LOGICAL CONSEQUENCE: A TURN IN STYLE

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This talk summarizes some of the things that contemporary logic and, in particular, proof theory stemming from Gentzen have to say about the notion of consequence. It starts from very elementary facts, the understanding of which doesn't require any technical knowledge, to reach the more specialized areas of substructural logics and categorial proof theory. There, one may turn to a style of proof-theoretical investigation whose goal is not just the elimination of cut. Some tentative philosophical suggestions are drawn from this summary.

In connection with the notion of consequence logicians make two distinctions that other, less philosophically minded, mathematicians usually fail to make, or at least don't find so important. The first is the distinction between deduction and implication, the second the distinction between syntactical and semantical notions of consequence. We shall first briefly examine these two distinctions.

1 Deduction and implication

The word 'consequence' belongs to a family of words in which one also finds 'deduction', 'derivation', 'inference', 'implication' and 'entailment'. The corresponding verbs are 'to deduce', 'to derive', 'to infer', 'to imply' and 'to entail', to which one should add 'to conclude' and 'to follow from', as well as the less formal 'to obtain from', 'to get from', 'to yield' and 'to give'. All these words have something to do with the activity of passing from sentences to sentences so that truth, or some related property like provability, is preserved. In case preservation of provability is envisaged, we presumably have to conform to some rules when we pass from sentences to sentences. We are then in the domain of syntax. In case preservation of truth is envisaged, we are in the domain of semantics.

The words 'deduction', 'derivation' and 'inference' may perhaps be distinguished by nuances — 'inference', for example, stressing an immediate step, and 'deduction' rather a chain of steps — but, otherwise, all three words are used in pretty much the same way. There are also no essential differences among the verbs 'to deduce', 'to derive', 'to infer', 'to conclude', 'to obtain from' and 'to get from'. So we may concentrate on one of these words, and let that be 'deduction'. A deduction is either the aforementioned activity or the result of this activity, which consists in having an array of sentences. Sentences coming later in the array are usually introduced with 'so', 'hence' or 'therefore', or are written beneath previously written sentences, sometimes with a line separating them. The sentences above are called 'premises' and the sentences below 'conclusions'; i.e., in a deduction, we pass from premises to a conclusion. There may be several premises from which we deduce a conclusion. Syntactically, a deduction is correct when it is done in conformity to some prescribed rules; semantically, a correct deduction, often called 'valid deduction', is one that preserves truth when passing from the premises to the conclusion. However, when one talks about 'deductions' in logic, syntactical aspects of the matter are usually more prominent.

In logic, the word 'implication' is reserved for the connective 'if __, then __' and sentences of the form 'if A, then B', usually written with symbols as '$A \rightarrow B$', or '$A \Rightarrow B$', or '$A \supset B$'. The verbs 'to imply', 'to follow from', 'to yield' and 'to give' are all tied to implication (of course, 'to follow from' is converse to the others). 'Entailment' and 'to entail' are used pretty much as 'implication' and 'to imply', except in those quarters of logic where they are reserved for a special, relevant, kind of implication. In the implication 'if A, then B', the sentence A is called the 'antecedent' and B the 'consequent'. (We also say that A is a sufficient condition for B, and B a necessary condition for A.) Conditionals, i.e. hypothetical sentences of the form 'if A, then B', need not all be implications in the logicians' sense: they may reflect relationships from the realm of the laws of nature or human behaviour, they may have to do with explanation, causation and dispositions. Among these other conditionals, a special place is occupied by counterfactual conditionals, whose if-clauses are known to be false. These matters may be germane to implication, but are nevertheless separate.

In classical logic, an implication is true when it is not the case that the antecedent is true and the consequent false. When an implication is true, the corresponding deduction, in which the antecedent of the implication is the premise and the consequent the conclusion, is a correct deduction (in a semantical sense; when the truth of the implication may be recognized by appealing only to the understanding of the language, the corresponding deduction is syntactically correct). Conversely, the correctness of a deduction should guarantee that the corresponding implication is true. When there are several premises in the deduction, we make a single sentence out of them with the help of the connective 'and', called 'conjunction', and we put this sentence as the antecedent of the corresponding implication. So we may take that premises in a deduction are joined by an implicit conjunction.

Although deduction and implication correspond to each other, as we have just indicated, they are separated by the essential difference that exists between, on the one hand, the activity of passing from sentences to sentences and, on the other, a connective, or sentences made with this connective. The purpose of Lewis Carroll's piece about the Tortoise and Achilles [1895] is presumably to stress this difference. The lesson is that there is no deduction without rules of deduction: these rules cannot be entirely replaced by implications (we need at least the rule that enables us to pass from an implication to the corresponding deduction, i.e. *modus ponens*).

The distinction between deduction and implication is not only philosophical or, perhaps, psychological. It has technical repercussions, too, as witnessed by Gentzen's mathematical achievements, which we shall examine below.

Close relatives of the words in our family are ‘proof’ and ‘demonstration’, which refer to a correct deduction in which the premises are true or, in more general terms, acceptable. A somewhat more distant relative is ‘argument’. An argument is meant to convince somebody, and it may, but need not, be of a deductive sort.

The word ‘consequence’ refers either to the conclusion of a correct deduction or to the consequent of a true implication. Being a consequence is a relative property: in a correct deduction we say that the conclusion is a consequence *of* the premises, and in a true implication the consequent is a consequence *of* the antecedent. So a more basic notion than the property of being a consequence is the relation that obtains between the premises and the conclusion of a correct deduction, or between the antecedent and the consequent of a true implication. These relations are called ‘consequence relations’.

2 Syntactical and semantical logical consequences

In logic, a consequence relation arising from a syntactically correct deduction is called a ‘syntactical consequence relation’. If Γ is the collection of premises of a deduction and A the conclusion, one usually writes $\Gamma \vdash A$ to express that A is a syntactical consequence of Γ. However, when in logic one says ‘consequence relation’ *simpliciter*, one usually means ‘semantical consequence relation’ rather than ‘syntactical consequence relation’. A semantical consequence relation corresponds to generalized implication explained by reference to truth values, as in classical logic. For a sentence A to be a semantical consequence of a collection of sentences Γ, which is usually written $\Gamma \models A$, we must have that if every sentence in Γ is true, then A is true.

If Γ consists of a single sentence, then a semantical consequence relation is the relation that exists between the antecedent and the consequent of a true implication. If Γ is a finite collection of sentences, then we may put all these sentences into a conjunction and have consequence corresponding to the implication whose antecedent is this conjunction and whose consequent is A. Short of introducing infinitary connectives (which indeed may be done in logic), we cannot reduce to implications consequence relations involving infinite collections Γ. We have such a consequence relation, for example, with Γ being made of the sentences ‘0 has a successor’, ‘1 has a successor', '2 has a successor', etc. for each natural number; as a consequence of Γ we have the sentence 'Every natural number has a successor'. On a rather abstract level of logic, one may envisage a deduction corresponding to the consequence relation in this example (the rule justifying this deduction is called the ω-*rule*), but syntactical consequence relations, unlike this one, usually satisfy a property called 'compactness', which says that if A is a consequence of Γ, then A is a consequence of a finite collection of sentences included in Γ.

Besides studying matters that pertain to any notion of consequence, logic concentrates in particular upon consequence relations that hold in virtue of logical form. Logical form is obtained by keeping constant in sentences just certain words called 'logical constants' (they are also sometimes called 'logical words', or 'logical particles'). Everything else is replaced by schematic letters or variables. The resulting expressions are not called 'sentences' any more, but 'formulae'. Logical consequence relations are those that hold for these formulae, and hence for any of their instances. The study of the syntactical aspects of consequence belongs to proof theory, whereas semantical aspects are studied in model theory.

