

General Proof Theory

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By the end of the last century Saunders Mac Lane wrote:

So “proof” is the central issue in mathematics. There ought to be a vibrant specialty of “proof theory”. There is a subject with this title, started by David Hilbert in his attempt to employ finitistic methods to prove the correctness of classical mathematics. This was used essentially by Gödel in his famous incompleteness theorem, carried on further by Gerhard Gentzen with his cut elimination theorem. In 1957, at a famous conference in Ithaca, proof theory was recognized as one of the four pillars of mathematical logic (along with model theory, recursion theory and set theory). But the resulting proof theory is far too narrow to be an adequate pillar;... ([1], p. 152)

General proof theory—the term is due to Dag Prawitz—should lead to the proof theory Mac Lane was looking for. It addresses the philosophically-looking question “What is a proof?” by dealing with technical questions related to normal forms of proofs, and in particular with the question of identity criteria for proofs. It follows Gentzen more than Gödel, and in doing that it deals with the structure of proofs, as exhibited for example by the Curry-Howard correspondence, rather than with their strength measured by ordinals.

Much of general proof theory is the field of categorial proof theory, opened up by Joachim Lambek. Fundamental notions of category theory like the notion of adjoint functor, and very important structures like cartesian closed categories, came to be of central concern for logic in that field. William Lawvere’s contribution here is decisive. Besides that, results of the kind categorists call coherence results provide a model theory for equality of proofs. For example, the coherence theorem for symmetric monoidal closed categories, which antedates considerably the advent of linear logic, is about equality of proofs in a fragment of this logic. Much of Philip Scott’s work in categorial proof theory is inspired by linear logic. Mac Lane, who with Max Kelly proved this coherence theorem by relying on Gentzen’s cut elimination, introduced the subject of coherence in

category theory, through which logic finds new ties with geometry, topology and algebra.

Mac Lane wrote a doctoral thesis in proof theory under the supervision of Paul Bernays. In that thesis, contrary to the spirit of the proof theory that came to dominate the twentieth century, and in accordance with the spirit of general proof theory, he concentrated on the justification of the inference steps, rather than on the propositions that make the proofs. In general proof theory one looks for an algebra of proofs, and for that one should concentrate on the operations of this algebra, which come with the inference rules. The propositions that make the proofs are secondary. Usually, however, the propositions are in the forefront, and not the rules, which are noted only in the margins (and this marginal “bureaucracy” may even be found undesirable).

As an equational theory, the algebra of proofs involves the question of identity criteria for proofs, the central question of general proof theory. This is a question that may be found, at least implicitly, in David Hilbert’s discarded 24th problem:

Überhaupt, wenn man für einen Satz zwei Beweise hat, so muss man nicht eher ruhen, als bis man die beide aufeinander zurückgeführt hat oder genau erkannt hat, welche verschiedenen Voraussetzungen (und Hilfsmittel) bei den Beweisen benutzt werden:... [In general, if one has two proofs for a proposition, one must keep going until one has derived each of them from the other, or until one has discerned clearly enough what different conditions (and means) have been used in these proofs:...] ([2], Section 10.4, pp. 280-282)

References

- [1] S. MAC LANE, *Despite physicists, proof is essential in mathematics*, *Synthese*, vol. 111 (1997), pp. 147-154
- [2] R. THIELE, *Hilbert and his twenty-four problems*, *Mathematics and the Historian’s Craft: The Kenneth O. May Lectures* (G. Van Brummelen and M. Kinyon, editors), Canadian Mathematical Society, Springer, New York, 2005, pp. 243-295