the lexicographical order.

Suppose \((W_1, W_3)\) is in \(t'\). Then for some \(W_4\) we need to have \((W_1, W_4)\) in \(h_{1,k_1}^n \cdots h_{j-1,k_{j-1}}^n\) and \((W_4, W_3)\) in \(h_{j,k_j}^n\). It is excluded that \(W_4\) is \(W_2\), as we remarked above.

Suppose \(k_j = 2l + 1\), and suppose \((W_2, W_3)\) is \((V(LR)^j LLW, VRR(LR)^j W)\) for \(|W| = n - j - 1\). By Lemma 3, this is one possible case. Then, by Remark 5, we must have that \(W_4\) is \(W_4 S_q W\) for \(S_q\) different from \(L\), since \(W_4\) cannot be \(W_2\). By Remark 7, we have that \(W_4 \neq \) the form \(W_4 LW\). By Remark 6, we cannot have \((W_1 LW, W_4 S_q W)\) in \(h_{1,k_1}^n \cdots h_{j-1,k_{j-1}}^n\). So \((W_1, W_3)\) is not in \(t'\). We proceed analogously in other possible cases.

As a corollary of Lemma 8 and of Theorem 9, we obtain that every matrix in \(J_n\) has an entry with 1 (one of its “diagonals”) where all the matrices preceding it in the lexicographical order have 0. This implies that the matrices assigned to the elements of \(J_n\) make a linearly independent list, and hence Brauer’s representation of Temperley–Lieb algebras in matrices, which we have considered here, is faithful.

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References