Consequence was studied in logic since ancient times, and a big corpus of medieval logic is devoted to the study of *consequentiae*. However, it is rather difficult to find in these old works a clear distinction between syntactical and semantical aspects of the matter, or between the activity of deduction and the connective of implication. As a matter of fact, these distinctions have not been quite clear even at the inception of modern logic. They don't appear in a form that would satisfy today's standards before the 1930s.

In [1936], Alfred Tarski gave a definition of the semantical notion of logical consequence (Bernhard Bolzano is usually credited of having anticipated this definition a hundred years earlier). The novelty of Tarski's definition is not the understanding of consequence in terms of a generalized implication — this is something everybody always had in mind. What he made precise is the notion of truth in a model, which replaces the intuitive notion of truth in the definition of semantical consequence given at the beginning of this section.

At roughly the same time, in a series of papers (see in particular [1930, 1930a]), Tarski studied logical consequence in an abstract axiomatic way, where purely syntactical aspects of the matter are prominent. In his syntactical studies, Tarski's basic notion was not a consequence relation between a collection of formulae and a formula, but a consequence operation **Cn**, which assigns to a set of formulae Γ the set of its consequences **Cn**(Γ). Properties satisfied by **Cn** are reminiscent of properties of a topological closure operation. From Tarski's axioms we can deduce:

$$\Gamma \subseteq \mathbf{Cn}(\Gamma);$$
$$\mathbf{Cn}(\mathbf{Cn}(\Gamma)) \subseteq \mathbf{Cn}(\Gamma);$$
$$\textit{if}\ \ \Gamma \subseteq \Delta,\ \textit{then}\ \ \mathbf{Cn}(\Gamma) \subseteq \mathbf{Cn}(\Delta);$$
$$\mathbf{Cn}(\Gamma) \subseteq \cup_{\Gamma_{\mathrm{fin}} \subseteq \Gamma} \mathbf{Cn}(\Gamma_{\mathrm{fin}}),$$

where Γ_{fin} is a finite set; the last inclusion expresses the compactness of consequence. Tarski's style of studying logical consequence via **Cn** has had a big influence in his native Poland (see [Wójcicki 1988]), but can hardly count as a universally accepted standard.

A more widely accepted style in the syntactical study of logical consequence was inaugurated by Gerhard Gentzen in his dissertation 'Investigations into Logical Deduction' [1935]. This work is the cornerstone of proof theory.

3 Sequents and structural principles

Gentzen's basic notion is a relation $\vdash$ between a sequence of formulae Γ and a formula A (Paul Hertz has studied this notion before Gentzen and has influenced him). He and a number of other authors write an arrow $\rightarrow$ instead of the turnstile $\vdash$. Expressions of the form $\Gamma \vdash A$ are called 'sequents', and in proof theory done in the style of Gentzen talk about sequents is more common than talk about consequence relations, though most logicians should be aware that asserting a sequent $\Gamma \vdash A$ amounts to asserting that the formula A is a consequence of the formulae in Γ. (The term 'sequent' could perhaps enter into the philosophical jargon to designate what remains in a deduction when it is disengaged from the activity of deducing, as the term 'proposition' is used for what remains in an assertion when it is disengaged from the activity of asserting.)

In $\Gamma \vdash A$, Gentzen takes Γ to be a sequence of formulae, but he assumes *structural* rules that permit him to transform $\Gamma \vdash A$ into $\Gamma' \vdash A$ where Γ' is obtained from Γ by permuting members of Γ or by omitting repetitions among these members. Following Gentzen, a logical principle concerning sequents is called 'structural' if and only if it can be formulated without mentioning any constant of the language to which belong the formulae we write on the left-hand side and right-hand side of the turnstile. (In the Polish logic literature, structural rules are those whose instances are closed under substitution; this notion should not be confused with Gentzen's notion of a structural rule.) The structural rule that permits permuting members of a sequence Γ is called 'permutation' (sometimes also 'interchange', or 'exchange') and the structural rule that permits omitting repetitions is called 'contraction'. Assuming permutation and contraction amounts to reducing sequences Γ in $\Gamma \vdash A$ to sets.

Gentzen's reason for working with sequences rather than sets is probably that he wanted sequents to be expressions of a language, i.e. strings of symbols. For the same reason, Gentzen's sequents $\Gamma \vdash A$ always have Γ finite. (Gentzen was a pupil of David Hilbert and wanted to reason about sequents in a finitistic way.) Because of the compactness property, which syntactical consequence relations usually have, this is not such an essential limitation. But, in general, it makes sense to extend Gentzen's approach to sequents $\Gamma \vdash A$ with Γ infinite.

Most logicians shun such infinitary sequents, as if they had finitistic scruples. However, one may expect to find such sequents useful in second-order or higher-order logic, in logic with infinitary connectives, in arithmetic with the ω-rule, or whenever finite axiomatizability fails. Of course, sequents are then something we talk *about*, rather than something we talk *in*.* But such are most object languages of logicians: they are studied at the metalevel, rather than actually used.

By taking sequents $\Gamma \vdash A$ where Γ is a set of formulae, perhaps infinite, and by taking that $\Gamma \vdash A$ is equivalent to $A \in \mathbf{Cn}(\Gamma)$, the principles concerning **Cn** mentioned in section 2 yield further structural principles for sequents:

(identity) $\{C\} \vdash C$;
(cut) *if* $\Gamma \vdash C$ *and* $\{C\} \cup \Delta \vdash B$, *then* $\Gamma \cup \Delta \vdash B$;
(thinning) *if* $\Gamma \vdash B$, *then* $\Gamma \cup \Delta \vdash B$;
(compactness) *if* $\Gamma \vdash B$, *then for some* $\Gamma_{\mathrm{fin}} \subseteq \Gamma$, $\Gamma_{\mathrm{fin}} \vdash B$.

These principles correspond to the four principles for **Cn**, but the correspondence is not exact; for example, to obtain $\Gamma \subseteq \mathbf{Cn}(\Gamma)$ from identity we need thinning (also called 'weakening'). One should note that in deductions corresponding to the implications of the last three principles, as well as in deductions arising from the structural rules of permutation and contraction, the premises and conclusions are themselves about deductions. For example, in cut, the premises $\Gamma \vdash C$ and $\{C\} \cup \Delta \vdash B$, and the conclusion $\Gamma \cup \Delta \vdash B$ are themselves about deductions. So we have here something we may conceive as a deduction of higher level, whose premises and conclusion are not sentences, but deductions.

If we have a noncompact consequence relation, we may envisage replacing cut by the following generalized cut

$$\textit{if}\ (\forall C \in \Theta)\, \Gamma \vdash C\ \ \textit{and}\ \ \Theta \cup \Delta \vdash B,\ \textit{then}\ \ \Gamma \cup \Delta \vdash B.$$

* I owe this remark to Professor Dag Prawitz.

If Θ is empty, $(\forall C \in \Theta)\, \Gamma \vdash C$ holds for every Γ, and we can derive thinning from our generalized cut. In fact, the structural principles of identity, generalized cut and thinning are all summed up by the equivalence

$$(\forall C \in \Theta)\, \Gamma \vdash C \;\; \textit{if and only if} \;\; \forall \Delta \forall B(\textit{if} \;\; \Theta \cup \Delta \vdash B, \; \textit{then} \;\; \Gamma \cup \Delta \vdash B),$$

while identity and ordinary cut are summed up by the equivalence

$$\Gamma \vdash C \;\; \textit{if and only if} \;\; \forall \Delta \forall B(\textit{if} \;\; \{C\} \cup \Delta \vdash B, \; \textit{then} \;\; \Gamma \cup \Delta \vdash B)$$

(other such equivalences, characterizing various sorts of consequence relations, are considered in [D. 1994]). At the bottom of these equivalences stands the fact that the reflexivity and transitivity of a binary relation R are converse to each other when these two properties are summed up by the equivalence

$$xRy \;\; \textit{if and only if} \;\; \forall z(\textit{if} \;\; yRz, \;\; \textit{then} \;\; xRz).$$

All these equivalences show that the structural principle of identity may be conceived as converse to cut. (A related duality between identity and cut is explored in [Girard 1987, I.3] and [Hösli & Jäger 1994]; in this approach, identity is taken as saying that C as a premise yields C as a conclusion, whereas cut says that C as a conclusion yields C as a premise.)

4 Cut elimination

Among structural principles, the cut rule has played by far the most important role in Gentzen's work and in the proof-theoretical tradition stemming from it. Cut is specific because, unlike all the other rules in Gentzen's formulation of logic, it has a formula (the formula C in the formulation of cut in the previous section), called the 'cut formula', which is in the premises but can disappear in the conclusion. Without cut we can establish the *subformula property*, which says that every provable sequent has a proof made of sequents in which occur only subformulae of the formulae occurring in our provable sequent. By constructing systems of sequents where the cut rule, or an analogue of it, is not primitive, but is nevertheless admissible (i.e. can be added to the system without increasing the stock of provable sequents), Gentzen produced a beautiful and extremely useful logical tool. This technique of *cut elimination* can serve to establish many important properties of systems, like decidability and completeness. Within the realm of Hilbert's program it serves for consistency proofs: Gentzen used it for his consistency proof of arithmetic. However, cut is a principle that should hold for most, if not all, reasonable notions of consequence: it expresses the transitivity of the consequence relation. To eliminate it does not mean to reject it. There is an advantage in not having it postulated as primitive, but it should nevertheless be admissible. And if we are not interested in exploiting cut elimination, or deal with a system where it cannot be accomplished, cut will figure among the postulates.

To obtain cut elimination in logical systems, Gentzen postulated sequent rules for logical constants in a symmetrical fashion. For each constant there are rules for introducing it on the left-hand side and on the right-hand side of the turnstile. If we read $\Gamma \vdash A$ as asserting that there is a *natural-deduction* proof of A with uncancelled hypotheses in Γ, rules for introducing logical constants on the left-hand side of the turnstile correspond to natural-deduction elimination rules, whereas rules for introducing them on the right-hand side correspond to natural-deduction introduction rules. A cut-free sequent proof gives instructions for building a natural-deduction proof in *normal form*, where, roughly speaking, eliminations are followed by introductions (see [Prawitz 1965]). If the structural principle of identity $\{C\} \vdash C$ is postulated only for atomic formulae C (so that for nonatomic formulae it has to be derived), the normal form will have only atomic formulae in between the eliminations and introductions (see [Prawitz 1971]). Actually, it is not necessary to eliminate all cuts for building natural-deduction proofs in such a normal form: it is enough if we eliminate cuts with nonatomic cut formulae. Besides this basic difference, there are others: cut elimination and conversion of natural-deduction proofs to normal form, i.e. *normalization*, are not exactly equivalent techniques. They are nevertheless closely related (see [Zucker 1974] and [Pottinger 1977]).

Advance in mathematics often consists in the invention of a good notation. Tarski and Gentzen often speak about the same matters concerning logical consequence, but Gentzen's notation was better chosen. It is doubtful whether one could have discovered cut elimination with Tarski's notation.

5 Plural consequence in classical logic

For classical logic Gentzen used sequents of the form $\Gamma \vdash \Delta$, where on the right-hand side of the turnstile we don't have necessarily a single formula: both Γ and Δ are sequences of formulae. Let us call these sequents *plural* (they are sometimes called 'multiple-conclusion sequents'), whereas sequents where Δ has a single member or is empty are *singular* (these are sometimes called 'single-conclusion sequents'). The commas in the sequence Γ correspond to conjunction, whereas the commas in the sequence Δ to disjunction. There are structural rules for dealing with the right-hand side of sequents analogous to the structural rules mentioned above, which deal with the left-hand side. This permitted Gentzen to give an extremely elegant symmetrical axiomatization of classical logic, doing full justice to the underlying Boolean dualities.

He passes from this axiomatization of classical logic to an axiomatization of intuitionistic logic not by changing rules for logical constants, but just by restricting the language to singular sequents (in both axiomatizations cut can be eliminated). The restriction to singular sequents is tantamount to restricting the structural rule of thinning on the right-hand side so that we never have as a conclusion a sequent that is not singular. (There are, however, plural versions of intuitionistic sequent systems, where only rules for introducing implication, negation and the universal quantifier on the right-hand side fall under the singular-sequent restriction.)

Although the notion of plural consequence that underlies plural sequents is perfectly in tune with the spirit of classical logic, one very rarely hears about it outside proof theory done in the style of Gentzen. A semantical interpretation of plural sequents, which reads $\Gamma \vdash \Delta$ as saying that if all the formulae in Γ are true, then at least one of the formulae in Δ is true, was championed by Dana Scott in several papers from the early 1970s. In [1974] Scott remarked that a singular consequence relation, whose corresponding **Cn** operation satisfies Tarski's axioms, may be extended conservatively to different plural consequence relations, and he characterized the minimal and maximal ones among these. The minimal one $\Gamma \vdash_{min} \Delta$ obtains when for a member A of Δ we have $\Gamma \vdash A$ with the singular consequence, whereas the maximal one $\Gamma \vdash_{max} \Delta$ obtains when we have something that amounts to $\Gamma \vdash \bigvee\Delta$ with the singular consequence, $\bigvee\Delta$ being a disjunction (not necessarily finite) of the formulae in Δ. It can be shown that $\Gamma \vdash_{min} \Delta$ amounts to

$$(\exists A \in \Delta)\forall\Theta\forall B(\textit{if } \{A\}\cup\Theta \vdash B, \textit{ then } \Gamma\cup\Theta \vdash B),$$

whereas $\Gamma \vdash_{max} \Delta$ amounts to

$$\forall\Theta\forall B(\textit{if } (\forall A \in \Delta)\ \{A\}\cup\Theta \vdash B, \textit{ then } \Gamma\cup\Theta \vdash B),$$

which for Δ nonempty is equivalent to

$$\forall\Theta\forall B(\exists A \in \Delta)(\textit{if } \{A\}\cup\Theta \vdash B, \textit{ then } \Gamma\cup\Theta \vdash B)$$

(cf. [D. 1994]). Gentzen's plural sequents for classical logic correspond to the maximal plural consequence relation of classical logic.

The semantical reading of plural sequents is much more natural than some proof-theoretical interpretations that have been proposed. In one of these proof-theoretical interpretations plural sequents are supposed to talk about natural-deduction proofs that are not trees: they branch disjunctively towards the conclusions, and not only conjunctively towards the premises, like ordinary natural-deduction trees (see

[Shoesmith & Smiley 1978]). In another, more obvious, interpretation we have natural-deduction trees whose nodes are not made of single formulae but of sequences of formulae (see [Boričić 1985] and [Celluci 1992]). As suggested in [1956] by William Kneale (who was influenced both by [Gentzen 1935] and [Carnap 1943, § 32]), the philosophy behind these peculiar forms of deduction might be that in classical logic, instead of deducing single sentences, we should be deducing where is the field where truth lies. The problem is that it is doubtful whether this style of deduction could, or indeed should, be imposed on classical logicians, whereas nothing save habit seems to prevent them from accepting a plural notion of semantical consequence.

The lack of symmetry between premises and conclusions in ordinary deduction, which consists in having possibly several premises, but a single conclusion, may explain the lack of symmetry in intuitionistic logic, where conjunction is not dual to disjunction and the universal quantifier is not dual to the existential quantifier. Intuitionistic logic is truly the logic of ordinary, asymmetrical, deduction. This is demonstrated by the connection between singular sequents and natural deduction. The ordinary natural-deduction introduction and elimination rules for implication, which correspond to the Deduction Theorem and *modus ponens*, don't produce classical implication but intuitionistic implication. Classical logic, seen through the prism of plural sequents and the underlying symmetrical notion of plural consequence, seems to be rather about semantical matters involving the notion of truth.

The orientation of classical logic towards truth rather than deduction is manifested in the kind of formal system with which logicians are chiefly concerned. These systems, whose style is often associated with the name of Hilbert, concentrate on theoremhood rather than consequence. They have many axioms and a very few rules of inference (often only *modus ponens* and universal generalization). Syntactical consequence in these formal systems is only a derived notion, upon which one comes when one introduces proofs from hypotheses. In the model-theoretical tradition, which has been dominant in modern logic, one hardly feels the need for other formal systems save those in the style of Hilbert.

There is a gain in model theory in proving *strong completeness*, which is an equivalence between syntactical and semantical notions of consequence, over proving only ordinary completeness, which is the equivalence between theoremhood and validity, i.e. universal truth. With strong completeness, we can derive the model-theoretical Compactness Theorem from the compactness of the syntactical consequence relation of classical logic. However, once strong completeness for first-order classical logic has been proved, there don't seem to be many results in model theory that need to be stated in a strong form involving consequence relations, rather than an ordinary form involving theoremhood and validity. With compactness for our consequence relations and with the Deduction Theorem, the strong form of the result need not be more difficult to prove than the ordinary form: one can be inferred from the other. Consequence relations become interesting when completeness and compactness fail, as in second-order logic, or in the study of nonclassical logics, where one often finds that formal systems should axiomatize a particular notion of consequence rather than simply theoremhood.

6 Substructural consequence relations

Some nonclassical logics differ from classical logic, and from one another, not in what they assume about logical constants, but in what they assume about structural principles for their consequence relations. It is as if the structural part of logic were more fundamental: we change logic by changing this part. Logical constants are secondary: they are invariant, they play the same role in different structural contexts. These nonclassical logics have started being called ‘substructural’ in 1990. Individual substructural logics are much older (intuitionistic and relevant logic were first introduced in the 1920s). The term ‘substructural’ may convey that something has been subtracted in the structural part of logic, but it should also convey, according to the standard meaning of the word, that these logics are the product of interventions in the supporting structures of logic. (For references about substructural logics see [D. & Schroeder-Heister 1993].)

The most important substructural logic is intuitionistic logic, which, as was remarked in the previous section, may be obtained from classical logic by restricting thinning on the right-hand side. Rejecting thinning altogether produces a variant of relevant (or relevance) logic. The notions of consequence of these two logics are still based on *sets* of formulae Γ in $\Gamma \vdash A$. If, however, we reject the structural rule of contraction, as this is done in BCK logic, or both thinning and contraction, as in linear logic, the corresponding notions of consequence are based on *multisets* Γ in $\Gamma \vdash A$, a multiset being a set of occurrences of formulae, i.e. a set with possible repetitions. Finally, there is a logic derived from Joachim Lambek’s calculus of syntactic categories, where permutation is rejected, too. In this logic *sequences* Γ in $\Gamma \vdash A$ are taken seriously. The weakest logic in this hierarchy is a nonassociative variant of Lambek’s calculus, where in $\Gamma \vdash A$ the collection Γ is not a sequence but a *term* made with a binary nonassociative operation joining formulae. This operation replaces the associative comma of sequences. In this logic we have rejected the structural rule of association, which is only implicit in Gentzen’s sequents based on sequences.

These logics have various inspirations. Intuitionistic logic is a product of the constructivist side in the great debate on the foundations of mathematics. Relevant logic, though its earliest roots are mixed with the roots of intuitionistic logic, has a more philosophical inspiration: it grew out of attempts to formalize an implication devoid of the unintuitive features of material implication. The first reason for introducing the older contractionless logic, BCK logic, was to evade set-theoretical paradoxes while keeping unrestricted comprehension (systems of many-valued logic, where contraction is likewise restricted, have been used for the same purpose). The younger contractionless logic, linear logic, draws its inspiration and much of its appeal from quarters on the borderline of computer science. Finally, Lambek's calculus belongs primarily to mathematical linguistics, where it serves to build formal grammars called 'categorial grammars'. However, it has a natural algebraic inspiration, too, and it may be understood as a logic if we envisage deductions where we don't want to lose sight of the order of premises or of their exact arrangement on the top of deduction trees (in the case of Lambek's nonassociative calculus such trees are binary).

This information about the premises, which for most purposes we don't need, is lost by adding the structural rules of association and permutation. By adding contraction we lose information about how many times a premise was used. Finally, with thinning we introduce phantom premises that are not used at all, and we lose information about which premises were really used. It is presumably easier to work with less information, if we don't really need it. Substructural logics may be motivated by notions of deduction where we pay attention to information that we usually prefer to forget. So they are logics of deduction, and as such should be included in intuitionistic logic. However, there are also plural variants of substructural logics (indeed, in relevant, BCK and linear logic more attention has been paid to such variants).

In the production of substructural logics there has also been, no doubt, a certain amount of experimentation, of free play, with the infinitely many possibilities of Gentzen's apparatus. Interest in a particular logic, like linear logic, is then justified by its mathematical regularity rather than by applications. Like other mathematicians, logicians produce tools that may, but need not, find a user. In spite of what they claim when they go to the market, they are more concerned with the intrinsic than with the instrumental value of their products.

7 Logical constants

In classical logic and all substructural logics mentioned in the last section we can give the same postulates for the main logical constants. If we don't care about eliminating cut, the rules that stand behind the following equivalences will do for logics with association:

$$A, \Gamma \vdash \Delta, B, \Theta \quad \textit{if and only if} \quad \Gamma \vdash \Delta, A \rightarrow B, \Theta;$$

$$(\Gamma \vdash \Delta, A, \Theta \ \ \textit{and} \ \ \Gamma \vdash \Delta, B, \Theta) \ \ \textit{if and only if} \ \ \Gamma \vdash \Delta, A \wedge B, \Theta;$$
$$(\Delta, A, \Theta \vdash \Gamma \ \ \textit{and} \ \ \Delta, B, \Theta \vdash \Gamma) \ \ \textit{if and only if} \ \ \Delta, A \vee B, \Theta \vdash \Gamma;$$
$$\Gamma \vdash \Delta, A, \Theta \ \ \textit{if and only if} \ \ \Gamma \vdash \Delta, \forall xA, \Theta;$$
$$\Delta, A, \Theta \vdash \Gamma \ \ \textit{if and only if} \ \ \Delta, \exists xA, \Theta \vdash \Gamma;$$

provided x does not occur free in the sequents on the right-hand sides of the last two equivalences. (We have something similar without association, but at the cost of some technical complications.) In the present context we need the structural rule of *substitution*, which enables us to pass from $\Gamma \vdash \Delta$ to $(\Gamma \vdash \Delta)^x_a$, obtained from $\Gamma \vdash \Delta$ by uniformly replacing the free variable x by the term a, with the usual provisos preventing the free variables of a to become bound after the substitution. Equality too can be completely characterized by an equivalence:

$$(\Delta, \Theta \vdash \Gamma)^x_a \ \ \textit{if and only if} \ \ \Delta, x{=}a, \Theta \vdash \Gamma.$$

The connectives of conjunction and disjunction, $\wedge$ and $\vee$, behave like finite meet and join in a lattice, and the universal and existential quantifiers, $\forall$ and $\exists$, like infinite meet and join, the lattice ordering corresponding to $\vdash$ (however, in the absence of thinning or contraction, $\wedge$ will not be distributive over $\vee$). The propositional constants **T** and **F**, which behave like greatest and least element in a lattice, would satisfy the axioms $\Gamma \vdash \Delta, \mathbf{T}, \Theta$ and $\Delta, \mathbf{F}, \Theta \vdash \Gamma$. Lattice connectives are called 'additive' in linear logic and 'extensional' in relevant logic. In the absence of thinning or contraction, $\wedge$ will be distinguished from the product connective $\bullet$, which satisfies

$$\Delta, A, B, \Theta \vdash \Gamma \ \ \textit{if and only if} \ \ \Delta, A \bullet B, \Theta \vdash \Gamma,$$

whereas in classical and intuitionistic logic these two connectives coincide. In plural variants of some substructural logics there may be a connective $+$ dual to $\bullet$, which replaces the comma on the right-hand side of the turnstile and is distinguished from $\vee$. Similarly, we may have a propositional constant **t**, distinguished from **T**, replacing the empty collection of formulae on the left-hand side, and **f**, distinguished from **F**, replacing the empty collection of formulae on the right-hand side of the turnstile. Algebraically, **t** and **f** behave like units for the multiplications $\bullet$ and $+$, respectively. We may define a negation $\neg A$ of an intuitionistic bent by $A \to \mathbf{f}$ or $A \to \mathbf{F}$, but the main proponents of relevant and linear logic prefer an involutive negation that satisfies De Morgan's laws, with respect to $\wedge$ and $\vee$, as well as with respect to $\bullet$ and $+$. The connectives $\bullet$, $+$, **t** and **f**, together with implication $\to$, are called 'multiplicative' in linear logic and 'intensional' in relevant logic.

The essential algebraic fact concerning $\bullet$ and $\to$ is that we have

$$A \bullet C \vdash B \ \ \textit{if and only if} \ C \vdash A \to B,$$

which means that algebraically implication behaves like a residual of the product. The product connective $\bullet$ stands for the comma on the left-hand side of the turnstile, and matches the way premises are joined, which in substructural logics is not necessarily an implicit conjunction corresponding to the connective $\wedge$. It is indistinguishable from such a conjunction in classical and intuitionistic logic. The implication connective $\rightarrow$ stands for the turnstile, and matches most directly the consequence relation. This explains why implication may be taken as the central connective of logic. The equivalences above suggest that all logical constants match some features of sequents, i.e. of the consequence relation. (Similar equivalences characterizing logical constants may be found in [Kneale & Kneale 1962, IX.4], [Bernays 1965], [Scott 1971, 1974] and [D. 1985, 1988]; a philosophy suggested by them is expounded in [D. 1989].)

8 Categorial proof theory

If we concentrate upon simple singular sequents of the form $A \vdash B$, where on both sides we have a single formula, then a sequent system may be conceived as a directed graph (i.e. directed multigraph with loops) whose vertices are formulae and where between a pair of vertices A and B there is an *arrow* if and only if $A \vdash B$ is provable in the system. Then, corresponding to different proofs of the sequent $A \vdash B$, we may distinguish between different arrows joining the same formulae A and B. We may introduce terms to code these proofs: atomic terms will correspond to axiomatic sequents and operations on terms to rules of inference. For example, the term $\mathbf{1}_A$ will code the axiomatic structural sequent of identity $A \vdash A$, which we write $\mathbf{1}_A$: $A \vdash A$, and for f: $A \vdash B$ and g: $B \vdash C$, by applying the binary partial operation of composition to g and f we obtain gf: $A \vdash C$. Composition corresponds to a simple form of the structural rule cut. Identity and cut are present in all the substructural logics of section 6.

This way we have come quite close to the mathematical notion of a category. It remains to replace formulae by arbitrary *objects*, to think about connectives as operations on objects, and to impose the categorial equations between arrows:

(1) For f: $A \vdash B$, $f\mathbf{1}_A = f$ and $\mathbf{1}_B f = f$.

(2) For f: $A \vdash B$, g: $B \vdash C$ and h: $C \vdash D$, $h(gf) = (hg)f$.

These equations have a clear proof-theoretical meaning. The equations of (1) correspond to the elimination of cuts with identity axioms, something we find in Gentzen's cut-elimination procedure. The equation (2) corresponds to the permutation of one cut with another cut; in $h(gf)$ we first apply cut with the cut formula B and then with the cut formula C, whereas in $(hg)f$ it is the other way round. Part of Gentzen's cut-elimination procedure consists in permuting cut with other rules, pushing it upwards where it will eventually disappear. Gentzen eschewed permuting cut with other cuts, because he starts, wisely, with a cut above which there are no other cuts. But he may as well have started with an arbitrary cut, which would require a permutation corresponding to (2). The exact moves in Gentzen's cut-elimination procedure are better described with the help of *multicategories.* In multicategories, instead of arrows, we have *multiarrows* of type $\Gamma \vdash A$, where, exactly as in Gentzen's sequents, Γ is a sequence of objects (see [Lambek 1989, 1993]).

In a category or multicategory for a system of sequents we don't have just a consequence *relation,* i.e. a *set of ordered pairs,* but rather an *indexed family of ordered pairs.*

The application of Gentzen's proof theory to category theory, and vice versa, was inaugurated by Lambek in the 1960s, and afterwards pursued mainly by him and his pupils. (Such an endeavour is suggested in [Lawvere 1969], and there is a Russian school of the same inspiration led by Mints since the 1970s; see references in [Lambek & Scott 1986] and [Szabo 1978]. One usually hears about 'categorical' proof theory and 'categorical' logic, rather than 'categorial', but 'categorical theory' is already reserved for a quite different notion in model theory.) Some important sorts of categories — cartesian, cartesian closed and bicartesian closed categories — correspond to fragments of intuitionistic logic. If these categories are formulated equationally, their equations between arrows are related to normalization of proofs in intuitionistic logic. Category theory enables us to get a unifying view of normalization through the concept of adjointness. This concept may explain the success of cut elimination and normalization. Adjoint functors are behind equivalences characterizing logical constants, like those of the preceding section.

The symmetric monoidal closed categories of Saunders Mac Lane [1971, VII], whose roots are in multilinear algebra, correspond to the multiplicative fragment of intuitionistic linear logic. Gentzen's methods were applied in the study of the coherence problem for these categories, i.e. the problem whether all diagrams of a particular sort commute (see [Kelly & Mac Lane 1971]). Quite natural sorts of categories correspond to other substructural logics, too, and clarify the properties of their consequence relations (see [D. & Petrić 1993]).

Natural-deduction proofs may also be coded by typed lambda terms in such a way that lambda conversion corresponds to normalization of proofs. This technique, called the 'formulae-as-types interpretation' or 'Curry-Howard correspondence', is closely related to coding of proofs in categories (see [Lambek & Scott 1986, part I], [Girard, Lafont & Taylor 1989] and [Stenlund 1972]). The connection between proof theory and combinatory logic is also manifested in the correspondence between the structural principles of identity, association, permutation, contraction and thinning, on the one hand, and the combinators I, B, C, W and K, on the other.

All these tools from category theory, the lambda calculus and combinatory logic are well adapted to intuitionistic logic and its substructural fragments, but not to classical logic. This may be explained by repeating something suggested in section 5: the consequence relations of the former logics are syntactical in inspiration, whereas the consequence relation of classical logic is semantical.

9 Proof theory beyond cut elimination

Many authors seem to think that cut elimination is not just a tool, but a goal — in fact, *the* goal of proof theory. This is presumably why they expend considerable effort to prove cut elimination even in cases where in the most natural sequent formulation cut is not eliminable, and where the system in question is already mastered with handier tools. Such is exactly the case with the modal logic S5, whose ordinary plural-sequent formulation has the modal rules

$$(\Box\vdash)\ \frac{A,\ \Gamma\ \vdash\Delta}{\Box A,\ \Gamma\ \vdash\Delta} \qquad\qquad (\vdash\Box)\frac{\Gamma\vdash\Delta,\ A}{\Gamma\vdash\Delta,\ \Box A}$$

provided that in $(\vdash\Box)$ every formula in Γ and every formula in Δ is prefixed with $\Box$. With these rules we have the proof

$$\dfrac{\dfrac{\dfrac{\Box A\ \vdash\Box A}{\vdash\Box A,\ \neg\Box A}}{\vdash\Box A,\ \Box\neg\Box A}\ (\vdash\Box) \qquad (\Box\vdash)\ \dfrac{A\ \vdash A}{\Box A\ \vdash A}}{\vdash A,\ \Box\neg\Box A}\ (\text{cut})$$

and the sequent proved cannot be proved without cut. However, other, very efficient, syntactical and semantical methods, in particular Kripke models, are available for the investigation of S5. It seems that with the help of these methods no important question concerning S5 has been left unanswered. In spite of that, a number of authors have pursued, and are still pursuing, a cut-free formulation of S5, producing sequent systems of a fairly complicated kind.

It is probably more interesting to learn that in the ordinary sequent formulation of S5 cut is not eliminable than to be given another complicated cut-free formulation of this logic. To find that cut is not eliminable in a natural sequent system is not a failure but a discovery. It is also quite interesting that, although in the ordinary sequent formulation of S5 cut cannot be eliminated, the subformula property can still be established (the point is that we can restrict all applications of cut to those where the cut formula is a subformula of the conclusion; see [Takano 1992]; cf. [Fitting 1983, chapter 5, # 8], [Smullyan 1968] and [Mints 1992, Theorem 2, p. 238]).

If the pursuit of cut elimination is not always justified on practical grounds, it can perhaps be justified on theoretical grounds (which are also sometimes aesthetical grounds). One may take that cut elimination and normalization are testimony of a ‘harmony’ between introduction and elimination rules: following Gentzen [1935, section 5.13], the premises of the former should equal in strength the conclusions of the latter (this harmony may be explicated by adjointness, as suggested in the previous section). If a formulation of a system lacks harmony, it would not characterize properly the constants of the system. Jean-Yves Girard has other suggestive things to say about the philosophy of cut elimination. For one, he says that cut-free proofs are for machines, whereas an intelligent being works with cut: he introduces lemmata in his proofs, and lemmata are cut formulae (cf. [1987, p. 95]). With cut, proofs become more manageable and shorter.

Granted that every ‘good’ system should have a cut-free formulation (and is hence ‘good for machines’), we might wish to ascertain that some systems are not good because they cannot have such a formulation. If everything is good, goodness isn’t worth much. However, it is more difficult to establish that a system cannot have a cut-free formulation than to find such a formulation if the system can have it. One can never be sure that one trick or another, perhaps extending the notion of sequent in some complicated way, will not produce the desired cut-free sequent system. There should be limits to what is permited as a sequent system if we want to be able to establish that cut is not always eliminable.

The question is whether there is anything else to do in proof theory besides eliminating cut, or proving related normalization theorems. Such has been the preponderance of this theme in proof theory that one may well doubt that anything else exists. Many authors take cut elimination to be such a bliss that after it is established there is hardly anything else to say (as in fairy tales there is nothing more to say after the young lady got married). Sometimes, but not always, a few corollaries may be drawn: the subformula property, decidability, interpolation... (We have primarily in mind here proof theory of logical systems in the tradition of Gentzen, and not proof theory of formalized fragments of mathematics in the tradition of Hilbert, which concentrates on consistency proofs.) Without pretending that they equal cut elimination in importance, let us mention a few other problems that have been or may be treated by proof-theorists.

There are, first, things with the flavour of universal algebra, like Tarski’s results about **Cn** from the 1930s or Scott’s results about plural consequences mentioned in section 5. (For results of a similar sort about definability and interpolation see [D. & Schroeder-Heister 1985, 1988].) These results don’t depend on cut elimination.

Next, one may produce symmetrical sequent axiomatizations, like those with the equivalences characterizing logical constants in section 7, in which cut is not eliminable. Such axiomatizations should make clearer the articulation of logical systems, as the fact that the same equivalential principles for logical constants can be given for various substructural logics makes clearer the articulation of these logics. When tied to adjointness, this matter has also a categorial side.

It is especially in categorial proof theory that one may expect to find results that go beyond cut elimination and normalization. Composition, the simple form of cut found in categories, is not usually eliminable in categories corresponding to logical systems, but normalization is incorporated in the equations that hold between arrows. So results in categorial proof theory depend on something like cut elimination, but they may give more than what is usually inferred from cut elimination.

One such result is Lambek's *functional completeness* theorem for categories corresponding to fragments of intuitionistic logic (see [Lambek 1974] and [Lambek & Scott 1986, pp. 41-62]). This important theorem, whose applications one may study in [Lambek & Scott 1986], is a refinement of the deduction theorem. Roughly speaking, the deduction theorem says that the system is strong enough to express its own deductive metatheory: to deductions from hypotheses correspond arrows (i.e. sequents or implications) in the system. Functional completeness says that the deductive metatheory can be embedded in the system: there is a structure-preserving one-one map assigning arrows in the system to deductions from hypotheses (cf. [D. 1996]).

Functional completeness for categories corresponding to substructural logics and their modal extensions is treated in [D. & Petrić 1993], which follows [D. 1985, 1992]. In these papers, functional completeness is connected with systems of sequents of higher level. Such sequents, in which on the left-hand side and right-hand side of the turnstile one has sequents, were first introduced to analyze modal operators, by developing an idea of [Scott 1971]. They enable us to get a unique characterization of modal operators like the exponentials of linear logic, i.e. operators of the S4 or S5 type, which moreover may deliver the missing structural principles for formulae prefixed with a modal operator.

Functional completeness is a property of the same kind as what Haskell Brooks Curry called *combinatorial completeness* for systems of combinators; namely, the possibility of defining lambda abstraction in terms of combinators (see [Curry & Feys 1958, pp. 5, 186-194]). To prove versions of this property for categories corresponding to substructural logics (which is analogous to having restricted forms of lambda abstraction in systems of combinators where some of the combinators K, W, C or B may be missing), cartesian categories are presented in [D. & Petrić 1993] in a nonstandard fashion, in which the combinatory 'building blocks' hidden in the usual assumptions about projection arrows and the pairing operation on arrows are exhibited explicitly. (Such a nonstandard, combinatorially dissected, presentation of cartesian categories is used in [D. & Petrić 1994] for another purpose, which is mentioned below.)

It is shown in [D. & Petrić 1993] that assumptions made for cartesian and other categories corresponding to substructural logics are not only sufficient but also necessary for proving functional completeness. So, these categories may be characterized by functional completeness. One needs the same assumptions (in particular, Mac Lane's pentagonal and hexagonal commuting diagrams of natural associativity and commutativity; see [1971, VII.1, p. 158, and VII.7, p. 180]) to prove coherence for symmetric monoidal categories. Such is the case too if we consider coherence for cartesian and other categories corresponding to substructural logics. This leads to the conjecture that coherence and functional completeness are in general properties that entail each other.

As the deduction theorem is refined by functional completeness, so we may refine the *interpolation* theorem in categorial proof theory. If interpolation says that for every arrow $f: A \mid\!- B$ there are arrows $g: A \mid\!- C$ and $h: C \mid\!- B$, the formula C being in the intersection of the languages of A and B, in categories we may wish to have also that $f = hg$. This refinement is proved for bicartesian closed categories, i.e. for intuitionistic propositional logic, in [Čubrić 1994].

Categorial proof theory does not only suggest refinements of old results, like the deduction theorem and interpolation, but also new results, based on concepts that were not previously envisaged in proof theory. Such a concept, interesting both from a logical and philosophical point of view, is *isomorphism* between formulae. A formula A is isomorphic to a formula B if and only if there is a deduction f from A to B and a deduction f^{-1} from B to A such that f composed with f^{-1} reduces via normalization to the identity deduction $\mathbf{1}_A$ from A to A and f^{-1} composed with f reduces via normalization to the identity deduction $\mathbf{1}_B$ from B to B (these reductions are represented in categories by equations between arrows). That two formulae are isomorphic is equivalent to the assertion that the deductions involving one of them, either as premise or as conclusion, can be extended to deductions where this formula is replaced by the other, the deductions involving the first formula being in one-one correspondence with the deductions involving the second. Roughly speaking, whatever you can do in deductions with one of these formulae you can do as well with the other. Isomorphism is an equivalence relation stronger than the usual mutual implication. So, for example, $C \wedge D$ is isomorphic to $D \wedge C$ in intuitionistic logic, while $C \wedge C$ only implies and is implied by C, but is not isomorphic to it. (The problem is that the composition of the deduction from $C \wedge C$ to C, which is justified by either 'from $C \wedge D$ infer C' or 'from

$D \wedge C$ infer C', with the deduction from C to $C \wedge C$ does not reduce via normalization to the identity deduction $\mathbf{1}_{C \wedge C}$ from $C \wedge C$ to $C \wedge C$.)

It seems reasonable to suppose that isomorphism analyzes propositional identity, i.e. identity of meaning for propositions: A and B stand for the same proposition, i.e., A means the same as B, if and only if A is isomorphic to B. This way we would base propositional identity upon identity of deductions, which is codified by equality between arrows.

Isomorphic formulae in the implication-conjunction fragment of intuitionistic logic, i.e. objects isomorphic in all cartesian closed categories, have been characterized in [Soloviev 1981] via the axiomatization of equations with exponentiation, multiplication and one, true for natural numbers. This fragment of arithmetic (upon which one comes in connection with Tarski's 'high-school algebra' problem) is finitely axiomatizable and decidable. (For related results about this and some other fragments of intuitionistic logic — in particular, the implication-conjunction fragment of second-order intuitionistic propositional logic — see the survey [Di Cosmo 1995] and references therein; [Soloviev 1993] and [D. & Petrić 1995] deal with the analogous product-implication fragment of linear logic, including the constant unit proposition **t**, i.e. with characterizing objects isomorphic in all symmetric monoidal closed categories.) The problem of characterizing isomorphic formulae in bigger fragments of intuitionistic first-order logic — in particular, in the whole intuitionistic propositional calculus, which corresponds to bicartesian closed categories — seems to be still open.

Another isomorphism problem is to characterize all arrows that are isomorphisms. This has been done in [D. & Petrić 1994] for the cartesian category that corresponds to the conjunctive fragment of intuitionistic logic (including the constant true proposition **T**, which in intuitionistic logic is indistinguishable from **t**, as conjunction $\wedge$ is indistinguishable from the product $\bullet$; this fragment of intuitionistic logic coincides with the corresponding singular-consequence fragment of classical logic). The isomorphisms of this category make exactly the symmetric monoidal category that corresponds to the product fragment of linear logic (including the constant unit proposition **t**). A procedure similar to cut elimination, in which we eliminate contraction and thinning, serves for deciding whether an arrow of the cartesian category is an isomorphism. All this may be taken as a justification of linear logic: it motivates the rejection of contraction and thinning by the wish to keep only structural principles that don't spoil isomorphism between premises (conjunction and product serve to join premises, while **T** and **t** replace the empty collection of premises).

By replacing talk about provable sequents with talk about arrows, categorial proof theory makes more tangible the things studied by proof theory since Gentzen's time. In a categorial setting we can express precisely and succinctly when two deductions are equal. This equality of deductions, which is a restatement of the reductions of cut elimination and normalization, happens to be related to important mathematical concepts, like adjointness. In categorial proof theory we are given more than a powerful technique like cut elimination: we are also led to find an explanation why this technique works. According to Descartes (*Regulae* XVI, 458), a mathematician differs from a computer by not being content with efficiency. The goal of mathematics is not power, but understanding and beauty — power is only a by-product.

10 Consequence in the philosophy of language

Philosophy is as much concerned with understanding as mathematics. To reach this goal in its proper domain, twentieth-century philosophy has concentrated upon language, and has selected for particular attention two essential activities of language: referring and asserting. Gottlob Frege and Ludwig Wittgenstein found that when it comes to explaining how language functions, asserting is more basic than referring; i.e., sentences are more basic than nouns. A tangible reflection of that in logic is that the propositional calculus precedes the predicate calculus.

Philosophy has paid much less attention to the third essential activity of language — deducing, though the greater part of philosophy, a great part of mathematics and a considerable part of science consist in this activity. It is hard to overestimate the influence of Frege and Wittgenstein in this matter. Michael Dummett, inspired by the natural-deduction style of introduction and elimination rules, has tried stressing the role of deduction in explaining key features of language, such as meaning (see [Dummett 1975]). However, a much more typical attitude in the philosophy of language of the twentieth century is exemplified by Willard Van Orman Quine's 'Two Dogmas' picture of language of [1951], which doesn't say much about deduction. Quine put logic in a body of sentences in the inner regions of language, far from the periphery of empirical sentences, rather than in the links of deduction, which bind sentences together in all regions of language and are not on the same level as these sentences. (In this respect Quine resembles Carroll's Tortoise.)

It seems that philosophy has not yet fully exploited ideas suggested by proof theory done in the footsteps of Gentzen. We have come upon a rather concrete suggestion of this kind in the previous section when we considered the analysis of propositional identity in terms of isomorphism. A less concrete suggestion may come from realizing that the part of logic that deals only with structural principles of deduction precedes both the propositional and predicate calculus. We might then be lead to speculate about giving deducing precedence over all other activities of language, or at least over asserting and referring. (Identity of deductions precedes identity of propositions in the analysis of the latter notion in terms of isomorphism; i.e., deducing precedes asserting.) This would be a precedence in the order of explaining language and not in the order of learning language (as the precedence of asserting over referring should not mean that we learn to assert before we learn to refer when we learn to speak). In such a philosophical conception, consequence relations would be more basic than sentences and nouns for explaining how language functions.

It seems indisputable that, in the order of explaining, a science that seeks laws accounting for phenomena has precedence over a taxonomical science (for example, physiology precedes anatomy, and theoretical linguistics precedes descriptive linguistics). This precedence is parallel to the precedence of asserting over referring. The precedence of logic over all sciences, championed by many philosophers, and in particular the precedence of logic over the rest of mathematics in foundational studies (which is not the same as the logicists' endeavour to reduce mathematics to logic) are parallel to the precedence of deducing over asserting, granted that, in contrast to other bodies of knowledge, logic draws its essence from deduction.

References

Bernays, P. [1965] Betrachtungen zum Sequenzen-Kalkul, in: A.-T. Tymieniecka ed., *Contributions to Logic and Methodology in Honor of J.M. Bocheński*, North-Holland, Amsterdam, pp. 1-44.

Boričić, B. [1985] On sequence-conclusion natural deduction, *Journal of Philosophical Logic* vol. 14, pp. 359-377.

Carnap, R. [1943] *Formalization of Logic*, Harvard University Press, Cambridge, Mass.

Carroll, L. [1895] What the tortoise said to Achilles, *Mind* N.S. vol. 4, pp. 278-280.

Celluci, C. [1992] Existential instantiation and normalization in sequent natural deduction, *Annals of Pure and Applied Logic* vol. 58, pp. 111-148.

Čubrić, D. [1994] Interpolation for bicartesian closed categories, *Archive for Mathematical Logic* vol. 33, pp. 291-319.

Curry, H.B., and Feys, R. [1958] *Combinatory Logic, Vol. I*, North-Holland, Amsterdam.

Di Cosmo, R. [1995] *Isomorphisms of Types*: *From λ-Calculus to Information Retrieval and Language Design*, Birkhäuser, Boston.

Došen, K. [1985] Sequent systems for modal logic, *The Journal of Symbolic Logic* vol. 50, pp. 149-168.

___ [1988] Sequent systems and groupoid models, I, *Studia Logica* vol. 47, pp. 353-385 (Addenda and corrigenda, *Studia Logica* vol. 49, 1990, p. 614).

___ [1989] Logical constants as punctuation marks, *Notre Dame Journal of Formal Logic* vol. 30, pp. 362-381 (slightly amended version in: D.M. Gabbay ed., *What is a Logical System?*, Oxford University Press, Oxford, 1994, pp. 273-296).

___ [1992] Modal logic as metalogic, *Journal of Logic, Language and Information* vol. 1, pp. 173-201.

___ [1994] On passing from singular to plural consequences, in: E. Orlowska ed., *Logic at Work: Essays Dedicated to the Memory of Helena Rasiowa*, Physica-Verlag, Heidelberg, 1999, pp. 533-547.

___ [1996] Deductive completeness, *The Bulletin of Symbolic Logic* vol. 2, pp. 243-283, Errata, *ibid.* p. 523.

Došen, K., and Petrić, Z. [1993] Modal functional completeness, in: H. Wansing ed., *Proof Theory of Modal Logic*, Kluwer, Dordrecht, 1996, pp. 167-211.

___ [1994] Cartesian isomorphisms are symmetric monoidal: A justification of linear logic, *The Journal of Symbolic Logic* vol. 64 (1999), pp. 227-242.

___ [1995] Isomorphic objects in symmetric monoidal closed categories, *Mathematical Structures in Computer Science* vol. 7 (1997), pp. 639-662.

Došen, K., and Schroeder-Heister, P. [1985] Conservativeness and uniqueness, *Theoria* vol. 51, pp. 159-173 (Errata, *Theoria* vol. 52, 1986, p. 127).

___ [1988] Uniqueness, definability and interpolation, *The Journal of Symbolic Logic* vol. 53, pp. 554-570.

___, eds [1993] *Substructural Logics*, Oxford University Press, Oxford.

Dummett, M.A.E. [1975] The philosophical basis of intuitionistic logic, in: H.E. Rose and J.C. Shepherdson eds, *Logic Colloquium '73*, North-Holland, Amsterdam, pp. 5-40.

Fitting, M. [1983] *Proof Methods for Modal and Intuitionistic Logics*, Reidel, Dordrecht.

Gentzen, G. [1935] Untersuchungen über das logische Schließen, *Mathematische Zeitschrift* vol. 39, pp. 176-210, 405-431 (English translation in *The Collected Papers of Gerhard Gentzen*, M.E. Szabo ed., North-Holland, Amsterdam, 1969).

Girard, J.-Y. [1987] *Proof Theory and Logical Complexity, Vol. I*, Bibliopolis, Naples.

Girard, J.-Y., Lafont, Y., and Taylor, P. [1989] *Proofs and Types*, Cambridge University Press, Cambridge.

Hösli, B., and Jäger, G. [1994] About some symmetries of negation, *The Journal of Symbolic Logic* vol. 59, pp. 473-485.

Kelly, G.M., and Mac Lane, S. [1971] Coherence in closed categories, *Journal of Pure and Applied Algebra* vol. 1, pp. 97-140.

Kneale, W. [1956] The province of logic, in: H.D. Lewis ed., *Contemporary British Philosophy*, Allen and Unwin, London, pp. 237-261.

Kneale, W., and Kneale, M. [1962] *The Development of Logic*, Oxford University Press, Oxford.

Lambek, J. [1974] Functional completeness of cartesian categories, *Annals of Mathematical Logic* vol. 6, pp. 259-292.

___ [1989] Multicategories revisited, in: J.W. Gray and A. Scedrov eds, *Categories in Computer Science and Logic*, American Mathematical Society, Providence, pp. 217-239.

___ [1993] Logic without structural rules (Another look at cut elimination), in [Došen & Schroeder-Heister 1993], pp. 179-206.

Lambek, J., and Scott, P.J. [1986] *Introduction to Higher-Order Categorical Logic*, Cambridge University Press, Cambridge.

Lawvere, F.W. [1969] Adjointness in foundations, *Dialectica* vol. 23, pp. 281-296.

Mac Lane, S. [1971] *Categories for the Working Mathematician*, Springer, Berlin.

Mints, G. [1992] Lewis' systems and system T (1965-1973), in *Selected Papers in Proof Theory*, Bibliopolis, Naples, pp. 221-294 (English translation, with additions, of a paper published in Russian as an appendix to the Russian translation of R. Feys, *Modal Logic*, Nauka, Moscow, 1974, pp. 422-509).

Pottinger, G. [1977] Normalization as a homomorphic image of cut elimination, *Annals of Mathematical Logic* vol. 12, pp. 323-357.

Prawitz, D. [1965] *Natural Deduction: A Proof-Theoretical Study*, Almqvist & Wiksell, Stockholm.

___ [1971] Ideas and results in proof theory, in: J.E. Fenstad ed., *Proceedings of the Second Scandinavian Logic Symposium*, North-Holland, Amsterdam, pp. 235-307.

Quine, W.V.O. [1951] Two dogmas of empiricism, *Philosophical Review* vol. 60, pp. 20-43.

Scott, D.S. [1971] On engendering an illusion of understanding, *The Journal of Philosophy* vol. 68, pp. 787-807.

___ [1974] Completeness and axiomatizability in many-valued logic, in: L. Henkin et al. eds, *Proceedings of the Tarski Symposium*, American Mathematical Society, Providence, pp. 411-435.

Shoesmith, D.J., and Smiley, T.J. [1978] *Multiple-Conclusion Logic*, Cambridge University Press, Cambridge.

Smullyan, R.M. [1968] Analytic cut, *The Journal of Symbolic Logic* vol. 33, pp. 560-564.

Soloviev, S.V. [1981] The category of finite sets and cartesian closed categories (in Russian), *Zapiski nauchnykh seminarov LOMI* vol. 105, pp. 174-194 (English translation in *Journal of Soviet Mathematics* vol. 22, 1983, pp. 1387-1400).

___ [1993] A complete axiom system for isomorphism of types in closed categories, in: A. Voronkov ed., *Logic Programming and Automated Reasoning*, Lecture Notes in Artificial Intelligence no 698, Springer, Berlin, pp. 360-371.

Stenlund, S. [1972] *Combinators, Lambda Terms and Proof Theory*, Reidel, Dordrecht.

Szabo, M.E. [1978] *Algebra of Proofs*, North-Holland, Amsterdam.

Takano, M. [1992] Subformula property as a substitute for cut elimination in modal propositional logics, *Mathematica Japonica* vol. 37, pp. 1129-1145.

Tarski, A. [1930] Über einige fundamentale Begriffe der Metamathematik, *Comptes Rendus des Séances de la Société des Sciences et des Lettres de Varsovie* vol. 23, pp. 22-29 (English translation in [Tarski 1956]).

___ [1930a] Fundamentale Begriffe der Methodologie der deduktiven Wissenschaften, I, *Monatshefte für Mathematik und Physik* vol. 37, pp. 361-404 (English translation in [Tarski 1956]).

___ [1936] On the concept of logical consequence (in Polish), *Przegląd Filozoficzny* vol. 39, pp. 58-68 (English translation in [Tarski 1956]).

___ [1956] *Logic, Semantics, Metamathematics: Papers from 1923 to 1938*, J.H. Woodger ed., Oxford University Press, Oxford (second edition, J. Corcoran ed., Hackett, Indianapolis, 1983).

Wójcicki, R. [1988] *Theory of Logical Calculi: Basic Theory of Consequence Operations*, Kluwer, Dordrecht.

Zucker, J. [1974] The correspondence between cut elimination and normalization, *Annals of Mathematical Logic* vol. 7, pp. 1-156.